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Abstract

We review the main results in the papers [17] and [18] where we discuss in a nonassociative setting the Rota-Strang joint spectral radius of bounded subsets of (associative) normed algebras [26], and the related notion of a topologically nilpotent (associative) normed algebra [20, Pages 515-517].

Keywords: Joint spectral radius, nilpotency, topologically nilpotent algebras.

1 Introduction

In an early paper [26], G.-C. Rota and W. G. Strang prove the following.

Theorem 1.1. *For each bounded and multiplicatively closed subset S of any associative normed algebra A, there exists an equivalent algebra norm $|\!|\!|.|\!|\!|$ on A such that $|\!|\!| s |\!|\!| \leq 1$ for every $s \in S$.*

Then, they define the "spectral radius" $r(S)$ of any bounded subset S of an associative normed algebra A by the equality

$$r(S) := \limsup_{n\to\infty} \sup\{\|s_1 \dots s_n\|^{\frac{1}{n}} : s_1, \dots, s_n \in S\}, \tag{1.1}$$

and apply Theorem 1.1 to show that

$$r(S) = \inf\{\sup\{|\!|\!| s |\!|\!| : s \in S\} : |\!|\!|.|\!|\!| \in \mathrm{En(A)}\}, \tag{1.2}$$

where En(A) denotes the set of all equivalent algebra norms on A.

The Rota-Strang paper remained forgotten for many years. Nevertheless, the idea of the spectral radius of a bounded subset of an associative normed algebra underlines the definition of the so-called "topologically nilpotent" associative normed algebras. Indeed, such algebras can be introduced as those associative normed algebras such that the spectral radius of their closed unit balls is equal to zero. Following Palmer's review in [20, 4.8.8], "These [algebras] were introduced by J. K. Miziolek, T. Müldner and A. Rek [16] as a class of topological algebras. Recently their study was revived by Peter G. Dixon [8] and continued by Dixon and Vladimir Müller [9], Dixon and George A. Willis [10]."

We refer to Palmer's whole review of the papers just quoted [20, Pages 515-517] for a comprehensive view of the theory of topologically nilpotent associative normed algebras, noticing however that, in an excess of enthusiasm, an error creeps in the formulation of [20, Theorem 4.8.8]. To clarify this, let us consider the following conditions on an associative complex Banach algebra A:

1. A is topologically nilpotent.
2. There is some finite constant C satisfying
$$\sup\{\|a^n\|^{\frac{1}{n}} : a \in A, \|a\| \leq 1\} \leq Cn^{\frac{-3.2^n}{n}}$$
for all $n \in \mathbb{N}$.
3. For every element $a \in A$, there is some finite constant $C = C(a)$ satisfying $\|a^n\|^{\frac{1}{n}} \leq Cn^{\frac{-3.2^n}{n}}$ for all $n \in \mathbb{N}$.

Then conditions 2 and 3 are equivalent. Indeed, the implication $2 \Rightarrow 3$ is clear, whereas the converse implication is proved in [8, Theorem 2.1]. On the other hand, by [8, Theorem 3.2], 2 implies 1. However, contrarily to what asserted in [20, Theorem 4.8.8], 1 does not imply 2. The following counter-example has been communicated to us by V. Müller. Consider the associative complex Banach algebra A of those formal power series $\sum_{j=1}^{\infty} \alpha_j x^j$ (with one generator x and complex coefficients α_j) such that
$$\|\sum_{j=1}^{\infty} \alpha_j x^j\| := \sum_{j=1}^{\infty} |\alpha_j| \frac{1}{j^j} < \infty.$$

Then A is topologically nilpotent (that is, it satisfies 1) but does not fulfil 2. The details of the verification of this assertion can be seen in [18].

To conclude our review of topologically nilpotent associative normed algebras, let us say that they have turned out useful to prove significant positive answers to the question of splitting radical extensions of certain Banach algebras (see [3] for details).

The Rota-Strang spectral radius is rediscovered in the papers of V. S. Shulman [29] and Yu. V. Turovskii [32], where a special attention is payed to the spectral radius of finite subsets, and to those associative normed algebras whose finite subsets have zero spectral radius. For more information about the Rota-Strang spectral radius of finite subsets, the reader is referred to the papers of A. Soltysiak [31] and P. Rosenthal and A. Soltysiak [25].

The last word concerning the Rota-Strang spectral radius in the associative setting is provided by the impressive paper of V. S. Shulman and Yu. V. Turovskii [30], where a systematic study of this notion is made, and the connections between the invariant snbspace problem for operators semigroups, and the joint spectral radius, is investigated.

The aim of the present note is to review in some detail the results of [17], as well as to announce the main results obtained to now in [18]. Both papers are devoted to the nonassociative study of the material reviewed above.

In [17] we discuss the validity of Theorem 1.1 in the nonassociative setting. Although Theorem 1.1 does not remain true in general if the associativity of the algebra A is removed (Example 2.6), it remains valid depending on the goodness of the bounded and multiplicatively closed subset S (Theorem 2.3) and/or the goodness of the nonassociative normed algebra A. In its turn, the goodness of the normed algebra A could depend on either the purely algebraic structure of A (Theorem 2.5) or the behaviour of the norm (Proposition 2.8).

One of the key ideas in [18] is that, for a (possibly nonassociative) algebra A, the failure or success of A in relation to Theorem 1.1 can be quantified by means of a nonnegative extended real number $\beta(A)$ (Definition 3.1). The situation $\beta(A) = +\infty$ means that A becomes a complete disaster concerning Theorem 1.1, whereas the inequality $\beta(A) \leq 1$ can be interpreted as that Theorem 1.1 remains "approximately" true for A. On the other hand, we introduce in [18] the appropriate formal changes in the equality (1.1) in order to be provided with an understandable notion of spectral radius $r(S)$ of a bounded subset S of any (possibly nonassociative) normed algebra A (Definition 3.3). Then we show that the equality (1.2) is true for every bounded subset S of A if and only if $\beta(A) \leq 1$ (Corollary 3.5). Among the other results from [18] reviewed in the present note, we emphasize the following ones:

1. A (possibly nonassociative) normed algebra A is topologically nilpotent (with the same meaning as in the associative case) if and only if $\beta(A) = 0$ (Theorem 4.1).

2. An associative normed algebra A is topologically nilpotent if and only if so is the normed Jordan algebra obtained by symmetrization of its product (Theorem 4.2).

3. A finite-dimensional normed algebra A is topologically nilpotent if an only if it is nilpotent (Proposition 4.3).

4. A finite-dimensional normed Lie algebra A is nilpotent if an only if $\beta(A) < +\infty$ (a consequence of Theorem 4.4).

5. Every topologically nilpotent complete normed algebra coincides with its weak radical in the sense of [23] (Remark 4.6).

2 The norm-one boundedness property

By an algebra norm on a (possibly nonassociative) real or complex algebra A we mean a norm $\|\cdot\|$ on (the vector space of) A satisfying $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. By a normed algebra we mean a real or complex algebra endowed with an algebra norm. Let A be a normed algebra. We say that A satisfies the norm-one boundedness property (in short NBP) if, for each bounded and multiplicatively closed subset S of A, there exists an equivalent algebra norm $|\!|\!|\cdot|\!|\!|$ on A such that $|\!|\!|s|\!|\!| \leq 1$ for every $s \in S$. Now, the Rota-Strang Theorem 1.1, which can be seen also in [7, Theorem I.4.1], can be reformulated as follows.

Theorem 2.1. *Let A be an associative normed algebra. Then A satisfies the NBP.*

As an immediate consequence, we have the following.

Corollary 2.2. *Let A be an associative normed algebra, and let p be a nonzero idempotent in A. Then there exists an equivalent algebra norm $|\!|\!|\cdot|\!|\!|$ on A such that $|\!|\!|p|\!|\!| = 1$.*

It is easily realized that neither Theorem 2.1 nor even Corollary 2.2 remain true if the assumption of associativity is removed (see Example 2.6 below).

The main goal in [17] is to discuss the validity of Theorem 2.1 and Corollary 2.2 in the nonassociative setting.

We recall that the nucleus of an algebra A is defined as the set of those elements of A which associate with any two elements of A. A reasonable nonassociative generalization of Theorem 2.1 is the following.

Theorem 2.3. [17, Theorem 2.3] *Let A be a normed algebra, and let S be a bounded and multiplicatively closed subset of A contained in the nucleus of A. Then there exists an equivalent algebra norm $|\!|\!|\cdot|\!|\!|$ on A such that $|\!|\!|s|\!|\!| \leq 1$ for every $s \in S$.*

It follows from Theorem 2.3 that, if p is a nonzero nuclear idempotent in a normed algebra A, then there exists an equivalent algebra norm $|\!|\!|\cdot|\!|\!|$ on A such that $|\!|\!|p|\!|\!| = 1$ (a nonassociative generalization of Corollary 2.2). In particular, we have the following result, first proved by F. G. Ocaña [19].

Corollary 2.4. *Let A be a nonzero normed algebra with a unit $\mathbf{1}$. Then there exists an equivalent algebra norm $|\!|\!|\cdot|\!|\!|$ on A satisfying $|\!|\!|\mathbf{1}|\!|\!| = 1$.*

Different classes of nonassociative algebras which are "close" to the associative ones have appeared in the literature. Among them, the one of generalized standard algebras is specially relevant for our approach. Roughly speaking, this class is the smallest one containing all alternative algebras and all (commutative) Jordan algebras. We note that associative algebras are alternative, that generalized standard algebras are noncommutative Jordan, and that noncommutative Jordan algebras are power-associative. The reader is referred to [28] for the precise definition of generalized standard algebras, and to Schafer's book [27] for the definition of the remaining classes of algebras just quoted. Now, the following generalization of Corollary 2.2 has its own interest.

Theorem 2.5. [17, Theorem 3.2] *Let A be a normed generalized standard algebra, and let p be a nonzero idempotent in A. Then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||p||| = 1$.*

Let A be an algebra over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), and let λ be in $\mathbb{K}$. The λ-mutation of A, denoted by $A^{(\lambda)}$, is defined as the algebra whose vector space is that of A, and whose product (say $\Box$) is defined by $a\Box b := \lambda ab + (1-\lambda)ba$. We note that λ-mutations of noncommutative Jordan algebras are noncommutative Jordan algebras.

Theorem 2.5 does not remain true if the assumption that A is a generalized standard algebra is relaxed to the one that A is a noncommutative Jordan algebra. Indeed, we have the following.

Example 2.6. [17, Example 3.2] *Let λ be a real number with $\lambda > 1$. Then there exists a two-dimensional normed noncommutative Jordan algebra A with an idempotent p satisfying $\|p\| = \lambda$ and $|||p||| \geq \lambda$ for every algebra norm $|||\cdot|||$ on A.* Indeed, let p and q be the elements of the associative algebra $M_2(\mathbb{K})$ (of all 2×2 matrices over $\mathbb{K}$) given by $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, respectively, and let C denote the linear hull of $\{p, q\}$ in $M_2(\mathbb{K})$. Then C becomes a subalgebra of $M_2(\mathbb{K})$. Now take A equal to the λ-mutation of C, so that A is a two-dimensional noncommutative Jordan algebra, and the multiplication table of A is given by

	p	q
p	p	λq
q	$(1-\lambda)q$	0

Clearly p becomes an idempotent in A. In addition, define a norm on A by $\|\alpha p + \beta q\| := \lambda|\alpha| + |\beta|$. It is easily realized that $\|\cdot\|$ becomes an algebra norm on A satisfying $\|p\| = \lambda$. Moreover, since $pq = \lambda q$ in A, for every algebra norm $|||\cdot|||$ on A we have $\lambda|||q||| = |||pq||| \leq |||p|||\,|||q|||$, and hence $\lambda \leq |||p|||$.

In relation to Theorem 2.5, the following problem remains open.

Problem 2.7. *Does Theorem 2.1 remains true if the assumption that A is associative is relaxed to the one that A is generalized standard?*

Unfortunately, we do not know even if alternative normed algebras have to satisfy the NBP. Anyway, we realize in [17] that Theorem 2.1 is not characteristic of the associativity. Thus, for example, all nilpotent normed algebras satisfy the NBP [17, Proposition 5.8]. To be provided with more examples, we consider those normed algebras A fulfilling the "norm square equality" (in short, NSE) $\|a^2\| = \|a\|^2$ for every $a \in A$. Examples of normed algebras satisfying the NSE are absolute-valued algebras, JB-algebras, and smooth-normed algebras. The standard references for these objects are [24], [12], and [22, Section 3], respectively. Normed algebras satisfying the NSE enjoy a strong form of the NBP. Indeed, we have the following.

Proposition 2.8. [17, Proposition 5.1] *Let A be a normed algebra satisfying the NSE, and let S be a bounded and multiplicatively closed subset of A. Then we have that $\|s\| \leq 1$ for every $s \in S$.*

Remark 2.9. Let X be a normed space (with closed unit ball B_X, unit sphere S_X, and topological dual X^*), and let u be in S_X. The element u is said to be a strongly exposed point (of B_X) if there exists $g \in S_{X^*}$ with the property that, whenever (x_n) is a sequence in B_X such that $(g(x_n)) \to 1$, we have $(x_n) \to u$. When the functional g must be emphasized, we say that u is strongly exposed by g. It is well-known that u is strongly exposed by $g \in S_{X^*}$ if and only if $g(u) = 1$ and, for $0 < \delta < 1$, the diameter of the "slice"

$$S(X, g, \delta) := \{x \in B_X : \Re(g(x)) > 1 - \delta\}$$

tends to 0 as $\delta \to 0$. Therefore, if u is a strongly exposed point, then u is a denting point (of B_X), which means that there are slices of arbitrarily small diameter which contain u. On the other hand, if u is a denting point, then u is a strongly extreme point (of B_X), which means that, whenever (x_n) and (y_n) are sequences in B_X such that $(\frac{x_n+y_n}{2}) \to u$, we have $(x_n) \to u$ and $(y_n) \to u$ (see [14, Page 169]).

Now, *let A be a normed algebra, and let p be a nonzero idempotent in A.* We prove actually in [17, Corollary 3.4 and Proposition 4.2] that, *if either A is standard generalized or p is nuclear* (both requirements being automatically fulfilled whenever A is associative), *then there exists an equivalent algebra norm $|\!|\!|\cdot|\!|\!|$ on A satisfying $|\!|\!|p|\!|\!| = 1$ and such that p becomes a strongly exposed point of $B_{(A,|\!|\!|\cdot|\!|\!|)}$.*

Now, *let A be a normed algebra with a unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$. Then there exists an equivalent algebra norm $|\!|\!|\cdot|\!|\!|$ on A "arbitrarily close" to $\|\cdot\|$, satisfying $|\!|\!|\mathbf{1}|\!|\!| = 1$, and such that $\mathbf{1}$ becomes a strongly exposed point of $B_{(A,|\!|\!|\cdot|\!|\!|)}$* [17, Theorem 2.8].

Again, let A be a normed algebra with a unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$. Then $\mathbf{1}$ is a strongly extreme point of B_A. This result is well-known in the associative case [6, Theorem 4.5], and its easily generalized to the nonassociative case (see [17, Remark 2.9] for details). However, even if A is associative, $\mathbf{1}$ need not be a denting point (much less a strongly exposed point) of B_A. Indeed, the Banach algebra $\mathcal{L}(H)$ of all bounded linear operators on any infinite-dimensional Hilbert space H has no denting point [11]. Actually, all slices (and, more generally, all nonempty relatively weakly open subsets) of the closed unit ball of $\mathcal{L}(H)$ have diameter equal to 2 (see [5] and [4]). The reader is referred to [13] for quantitative versions of the fact that the units of norm-unital normed algebras are strongly extreme points, and to [6, 21, 22, 15, 2] for other interesting geometrical properties of the units of norm-unital normed algebras.

3 The joint spectral radius

For a more precise nonassociative discussion of Theorem 2.1, we introduce in [18] the following.

Definition 3.1. Let A be a normed algebra. Given a positive number k, we say that A satisfies the norm-k boundedness property if, for each bounded and multiplicatively closed subset S of A, there exists an equivalent algebra norm $\|\cdot\|$ on A such that $\|s\| \leq k$ for every $s \in S$. We put

$$\beta(A) := \inf\{k \in \mathbb{R}^+ : A \text{ satisfies the norm-}k \text{ boundedness property}\},$$

with the convention that $\inf \emptyset = +\infty$.

In [18] we prove the following.

Theorem 3.2. *For every normed algebra A, we have $\beta(A) \in \{0\} \cup [1, +\infty]$. Moreover, for each $\lambda \in \{0\} \cup [1, +\infty]$, there exists a normed two-dimensional noncommutative Jordan algebra A such that $\beta(A) = \lambda$.*

Let A be a normed algebra and let S be a bounded subset of A. When we do not know that A is associative, the definition of $r(S)$ in the equality (1.1) needs some formal changes. To be precise, consider the following.

Definition 3.3. Let A be an algebra, and let S be a subset of A. The words on S are defined inductively, according to their "degree". Indeed, the words on S of degree 1 are precisely the elements of S, and, for $1 < n \in \mathbb{N}$, the words on S of degree n are those elements of A which can be written as xy where x and y are words on S of degree $i \in \mathbb{N}$ and $j \in \mathbb{N}$, respectively, with $i + j = n$. Now assume that A is normed, and that S is bounded. For n in $\mathbb{N}$, let $M_n(S)$ stand for the least upper bound of the values of the norm at all words on S of degree n. Note that $M_n(S) \leq [\sup\{\|s\| : s \in S\}]^n$, and define the spectral radius, $r(S)$, of S by

$$r(S) := \limsup_{n \to \infty} [M_n(S)]^{\frac{1}{n}} \leq \sup\{\|s\| : s \in S\}.$$

It is easily realized that equivalent algebra norms on A give the same spectral radius for S, and hence that

$$r(S) \leq \inf\{\sup\{\|s\| : s \in S\} : \|\cdot\| \in \mathrm{En(A)}\}.$$

In [18], we show the following.

Proposition 3.4. *Let A be a normed algebra. Then the following conditions are equivalent:*

1. *$\beta(A) < +\infty$.*
2. *There exists a nonnegative real number k such that, for every bounded subset S of A, we have*

$$\inf\{\sup\{\|s\| : s \in S\} : \|\cdot\| \in \mathrm{En(A)}\} \leq \mathrm{k\, r(S)}.$$

Moreover, when these conditions hold, the minimum nonnegative real number k in condition 2 is equal to $\beta(A)$.

As a consequence, we derive the following.

Corollary 3.5. *Let A be a normed algebra. Then the following conditions are equivalent:*

1. *$\beta(A) \leq 1$ (that is, the conclusion in Theorem 2.1 is satisfied "approximately").*

2. *For every bounded subset S of A, we have*

$$r(S) = \inf\{\sup\{\|s\| : s \in S\} : \|\cdot\| \in \mathrm{En(A)}\}.$$

In relation to the above corollary, the following problem remains open.

Problem 3.6. *Is there a (necessarily nonassociative) normed algebra A failing to enjoy the NBP but satisfying $\beta(A) = 1$?*

4 Topologically nilpotent algebras

Let A be a normed algebra, and let S be a bounded subset of A. We say that S is quasi-nilpotent if $r(S) = 0$. Clearly, bounded nilpotent subsets of A are quasi-nilpotent. The normed algebra A is said to be topologically nilpotent if its closed unit ball is quasi-nilpotent.

In [18] we obtain the following characterization of topological nilpotency.

Theorem 4.1. *Let A be normed algebra. Then the following assertions are equivalent:*

1. *A is topologically nilpotent.*

2. *For every $\varepsilon > 0$, there exists an equivalent algebra norm $\|\cdot\|'$ on A such that $\|\cdot\|' \leq \varepsilon\|\cdot\|$.*

3. *$\beta(A) = 0$.*

If A is a normed algebra, then the algebra $A^{(\lambda)}$ will be considered without notice as a normed algebra under the norm $\sigma_\lambda\|\cdot\|$, where $\sigma_\lambda := |\lambda| + |1 - \lambda|$.

We prove in [18] the following.

Theorem 4.2. *Let A be a normed algebra over $\mathbb{K}$. Then we have*

1. *For λ in $\mathbb{K} \setminus \{\frac{1}{2}\}$, A is topologically nilpotent if and only if so is $A^{(\lambda)}$.*

2. *If A is topologically nilpotent, then so is $A^{(\frac{1}{2})}$.*

3. *There are choices of A such that $A^{(\frac{1}{2})}$ is topologically nilpotent, but A is not topologically nilpotent.*

4. *When A is associative, A is topologically nilpotent if and only if so is the Jordan normed algebra $A^{(\frac{1}{2})}$.*

A normed algebra is said to be finitely quasi-nilpotent if all its finite subsets are quasi-nilpotent. Clearly, every topologically nilpotent normed algebra is finitely quasi-nilpotent, but the converse assertion is not true, even if the normed algebra is associative, commutative, complex, and complete. Anyway, in the finite-dimensional case, we get the following result [18].

Proposition 4.3. *Let A be a finite-dimensional normed algebra. Then the following conditions are equivalent:*

1. *A is nilpotent.*

2. *A is topologically nilpotent.*

3. *A is finitely quasi-nilpotent.*

An algebra A is said to be algebraic if, for every $a \in A$, the operators L_a and R_a (of left and right, respectively, multiplication by a on A) are algebraic. This notion of algebraicity does not coincide with that of A. A. Albert [1] that every element of A generates a finite-dimensional subalgebra. Indeed, Albert's notion of algebraicity trivializes in the class of anti-commutative algebras, as is indeed useless in that class.

We prove in [18] that, if an anti-commutative complete normed algebraic algebra A satisfies that $\beta(A) < +\infty$, then there exists $n \in \mathbb{N}$ such that $L_a^n = 0$ for every $a \in A$. By invoking the implication $1 \Rightarrow 3$ in Theorem 4.1, and a celebrated theorem of E. I. Zel'manov [33] on the so-called Engel-Lie algebras, we derive the following.

Theorem 4.4. *Let A be a complete normed algebraic Lie algebra. Then the following assertions are equivalent:*

1. *A is topologically nilpotent.*

2. *A satisfies the NBP.*

3. $\beta(A) \leq 1$.

4. $\beta(A) < +\infty$.

5. *A is nilpotent.*

Let A be an algebra. A subalgebra B of A is called a full subalgebra of A if, whenever b is in B, a is in A, and $a + b - ab = b + a - ba = 0$, we have $a \in B$. Since the intersection of full subalgebras of A is another full subalgebra of A, it follows that, for any nonempty subset S of A, there is a smallest full subalgebra of A

which contains S. This subalgebra will be called the full subalgebra of A generated by S. Now, let $L(A)$ stand for the associative algebra of all linear operators on A. Following [23], we define the full multiplication algebra of A as the full subalgebra of $L(A)$ generated by all operators of left and right multiplication by elements of A on A, and denote it by $\mathcal{FM}(A)$.

In [18] we obtain the following.

Theorem 4.5. *Let A be a topologically nilpotent complete normed algebra. Then $\mathcal{FM}(A)$ is a radical algebra.*

Remark 4.6. Let A be any algebra over $\mathbb{K}$. The full multiplication algebra of A is introduced in [23] as an intermediate notion to define the so-called weak radical of A (denoted by w-Rad(A)), and to prove that, if w-Rad(A) $= 0$, then A has at most one complete algebra norm topology. The weak radical of A is actually defined as the largest $\mathcal{FM}(A)$-invariant subspace of A contained in the subspace

$$\{a \in A : \{L_a, R_a\} \subseteq \mathrm{Rad}(\mathcal{FM}(\mathrm{A}))\},$$

where Rad($\cdot$) means Jacobson radical. It is easily realized that $\mathcal{FM}(A)$ is a radical algebra if and only if $A = w$-Rad(A). Therefore, Theorem 4.5 can be reformulated by saying that, *if A is a topologically nilpotent complete normed algebra, then A is equal to its weak radical.* It is worth mentioning that the weak radical is "very small" (see [23, Proposition 2.3]), so that topologically nilpotent complete normed algebras are equal to their "radicals" for most familiar radicals.

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