# Non self-adjoint idempotents in $C^*$- and $JB^*$-algebras

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Abstract. We prove that, if a $JB^*$-algebra contains a non self-adjoint idempotent, then it also contains a nonzero self-adjoint idempotent. This is achieved through an "almost description" of $C^*$- and $JB^*$-algebras generated by a non self-adjoint idempotent.

## 1. Introduction

It is well-known that, if a $C^*$-algebra contains a non self-adjoint idempotent, then it also contains a nonzero self-adjoint idempotent (see [**3, 5**]). In the more general case of $JB^*$-algebras, a similar result seems to be previously unknown. As a matter of fact, although the $JB^*$-algebra generated by a non self-adjoint idempotent can be seen as a closed $*$-invariant Jordan subalgebra of a suitable $C^*$-algebra, the nonzero self-adjoint idempotents built by the associative methods need not lie in the given $JB^*$-algebra. Nevertheless, by introducing new techniques, we prove in this paper that, in fact, $JB^*$-algebras containing non self-adjoint idempotents also contain nonzero self-adjoint idempotents. The key tool is an "almost description" of $C^*$- and $JB^*$-algebras generated by a non self-adjoint idempotent, which is summarized in what follows.

Let $A$ be a $C^*$-algebra containing a non self-adjoint idempotent $e$. We show in Corollary 2.3 that $K := sp(\sqrt{e^*e}) \setminus \{0\}$ is a compact subset of $[1, \infty[$ whose maximum element is greater than 1, and that, in general, no more can be said about $K$. Then we construct a Banach $*$-algebra $\mathcal{A}(K)$, which consists of all $2 \times 2$ matrices over $C(K)$ with an unusual multiplication, and has a distinguished non self-adjoint idempotent $p$, and prove in Theorem 2.6 the existence of a unique continuous $*$-homomorphism $F : \mathcal{A}(K) \to A$ such that $F(p) = e$. As a consequence, a $C^*$-algebra contains a non self-adjoint idempotent if and only if it contains a non central self-adjoint idempotent (Corollary 2.7).

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As a transition between the $C^*$- and the $JB^*$- case, we note that, for an element $a$ in a $C^*$-algebra $A$, $sp(\sqrt{a^*a}) \setminus \{0\}$ can be determined in terms of the $JB^*$-algebra underlying $A$, and, even more, in terms of the $JB^*$-triple underlying $A$. Indeed, $sp(\sqrt{a^*a}) \setminus \{0\}$ coincides with the "triple spectrum" $\sigma(a)$ of $a$ (Lemma 3.1).

Now, let $J$ be a $JB^*$-algebra containing a non self-adjoint idempotent $e$. We prove in Theorem 3.4 that $K := \sigma(e)$ is a compact subset of $[1, \infty[$ whose maximum element is greater than 1, and that, for a canonical closed $*$-invariant Jordan subalgebra $\mathcal{J}(K)$ of $\mathcal{A}(K)$ containing the distinguished non self-adjoint idempotent $p \in \mathcal{A}(K)$, there exists a unique continuous $*$-homomorphism $G : \mathcal{J}(K) \to J$ such that $G(p) = e$. As a consequence, a $JB^*$-algebra contains a non self-adjoint idempotent if and only if it contains a non central self-adjoint idempotent (Corollary 3.5).

## 2. The case of $C^*$-algebras

Let $A$ be a $C^*$-algebra. In the case that $A$ has not a unit, we denote by $A_1$ the $C^*$-algebra obtained by adjoining a unit to $A$. Otherwise, we put $A_1 := A$. As usual, for $a \in A$, we define the spectrum of $a$ as the nonempty compact subset $sp(a)$ of $\mathbb{C}$ given by

$$sp(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not invertible in } A_1\}.$$

The following lemma exploits some ideas in page 28 of [**10**].

Lemma 2.1. *Let $A$ be a $C^*$-algebra, and let $e$ be a non self-adjoint idempotent in $A$. Then $sp(i(e - e^*))$ is a symmetric subset of the real line, and the mapping $\lambda \to 1 + \lambda^2$ becomes a surjection from $sp(i(e - e^*)) \setminus \{0\}$ onto $sp(e^*e) \setminus \{0, 1\}$. Consequently, we have:*

(1) $\|e\|^2 = 1 + \|e - e^*\|^2$.

(2) $\{0, \|e\|^2\} \subseteq sp(e^*e) \subseteq \{0\} \cup [1, \|e\|^2]$.

Proof. A straightforward computation shows that, for $\lambda \in \mathbb{C}$, we have

$$\lambda(1 + \lambda^2)[i(e - e^*) - \lambda] = (e^* - i\lambda)[e^*e - (1 + \lambda^2)](e + i\lambda).$$

On the other hand, if $\lambda$ is in $\mathbb{C} \setminus \{0, i, -i\}$, then $(e^* - i\lambda)$ and $(e + i\lambda)$ are invertible in $A_1$, and we have $\lambda(1 + \lambda^2) \neq 0$. It follows that, for such a $\lambda$, $i(e - e^*) - \lambda$ is invertible in $A_1$ if and only if so is $e^*e - (1 + \lambda^2)$. Now, keeping in mind that $sp(i(e-e^*))$ (respectively, $sp(e^*e)$) consists only of real (respectively, nonnegative real) numbers, we easily derive that $sp(i(e - e^*))$ is symmetric (relative to zero), and that the mapping $\lambda \to 1 + \lambda^2$ is a surjection from $sp(i(e - e^*)) \setminus \{0\}$ onto $sp(e^*e) \setminus \{0, 1\}$. The consequences, listed in the statement, are obvious. ■

The following corollary is well-known (see [**3, 5**]).

Corollary 2.2. *Let $A$ be a $C^*$-algebra, and let $e$ be a non self-adjoint idempotent in $A$. Then there exists a self-adjoint idempotent $p \in A$ such that $ep = e$.*

Proof. By Lemma 2.1, zero is an isolated point of $sp(e^*e)$, and hence the function $\chi : sp(e^*e) \to \mathbb{C}$, defined by $\chi(0) := 0$ and $\chi(t) := 1$ for $t \in sp(e^*e) \setminus \{0\}$, is continuous. Now $p := \chi(e^*e)$ is a self-adjoint idempotent in $A$ satisfying $e^*ep = e^*e \Rightarrow e^*e(1-p) = 0 \Rightarrow (1-p)e^*e(1-p) = 0 \Rightarrow e(1-p) = 0 \Rightarrow ep = p$. ∎

We will see in Corollary 2.7 below that the self-adjoint idempotent $p$ in the above proof is in fact non central.

Corollary 2.3. *For a subset $K$ of the complex plane, the following assertions are equivalent:*

1. *$K$ is a compact subset of $[1, \infty[$ whose maximum element is greater than 1.*
2. *There exists a $C^*$-algebra $A$, and a non self-adjoint idempotent $e \in A$, such that $sp(\sqrt{e^*e}) \setminus \{0\} = K$.*

Proof. (2) ⇒ (1).- By Lemma 2.1

(1) ⇒ (2).- Assume that (1) holds. Let $A$ denote the $C^*$-algebra of all continuous functions from $K$ to the $C^*$-algebra $M_2(\mathbb{C})$ (of all $2 \times 2$ matrices with entries in $\mathbb{C}$), and let $e$ stand for the element of $A$ defined by

$$e(t) := \begin{pmatrix} 1 & \sqrt{t^2 - 1} \\ 0 & 0 \end{pmatrix}$$

for every $t \in K$. Then, for $t \in K$, $e(t)$ is an idempotent in $M_2(\mathbb{C})$ different from 0 and 1. Moreover, since $e(t)e(t)^* = t^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we have $\|e(t)\| = t$, and hence $sp(\sqrt{e(t)^*e(t)}) \setminus \{0\} = \{t\}$. It follows that $e$ is a non self-adjoint idempotent of $A$ satisfying $sp(\sqrt{e^*e}) \setminus \{0\} = K$. ∎

Let $K$ be a compact subset of $\mathbb{C}$. We denote by $C(K)$ the $C^*$-algebra of all continuous complex valued functions on $K$. In the case that $0 \in K$, we denote by $C_0(K)$ the closed ideal of $C(K)$ consisting of those functions $f \in C(K)$ satisfying $f(0) = 0$.

Lemma 2.4. *Let $A$ be a $C^*$-algebra, and let $a$ be in $A$ such that $0 \in sp(a^*a)$. Then there exists a unique linear isometry*

$$\Phi : C_0(sp(\sqrt{a^*a})) \to A$$

*satisfying $\Phi(f) = ah(\sqrt{a^*a})$ for those $f \in C_0(sp(\sqrt{a^*a}))$ for which there exists $h \in C(sp(\sqrt{a^*a}))$ such that $f(t) = th(t)$ for every $t \in sp(\sqrt{a^*a})$. Moreover, for $f, g \in C_0(sp(\sqrt{a^*a}))$, we have*

$$\Phi(f)\Phi(\overline{g})^* = f(\sqrt{aa^*})g(\sqrt{aa^*}), \ \Phi(\overline{f})^*\Phi(g) = f(\sqrt{a^*a})g(\sqrt{a^*a}),$$

$$\Phi(f)g(\sqrt{a^*a}) = \Phi(fg), \ \textit{and} \ g(\sqrt{aa^*})\Phi(f) = \Phi(gf).$$

Proof. The first conclusion in the statement is nothing other than Lemma 8 of [**6**]. In view of the Stone-Weierstrass theorem, to prove the equality $\Phi(f)\Phi(\overline{g})^* = f(\sqrt{aa^*})g(\sqrt{aa^*})$ for $f, g \in C_0(sp(\sqrt{a^*a}))$, we can assume that $f$ and $g$ are of the form $t \to tP(t^2)$ and $t \to tQ(t^2)$, for suitable complex polynomials $P$ and $Q$, respectively. Then we have

$$\Phi(f)\Phi(\overline{g})^* = aP(a^*a)Q(a^*a)a^* = aa^*P(aa^*)Q(aa^*) = f(\sqrt{aa^*})g(\sqrt{aa^*}),$$

as desired. The proof of the equality $\Phi(\overline{f})^*\Phi(g) = f(\sqrt{a^*a})g(\sqrt{a^*a})$ is similar. To realize that $\Phi(f)g(\sqrt{a^*a}) = \Phi(fg)$ and $g(\sqrt{aa^*})\Phi(f) = \Phi(gf)$, take $f$ of the form $t \to tP(t^2)$ for a complex polynomial $P$, and $g$ of the form $t \to Q(t^2)$, for a complex polynomial $Q$ with $Q(0) = 0$. ■

Let $K$ be a compact subset of $[1, \infty[$. Let $u$ stand for the element of $C(K)$ defined by $u(t) := t$ for every $t \in K$. We denote by $\mathcal{A}(K)$ the complex Banach $*$-algebra whose vector space is that of all $2 \times 2$ matrices with entries in $C(K)$, whose (bilinear) product is determined by the equalities $(f[ij])(g[kl]) := (fg)[il]$ if $j = k$ and $(f[ij])(g[kl]) := (u^{-1}fg)[il]$ if $j \neq k$, whose norm is given by $\|(f_{ij})\| := \|f_{11}\| + \|f_{12}\| + \|f_{21}\| + \|f_{22}\|$, and whose (conjugate-linear) involution $*$ is determined by $(f[ij])^* := \overline{f}[ji]$. Here, as usual, for $f \in C(K)$ and $i, j \in \{1, 2\}$, $f[ij]$ means the matrix having $f$ in the $(i, j)$-position and 0's elsewhere. For later computations, it is useful to see $\mathcal{A}(K)$ as a $C(K)$-module in the natural manner, namely by defining the product of a function $f \in C(K)$ and a matrix $(f_{ij}) \in \mathcal{A}(K)$ by $f(f_{ij}) := (ff_{ij})$. In this regarding, we straightforwardly realize that $\mathcal{A}(K)$ becomes in fact an algebra over $C(K)$, i.e., the operators of left and right multiplication by arbitrary elements of $\mathcal{A}(K)$ are $C(K)$-module homomorphisms. Moreover, the symbol $f[ij]$ can now be read as the product of the function $f \in C(K)$ and the matrix $[ij] \in \mathcal{A}(K)$, where, for $i, j \in \{1, 2\}$, $[ij]$ stands for the matrix having the constant function equal to one in the $(i, j)$-position and 0's elsewhere.

Lemma 2.5. *Let $K$ be a compact subset of $[1, \infty[$, and let $u$ stand for the element of $C(K)$ defined by $u(t) := t$ for every $t \in K$. Then $\mathcal{A}(K)$ is generated by $u[21]$ as a Banach $*$-algebra.*

Proof. Put $p := u[21]$, and let $C$ denote the closed $*$-invariant subalgebra of $\mathcal{A}(K)$ generated by $p$. We have $u^2[11] = p^*p \in C$. Therefore, since $C(K)$ is bicontinuously algebra-isomorphic to $C(K)[11]$ by means of the mapping $f \to f[11]$, and $C(K)$ is generated by $u^2$ as a Banach algebra, we obtain that $C(K)[11] \subseteq C$, and hence that

$$C(K)[21] = uC(K)[21] = (u[21])(C(K)[11]) = p(C(K)[11]) \subseteq C.$$

Starting with the fact $u^2[22] = pp^* \in C$, a similar argument shows that $C(K)[22]$ and $C(K)[12]$ are contained in $C$. It follows that $\mathcal{A}(K) = C$. ■

Now, we are ready to prove the main result in this section.

Theorem 2.6. *Let $A$ be a $C^*$-algebra, and let $e$ be a non self-adjoint idempotent in $A$. Then $K := sp(\sqrt{e^*e}) \setminus \{0\}$ is a compact subset of $[1,\infty[$ whose maximum element (namely $\|e\|$) is grater than 1, and there exists a unique continuous $*$-homomorphism $F : \mathcal{A}(K) \to A$ such that $F(u[21]) = e$, where $u$ stands for the function $t \to t$ from $K$ to $\mathbb{C}$. Moreover we have:*

1. *The closure in $A$ of the range of $F$ coincides with the $C^*$-subalgebra of $A$ generated by $e$.*
2. *$F$ is injective if and only if either 1 does not belong to $K$ or 1 is an accumulation point of $K$.*
3. *If 1 is an isolated point of $K$, then $\ker(F)$ consists precisely of those matrices $(f_{ij}) \in \mathcal{A}(K)$ which vanish at every $t \in K \setminus \{1\}$ and satisfy*

$$f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1) = 0.$$

Proof. By Corollary 2.3, we have that $K := sp(\sqrt{e^*e}) \setminus \{0\}$ is a compact subset of $[1,\infty[$ whose maximum element (namely $\|e\|$) is grater than 1. As a consequence, the $C^*$-algebras $C_0(sp(\sqrt{e^*e}))$ and $C(K)$ can and will be identified in an obvious way. Let $\Phi : C(K) \to A$ be the linear isometry given by Lemma 2.4 when we take in such a lemma $a := e$. For $i, j \in \{1, 2\}$, consider the linear isometry $\Phi_{ij} : C(K) \to A$ defined, for $f \in C(K)$, by

$$\Phi_{11}(f) := f(\sqrt{e^*e}),\ \Phi_{22}(f) := f(\sqrt{ee^*}),\ \Phi_{21}(f) := \Phi(f),\ \Phi_{12}(f) := \Phi(\overline{f})^*.$$

We claim that, for $f, g \in C(K)$ and $i, j, k, l \in \{1, 2\}$, we have

$$\Phi_{ij}(f)\Phi_{kl}(g) = \Phi_{il}(fg) \text{ if } j = k, \text{ and } \Phi_{ij}(f)\Phi_{kl}(g) = \Phi_{il}(u^{-1}fg) \text{ if } j \neq k.$$

Indeed, the equality $\Phi_{ij}(f)\Phi_{kl}(g) = \Phi_{il}(fg)$ for $j = k$ follows from Lemma 2.4. To realize that $\Phi_{ij}(f)\Phi_{kl}(g) = \Phi_{il}(u^{-1}fg)$ if $j \neq k$, keep in mind that $e$ is an idempotent, and take $f$ (respectively $g$) of the form $t \to P(t^2)$ for a complex polynomial $P$ with $P(0) = 0$, if $i = j$ (respectively, $k = l$), and of the form $t \to tP(t^2)$ for some complex polynomial $P$, otherwise. Now that the claim is proved, it is clear that the mapping $F : \mathcal{A}(K) \to A$ defined by

$$F((f_{ij})) := \Phi_{11}(f_{11}) + \Phi_{12}(f_{12}) + \Phi_{21}(f_{21}) + \Phi_{22}(f_{22})$$

becomes a continuous $*$-homomorphism satisfying $F(u[21]) = e$. Moreover, both the uniqueness of $F$ under the above conditions, and that the closure in $A$ of the range of $F$ coincides with the $C^*$-subalgebra of $A$ generated by $e$, follow from Lemma 2.5.

Let $(f_{ij})$ be in $\mathcal{A}(K)$. Then we have:

$$[11](f_{ij})[11] = (f_{11} + u^{-1}f_{12} + u^{-1}f_{21} + u^{-2}f_{22})[11],$$

$$[12](f_{ij})[12] = (u^{-1}f_{11} + u^{-2}f_{12} + f_{21} + u^{-1}f_{22})[12],$$

$$[21](f_{ij})[21] = (u^{-1}f_{11} + f_{12} + u^{-2}f_{21} + u^{-1}f_{22})[21],$$

$$[22](f_{ij})[22] = (u^{-2}f_{11} + u^{-1}f_{12} + u^{-1}f_{21} + f_{22})[22].$$

Assume that $(f_{ij})$ is in $\ker(F)$. Then, since $\ker(F)$ is an ideal of $\mathcal{A}(K)$, and, for all $i,j \in \{1,2\}$, the restriction of $F$ to $C(K)[ij]$ is an isometry, we deduce:

$$\begin{aligned} f_{11} + u^{-1}f_{12} + u^{-1}f_{21} + u^{-2}f_{22} &= 0,\\ u^{-1}f_{11} + u^{-2}f_{12} + f_{21} + u^{-1}f_{22} &= 0,\\ u^{-1}f_{11} + f_{12} + u^{-2}f_{21} + u^{-1}f_{22} &= 0,\\ u^{-2}f_{11} + u^{-1}f_{12} + u^{-1}f_{21} + f_{22} &= 0. \end{aligned}$$

Therefore, for every $t \in K$ we have:

$$\begin{aligned} t^2 f_{11}(t) + tf_{12}(t) + tf_{21}(t) + f_{22}(t) &= 0,\\ tf_{11}(t) + f_{12}(t) + t^2 f_{21}(t) + tf_{22}(t) &= 0,\\ tf_{11}(t) + t^2 f_{12}(t) + f_{21}(t) + tf_{22}(t) &= 0,\\ f_{11}(t) + tf_{12}(t) + tf_{21}(t) + t^2 f_{22}(t) &= 0. \end{aligned}$$

As a first consequence, if 1 belongs to $K$, then

$$f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1) = 0.$$

On the other hand, keeping in mind that, for $t \in K \setminus \{1\}$, we have

$$\begin{vmatrix} t^2 & t & t & 1\\ t & 1 & t^2 & t\\ t & t^2 & 1 & t\\ 1 & t & t & t^2 \end{vmatrix} = -(t^2-1)^4 \neq 0,$$

for such a $t$ we deduce

$$f_{11}(t) = f_{12}(t) = f_{21}(t) = f_{22}(t) = 0.$$

Therefore, if either 1 does not belong to $K$ or 1 is an accumulation point of $K$, then

$$f_{11} = f_{12} = f_{21} = f_{22} = 0.$$

Thus $F$ is injective when either 1 does not belong to $K$ or 1 is an accumulation point of $K$.

Assume that 1 is an isolated point of $K$. Then the function $\chi : K \to \mathbb{C}$, defined by $\chi(1) := 1$ and $\chi(t) := 0$ for $t \in K \setminus \{1\}$, is continuous. Put $p := F(\chi[11])$, $q := F(\chi[22])$, and $r := F(\chi[12])$. Since, for $i,j,k,l \in \{1,2\}$ the equalities $(\chi[ij])^* = \chi[ji]$ and $(\chi[ij])(\chi[kl]) = \chi[il]$ hold, we have that $p$ and $q$ are self-adjoint idempotents of $A$ satisfying $pqp = p$ (equivalently, $p \leq q$) and $qpq = q$ (equivalently, $q \leq p$), and that $pq = r$. It follows

$$p = q = r = r^*.$$

Let $(f_{ij})$ be in $\mathcal{A}(K)$ vanishing at every $t \in K \setminus \{1\}$ and such that $f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1) = 0$. Then we have

$$(f_{ij}) = f_{11}(1)(\chi[11]) + f_{12}(1)(\chi[12]) + f_{21}(1)(\chi[21]) + f_{22}(1)(\chi[22]),$$

and hence

$$F((f_{ij})) = (f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1))p = 0.$$

■

Corollary 2.7. *Let $A$ be a $C^*$-algebra. Then the following assertions are equivalent:*

(1) *For every $s \in [1, \infty[$ there exists an idempotent $e \in A$ such that $\|e\| = s$.*
(2) *There exists a non self-adjoint idempotent in $A$.*
(3) *There exists a non central self-adjoint idempotent in $A$.*

Proof. The implication (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3).- Let $e$ be the non self-adjoint idempotent of $A$ whose existence is assumed. Let $K$ and $F : \mathcal{A}(K) \to A$ be the compact set and the $*$-homomorphism, respectively, given by Theorem 2.6. Put $p := [11] \in \mathcal{A}(K)$ and $q := [12] \in \mathcal{A}(K)$. Then $p$ is a self-adjoint idempotent, and we have $pq - qp = [12] - u^{-1}[11]$, where $u$ stands for the function $t \to t$ from $K$ to $\mathbb{C}$. Noticing that, by Theorem 2.6, $pq - qp$ does not belong to $\ker(F)$, it follows that $F(p)$ is a non central self-adjoint idempotent of $A$.

(3) $\Rightarrow$ (1).- Let $e$ be the non central self-adjoint idempotent of $A$ whose existence is assumed. Take $a \in A$ with $ea - ae \neq 0$. Then the mapping $D : A \to A$ defined by $D(b) := ba - ab$ for every $b \in A$ becomes a continuous derivation such that $D(e) \neq 0$. Since, for $z \in \mathbb{C}$, $\exp(zD)$ is a continuous automorphism of $A$, it follows that the mapping $f : z \to \exp(zD)(e)$ from $\mathbb{C}$ to $A$ is an entire function with $f'(0) = D(e) \neq 0$, and whose range consists only of nonzero idempotents of $A$. Now, since $\|f(0)\| = 1$, Liouville's theorem implies that $\{\|f(z)\| : z \in \mathbb{C}\} = [1, \infty[$. ■

## 3. The case of $JB^*$-algebras

We recall that a $JB^*$-triple is a complex Banach space $X$ with a continuous triple product $\{\cdot,\cdot,\cdot\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all $x$ in $X$, the mapping $y \to \{x,x,y\}$ from $X$ to $X$ is a hermitian operator on $X$ and has nonnegative spectrum.
(2) The main identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y, \}, z\} + \{x, y, \{a, b, z\}\}$$

holds for all $a, b, x, y, z$ in $X$.
(3) $\|\{x, x, x\}\| = \|x\|^3$ for every $x$ in $X$.

Concerning Condition (1) above, we also recall that a bounded linear operator $T$ on a complex Banach space $X$ is said to be hermitian if $\|\exp(irT)\| = 1$ for every $r$ in $\mathbb{R}$.

Examples of $JB^*$-triples are all $C^*$-algebras under the triple product $\{\cdot,\cdot,\cdot\}$ determined by $\{a, b, a\} := ab^*a$.

Let $X$ be a $JB^*$-triple, and let $x$ be in $X$. It is well-known that there is a unique couple $(K, \phi)$, where $K$ is a compact subset of $[0, \infty[$ with $0 \in K$, and $\phi$ is an isometric triple homomorphism from $C_0(K)$ to $X$, such that the range of $\phi$ coincides with the $JB^*$-subtriple of $X$ generated by $x$, and $\phi(u) = x$, where $u$ stands for the mapping $t \to t$ from $K$ to $\mathbb{C}$ (see [**8**, 4.8], [**9**, 1.15], and [**2**]). The locally compact subset $K \setminus \{0\}$ of $]0, \infty[$ is called the triple spectrum of $x$, and will be denoted by $\sigma(x)$. We note that $\sigma(x)$ does not change when we replace $X$ with any $JB^*$-subtriple of $X$ containing $x$.

LEMMA 3.1. *Let $A$ be a $C^*$-algebra, and let $a$ be in $A$ such that $0 \in sp(a^*a)$. Then we have $\sigma(a) = sp(\sqrt{a^*a}) \setminus \{0\}$.*

PROOF. Let $\Phi : C_0(sp(\sqrt{a^*a})) \to A$ be the linear isometry given by Lemma 2.4. It is enough to show that $\Phi$ is a triple homomorphism, and that the range of $\Phi$ coincides with the $JB^*$-subtriple of $A$ generated by $a$. In its turn, to verify the first fact, it is enough to prove that $\Phi(f\overline{g}f) = \Phi(f)\Phi(g)^*\Phi(f)$ for those $f, g \in C_0(sp(\sqrt{a^*a}))$ which are of the form $t \to tP(t^2)$ and $t \to tQ(t^2)$, for suitable complex polynomials $P$ and $Q$, respectively. But, for such $f, g$ we have

$$\Phi(f)\Phi(g)^*\Phi(f) = aP(a^*a)\overline{Q}(a^*a)a^*aP(a^*a) = \Phi(f\overline{g}f).$$

Let $X$ denote the $JB^*$-subtriple generated by $a$. Since $\Phi(u) = a$, where $u$ denotes the mapping $t \to t$ from $sp(\sqrt{a^*a})$ to $\mathbb{C}$, and $\Phi$ is an isometric triple homomorphism, we have that $X$ is contained in the range of $\Phi$. On the other hand, since $a(a^*a)^{n+1} = \{a, a(a^*a)^n, a\}$ for every $n \in \mathbb{N}$, an induction argument shows that $a(a^*a)^n$ belongs to $X$ for every $n \in \mathbb{N}$, and hence that $\Phi(f)$ lies in $X$ whenever $f \in C_0(sp(\sqrt{a^*a}))$ is of the form $t \to tP(t^2)$ for a suitable complex polynomial $P$. Since the set of such $f$'s is dense in $C_0(sp(\sqrt{a^*a}))$, the range of $\Phi$ is contained in $X$. ∎

Over fields of characteristic different from two, Jordan algebras are defined as those (possibly non associative) commutative algebras satisfying the identity $(x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2)$. For $a$ and $b$ in a Jordan algebra, we put $U_a(b) := 2a \cdot (a \cdot b) - a^2 \cdot b$. Let $A$ be an associative algebra. Then $A$ becomes a Jordan algebra under the Jordan product defined by

$$a \cdot b := \frac{1}{2}(ab + ba).$$

Moreover, for all $a, b \in A$ we have

$$U_a(b) := 2a \cdot (a \cdot b) - a^2 \cdot b = aba.$$

Jordan subalgebras of $A$ are, by definition, those subspaces $J$ of $A$ satisfying $J \cdot J \subseteq J$.

LEMMA 3.2. *Let $A$ be an associative algebra, let $a$ and $b$ be in $A$, and let $n$ be in $\mathbb{N}$. Then both $a(ba)^n$ and $(ab)^n + (ba)^n$ belong to the Jordan subalgebra of $A$ generated by $\{a, b\}$.*

PROOF. Let $C$ denote the Jordan subalgebra of $A$ generated by $\{a, b\}$. We argue by induction on $n$. The lemma is true for $n = 1$ because $aba = U_a(b)$ and $ab + ba = 2(a \cdot b)$. Assume that the lemma is true for some value of $n$ (say $m$). Then we have $a(ba)^{m+1} = U_a[b(ab)^m] \in C$ and $(ab)^{m+1} + (ba)^{m+1} = ab(ab)^m + b(ab)^m a = 2a \cdot [b(ab)^m] \in C$. ■

Let $K$ be a compact subset of $[1, \infty[$. Then the linear mapping $\Psi : \mathcal{A}(K) \to \mathcal{A}(K)$, determined by

$$\Psi(f[ij]) := f[ij] \text{ if } i \neq j,\ \Psi(f[11]) := f[22],\ \Psi(f[22]) := f[11]$$

for every $f \in C(K)$, becomes an isometric involutive $*$-antiautomorphism of $\mathcal{A}(K)$. Therefore, the set of fixed elements for $\Psi$ is a closed $*$-invariant Jordan subalgebra of $\mathcal{A}(K)$, and hence a Banach-Jordan $*$-algebra. Such a Banach-Jordan $*$-algebra will be denoted by $\mathcal{J}(K)$. Note that elements of $\mathcal{J}(K)$ are precisely those matrices $(f_{ij}) \in \mathcal{A}(K)$ satisfying $f_{11} = f_{22}$, or equivalently, those elements of $\mathcal{A}(K)$ of the form $f([11]+[22])+g[12]+h[21]$ with $f, g, h \in C(K)$.

Lemma 3.3. *Let $K$ be a compact subset of $[1, \infty[$, and let $u$ stand for the element of $C(K)$ defined by $u(t) := t$ for every $t \in K$. Then $\mathcal{J}(K)$ is generated by $u[21]$ as a Jordan-Banach $*$-algebra.*

Proof. Put $p := u[21] \in \mathcal{J}(K)$, and let $J$ denote the closed $*$-invariant subalgebra of $\mathcal{J}(K)$ generated by $p$. We have $u^2[11] = p^*p$ and $u^2[22] = pp^*$, which, in view of Lemma 3.2, implies for $n \in \mathbb{N}$ that $u^{2n+1}[21] = p(p^*p)^n \in J$, $u^{2n+1}[12] = p^*(pp^*)^n \in J$, and

$$u^{2n}([11] + [22]) = (p^*p)^n + (pp^*)^n \in J.$$

Therefore, for every complex polynomial $P$, $uP(u^2)[21]$ and $uP(u^2)[12]$ lie in $J$, and, if $P(0) = 0$, then also $P(u^2)([11] + [22])$ lies in $J$. It follows $C(K)[21] \subseteq J$, $C(K)[21] \subseteq J$, and $C(K)([11] + [22]) \subseteq J$. This implies $\mathcal{J}(K) = J$. ■

$JB^*$-algebras are defined as those Banach-Jordan $*$-algebras $J$ satisfying $\|U_a(a^*)\| = \|a\|^3$ for every $a \in J$. $C^*$-algebras are $JB^*$-algebras under their Jordan products. As in the particular case of $C^*$-algebras, already commented, $JB^*$-algebras are $JB^*$-triples under the triple product $\{\cdot, \cdot, \cdot\}$ determined by $\{a, b, a\} := U_a(b^*)$ (see [**1**] and [**12**]).

Theorem 3.4. *Let $J$ be a $JB^*$-algebra, and let $e$ be a non self-adjoint idempotent in $J$. Then $K := \sigma(e)$ is a compact subset of $[1, \infty[$ whose maximum element (namely $\|e\|$) is grater than 1, and there exists a unique continuous $*$-homomorphism $G : \mathcal{J}(K) \to J$ such that $G(u[21]) = e$, where $u$ stand for the function $t \to t$ from $K$ to $\mathbb{C}$. Moreover we have:*

(1) *The closure in $J$ of the range of $G$ coincides with the $JB^*$-subalgebra of $J$ generated by $e$.*
(2) *$G$ is injective if and only if 1 is not an isolated point of $K$.*
(3) *If 1 is an isolated point of $K$, then $\ker(G)$ consists precisely of those matrices $(f_{ij}) \in \mathcal{J}(K)$ which vanish at every $t \in K \setminus \{1\}$ and satisfy*

$$f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1) = 0.$$

Proof. Let $J_e$ denote the $JB^*$-subalgebra of $J$ generated by $e$. By [**12**] and [**11**], there exists a $C^*$-algebra $A$ containing $J_e$ as a $JB^*$-subalgebra. Therefore, by Lemma 3.1 and Theorem 2.6, $K := \sigma(e)$ is a compact subset of $[1, \infty[$ whose maximum element (namely $\|e\|$) is grater than 1, and there exists a unique continuous $*$-homomorphism $F : \mathcal{A}(K) \to A$ such that $F(u[21]) = e$. Let $G$ stands for the restriction of $F$ to $\mathcal{J}(K)$. Then, clearly, $G$ is a continuous $*$-homomorphism from $\mathcal{J}(K)$ to the $JB^*$-algebra underlying $A$, which satisfies $G(u[21]) = e$. Noticing that the $JB^*$-subalgebras of $A$ and $J$ generated by $e$ coincide, it follows from Lemma 3.3 that $G$ is unique under the above conditions, and that the closure of the range of $G$ is $J_e$. This last fact allows us to see $G$ as a continuous $*$-homomorphisms from $\mathcal{J}(K)$ to $J$. Finally, Properties (2) and (3) for $G$ in the present theorem follow from the corresponding ones for $F$ in Theorem 2.6. ∎

Let $J$ be a Jordan algebra. For $a, b, c \in J$, we put

$$[a, b, c] := (a \cdot b) \cdot c - a \cdot (b \cdot c).$$

The centre of $J$ is defined as the set of those elements $a \in J$ such that $[a, J, J] = 0$. It is well-known and easy to see that central elements $a$ of $J$ satisfy $[J, J, a] = [J, a, J] = 0$.

Corollary 3.5. *Let $J$ be a $JB^*$-algebra. Then the following assertions are equivalent:*

(1) *For every $s \in [1, \infty[$ there exists an idempotent $e \in J$ such that $\|e\| = s$.*
(2) *There exists a non self-adjoint idempotent in $J$.*
(3) *There exists a non central self-adjoint idempotent in $J$.*

Proof. The implication (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3).- Let $e$ be the non self-adjoint idempotent of $J$ whose existence is assumed. Let $K$ and $G : \mathcal{J}(K) \to A$ be the compact set and the $*$-homomorphism, respectively, given by Theorem 3.4. Put

$$p := \frac{1}{2} u(1+u)^{-1}([11] + [12] + [21] + [22]) \in \mathcal{J}(K),$$

where $u$ stands for the function $t \to t$ from $K$ to $\mathbb{C}$, and $q := [12] \in \mathcal{J}(K)$. Then $p$ is a self-adjoint idempotent, and we have

$$[p, q, q] = \frac{1}{8}(2[12] - u^{-1}([11] + [22])).$$

Noticing that, by Theorem 3.4, $[p, q, q]$ does not belong to $\ker(G)$, it follows that $G(p)$ is a non central self-adjoint idempotent of $J$.

(3) $\Rightarrow$ (1).- Let $e$ be the non central self-adjoint idempotent of $J$ whose existence is assumed. By Lemma 2.5.5 of [**4**], there exists $a \in J$ such that $U_e(a) \neq e \cdot a$ or, equivalently, $[e, e, a] \neq 0$. Then, by [**7**, page 34], the mapping $D : J \to J$ defined by $D(b) := [e, b, a]$ for every $b \in J$ becomes a continuous derivation of $J$, which clearly satisfies $D(e) \neq 0$. Now, arguing

as in the proof of the implication (3) $\Rightarrow$ (1) in Corollary 2.7, we realize that Assertion (1) in the present corollary holds. ∎

Let $J$ be a $JB^*$-algebra containing a non self-adjoint idempotent $e$. Then the non central self-adjoint idempotent $p \in J$ provided by the above proof can be explicitly given as follows. Denote by $J_e$ the $JB^*$-subalgebra of $J$ generated by $e$, and take a $C^*$-algebra $A$ containing $J_e$ as a $JB^*$-subalgebra. Then in $A$ we have

$$p = \frac{1}{2}[(e + \sqrt{e^*e})(1 + \sqrt{e^*e})^{-1} + (e^* + \sqrt{ee^*})(1 + \sqrt{ee^*})^{-1}].$$

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