NEARLY ABSOLUTE-VALUED ALGEBRAS

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0. INTRODUCTION

Throughout this paper $\mathbb{K}$ will denote the field of real or complex numbers. An *absolute-valued* algebra over $\mathbb{K}$ is an algebra A over $\mathbb{K}$ endowed with a norm $\|.\|$ satisfying $\|xy\| = \|x\|\|y\|$ for all x, y in A. If the above equality is relaxed to the inequality $\|xy\| \leq \|x\|\|y\|$, then we find the familiar concept of a *normed* algebra. In this paper we deal with normed algebras A over $\mathbb{K}$ such that there exists $\rho > 0$ satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for all x, y in A. Such algebras will be called *nearly absolute-valued* algebras over $\mathbb{K}$. All algebras considered in this paper will be assumed to be non-zero. Let A be a finite-dimensional normed algebra over $\mathbb{K}$. By the compactness of spheres, A is nearly absolute-valued if (and only if) there are no non-zero divisors of zero in A. If this is the case, then A is isomorphic to $\mathbb{C}$ if $\mathbb{K} = \mathbb{C}$, and the dimension of A is equal to $1, 2, 4$, or 8 if $\mathbb{K} = \mathbb{R}$ [1, Chapter 11]. Moreover, if $\mathbb{K} = \mathbb{R}$ and A is commutative, then the dimension of A is equal to 1 or 2 [1, p. 235].

Roughly speaking, our results show that a great part of the theory of absolute-valued algebras remains true for nearly absolute valued algebras whenever the number ρ in the definition of these last algebras is near to one. Let us specify this assertion. Consider the following three conditions on an arbitrary algebra A:

1) A is commutative.
2) A has a unit.
3) A is algebraic.

It is known that, if A is an absolute-valued real algebra, and if Condition n) holds for some $n = 1, 2, 3$, then A is finite-dimensional ([2], [3]]). We extend the above result by proving that, for $n = 1, 2, 3$, there exists a universal constant $0 \leq K_n < 1$ uniquely determined by the following two properties:

i) There is an infinite-dimensional normed real algebra B enjoying Condition n) and satisfying $\|xy\| \geq K_n\|x\|\|y\|$ for all x, y in B.
ii) If A is any normed real algebra enjoying Condition n) and satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for some $\rho > K_n$ and all x, y in A, then A is finite-dimensional.

For n equal to $1, 2$, and 3, the above result is shown in Theorems 1.5, 3.1, and 4.3, respectively. For $n = 1$, we are provided with the additional information that $2^{-1/2} \leq K_1$.

Absolute-valued real algebras with a left unit need not be finite-dimensional ([4], [5]), but are prehilbert spaces [5] (hence uniformly non-square normed spaces). We prove in Theorem 2.7 the existence of a universal constant $2^{-1} \leq K \leq 2^{-1/4}$ uniquely determined by the following two properties:

i) There is a normed real algebra B which is not uniformly non-square, has a left unit, and satisfies $\|xy\| \geq K\|x\|\|y\|$ for all x, y in B.
ii) If A is any normed real algebra with a left unit and satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for some $\rho > K$ and all x, y in A, then A is uniformly non-square.

The case of nearly absolute-valued complex algebras is also considered. Such algebras are isomorphic to $\mathbb{C}$ whenever either they have a left unit or are algebraic (Remarks 2.8 and 4.5). If they are commutative, then a result similar to the one obtained in the real case holds (Theorem 1.8).

To conclude this introduction, let us note that, by an old theorem of I. Kaplansky [6] (see also [7]), every nearly absolute-valued associative real algebra is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ (the algebra of Hamilton's quaternions), so that no much more can be said about such an algebra. The consequence that nearly absolute-valued associative algebras are isomorphic to absolute-valued algebras is no longer true if associativity is removed (even if the number ρ in the definition of nearly absolute-valued algebras is near one). For instance, if λ is in $\mathbb{R}\backslash\{\frac{1}{2}\}$ with $0 < \lambda < 1$, and if A denotes the normed real algebra consisting of the normed space of $\mathbb{H}$ and the product $\Box$ defined by $x \Box y := \lambda xy + (1-\lambda)yx$, then A is a nearly absolute-valued algebra which cannot be isomorphic to an absolute-valued algebra (indeed, A is not associative and has a unit, whereas every absolute-valued four-dimensional real algebra with a unit is isomorphic to $\mathbb{H}$ [8]).

1. Nearly absolute-valued commutative algebras

The commutative rbanik-Wright theorem [2, Theorem 3] asserts that every absolute-valued commutative real algebra is isometrically isomorphic to $\mathbb{R}, \mathbb{C}$, or $\overset{*}{\mathbb{C}}$. (Here $\overset{*}{\mathbb{C}}$ denotes the absolute-valued real algebra obtained by replacing the usual product of $\mathbb{C}$ by the one $\Box$ given by $x \Box y := x^* y^*$, where $*$ means the familiar conjugation of $\mathbb{C}$.) As the next example shows, even the consequence that absolute-valued commutative real algebras have dimension ≤ 2 does not remain true in the more general setting of nearly absolute valued algebras.

Example 1.1. Let B the absolute-valued algebra over $\mathbb{K}$ whose normed space is an arbitrary infinite dimensional Hilbert space over $\mathbb{K}$ (with orthonormal basis $\{u_i : i \in I\}$, say) and whose product is determined by

$$u_i u_j := u_{\varphi(i,j)} ,$$

where φ is any prefixed injective mapping from $I \times I$ into I (see [2] and [9]). Denote by A the commutative normed algebra over $\mathbb{K}$ obtained from

B by replacing the product xy given above by the one $\circ$ defined by $x \circ y := \frac{1}{2}(xy + yx)$. For $x = \sum_i \lambda_i u_i$ and $y = \sum_i \mu_i u_i$ in A, we have

$$xy = \sum_{i,j} \lambda_i \mu_j u_{\varphi(i,j)} \text{ and } yx = \sum_{i,j} \mu_i \lambda_j u_{\varphi(i,j)} .$$

Therefore

$$(xy|yx) = \sum_{i,j} \lambda_i \mu_j \mu_i^* \lambda_j^* = (\sum_i \lambda_i \mu_i^*)(\sum_j \mu_j \lambda_j^*) = (x|y)(y|x) = |(x|y)|^2 .$$

It follows

$$\begin{aligned} 4\|x \circ y\|^2 = \|xy + yx\|^2 &= \|xy\|^2 + \|yx\|^2 + 2Re(xy|yx) \\ &= 2(\|x\|^2\|y\|^2 + |(x|y)|^2) , \end{aligned}$$

and hence $\|x \circ y\| \geq 2^{-1/2}\|x\|\|y\|$.

Despite the above example, we will show in this section that the commutative rbanik-Wright theorem essentially holds for "many" nearly absolute valued real algebras. The proof will involve some elemental facts of the theory of normed ultraproducts [10], a summary of which is provided in the sequel.

Let I be a non-empty set, and $\mathcal{U}$ an ultrafilter on I. Given a family $\{X_i\}_{i\in I}$ of normed spaces, we may consider the normed space $(\oplus_{i\in I} X_i)_\infty$ ℓ_∞-sum of this family (consisting of all families $\{x_i\} \in \prod_{i\in I} X_i$ such that $\|\{x_i\}\| := \sup\{\|x_i\| : i \in I\} < \infty$) and the closed subspace $N_\mathcal{U}$ of $(\oplus_{i\in I} X_i)_\infty$ given by $N_\mathcal{U} := \{\{x_i\} \in (\oplus_{i\in I} X_i)_\infty : \lim_\mathcal{U} \|x_i\| = 0\}$. The (normed) *ultraproduct* of the family $\{X_i\}_{i\in I}$ relative to the ultrafilter $\mathcal{U}$ is defined as the quotient normed space $(\oplus_{i\in I} X_i)_\infty / N_\mathcal{U}$, and is denoted by $(X_i)_\mathcal{U}$. If we denote by (x_i) the element in $(X_i)_\mathcal{U}$ containing a given family $\{x_i\} \in (\oplus_{i\in I} X_i)_\infty$, then it is easy to verify that $\|(x_i)\| = \lim_\mathcal{U} \|x_i\|$. Because of this formula, if, for each i in I, Y_i is a subspace of X_i, then in a natural way we can identify $(Y_i)_\mathcal{U}$ with a subspace of $(X_i)_\mathcal{U}$.

Lemma 1.2. *Let $\{X_i\}_{i\in I}$ be a family of normed spaces over $\mathbb{K}$, $\mathcal{U}$ an ultrafilter on I, and m a natural number. Assume that $\dim(X_i) \geq m$ for every i in I. Then $\dim((X_i)_\mathcal{U}) \geq m$.*

Proof. We argue by induction on m. For $m = 1$, it is enough to choose, for each i in I, a norm-one element x_i in X_i, and consider the non-zero element (x_i) in $(X_i)_\mathcal{U}$. Assume that the lemma is true for some $m \in \mathbb{N}$. With I and $\mathcal{U}$ as in the lemma, let $\{X_i\}_{i\in I}$ be a family of normed spaces over $\mathbb{K}$ such that $\dim(X_i) \geq m + 1$ for every i in I. For i in I, choose a subspace Y_i of X_i with $\dim(Y_i) = m$, and (by Riesz's lemma) take a norm-one element x_i in X_i satisfying $\|x_i + Y_i\| \geq 1$. Then, by the induction hypothesis, we have $\dim((Y_i)_\mathcal{U}) \geq m$, and the element (x_i) of $(X_i)_\mathcal{U}$ clearly satisfies $\|(x_i) + (Y_i)_\mathcal{U}\| \geq 1$. It follows $\dim((X_i)_\mathcal{U}) \geq m + 1$. ■

Let $\{A_i\}_{i\in I}$ be a family of normed algebras over $\mathbb{K}$, and $\mathcal{U}$ an ultrafilter on I. Then the normed space $(A_i)_{\mathcal{U}}$ is usually considered as a new normed algebra over $\mathbb{K}$ under the (well-defined) product $(x_i)(y_i) := (x_i y_i)$ [11]. In this situation, if, for each I in I, B_i is a subalgebra of A_i, then, up to the natural identification, $(B_i)_{\mathcal{U}}$ becomes a subalgebra of $(A_i)_{\mathcal{U}}$.

Theorem 1.3. *There exists a universal constant* $2^{-1/2} \leq K_0 < 1$ *uniquely determined by the following two properties:*

- i) *There is a normed real algebra* B *with* $\dim(B) \geq 3$ *and satisfying* $\|xy\| \geq K_0\|x\|\|y\|$ *and* $\|xy - yx\| \leq (1 - K_0)\|x\|\|y\|$ *for all* x, y *in* B.
- ii) *If* A *is any normed real algebra satisfying* $\|xy\| \geq \rho\|x\|\|y\|$ *and* $\|xy - yx\| \leq (1 - \rho)\|x\|\|y\|$ *for some* $\rho > K_0$ *and all* x, y *in* A, *then* $\dim(A) \leq 2$.

Proof. Let S denote the set of those elements ρ in the closed real interval $[0, 1]$ such that there exists a normed real algebra B with $\dim(B) \geq 3$ and satisfying $\|xy\| \geq \rho\|x\|\|y\|$ and $\|xy - yx\| \leq (1 - \rho)\|x\|\|y\|$ for all x, y in B. We claim that S is closed in $\mathbb{R}$. Let $\{\rho_n\}$ be a sequence in S convergent to some $\rho \in [0, 1]$. Then, for every n in $\mathbb{N}$, there exists a normed real algebra B_n satisfying $\|xy\| \geq \rho_n\|x\|\|y\|$ and $\|xy - yx\| \leq (1 - \rho_n)\|x\|\|y\|$ for all x, y in B_n, and such that $\dim(B_n) \geq 3$. Take an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ refining the Fréchet filter. Then, for all x, y in the normed real algebra $(B_n)_{\mathcal{U}}$ we have $\|xy\| \geq \rho\|x\|\|y\|$ and $\|xy - yx\| \leq (1 - \rho)\|x\|\|y\|$, and from Lemma 1.2 we deduce $\dim((B_n)_{\mathcal{U}}) \geq 3$. Therefore ρ lies in S. Now that the claim is proved, we define K_0 as the maximum element of S. With such a definition of K_0, certainly properties i) and ii) in the theorem hold. On the other hand, in view of Example 1.1, we have $2^{-1/2} \leq K_0$. Finally, since 1 does not belong to S (thanks to the commutative rbanik-Wright theorem), we obtain $K_0 < 1$. ∎

Corollary 1.4. *Let* A *be an absolute-valued real algebra satisfying* $\|xy - yx\| \leq (1 - \rho)\|x\|\|y\|$ *for some* $\rho > K_0$ *and all* x, y *in* A. *Then* A *is isometrically isomorphic to* $\mathbb{R}, \mathbb{C}$, *or* $\overset{*}{\mathbb{C}}$.

Proof. By Theorem 1.3, we have $\dim(A) \leq 2$. But it is well-known (see for instance [8] and [12]) that an absolute-valued real algebra with dimension ≤ 2 is isometrically isomorphic to either $\mathbb{R}, \mathbb{C}, \overset{*}{\mathbb{C}}, {}^*\mathbb{C}$, or $\mathbb{C}^*$. Here ${}^*\mathbb{C}$ and $\mathbb{C}^*$ stand for the algebras obtained from $\mathbb{C}$ by replacing its natural product by the ones $(x, y) \to x^*y$ and $(x, y) \to xy^*$, respectively. Since, in either ${}^*\mathbb{C}$ and $\mathbb{C}^*$, the commutator of the real and imaginary units of $\mathbb{C}$ has norm equal to 2 ($> 1 - \rho$), the result follows. ∎

The proof of the following theorem does not involve new ideas, and hence is omitted.

Theorem 1.5. *There exists a universal constant* $2^{-1/2} \leq K_1 < 1$ *uniquely determined by the following two properties:*

i) *There is a normed commutative real algebra* B $\dim(B) \geq 3$ *and satisfying* $\|xy\| \geq K_1 \|x\| \|y\|$ *for all* x, y *in* B.
ii) *If* A *is any normed commutative real algebra satisfying* $\|xy\| \geq \rho \|x\| \|y\|$ *for some* $\rho > K_1$ *and all* x, y *in* A*, then* $\dim(A) \leq 2$.

A consequence of the commutative rbanik-Wright theorem is that every absolute-valued commutative complex algebra is isometrically isomorphic to $\mathbb{C}$. Keeping this fact in mind, one can repeat the arguments in the proof of the results shown above in order to obtain their appropriate complex variants. Of course, we limit ourselves to formulate such variants.

Theorem 1.6. *There exists a universal constant* $2^{-1/2} \leq K_0^{\mathbb{C}} < 1$ *uniquely determined by the following two properties:*

i) *There is a normed complex algebra* B *with* $\dim(B) \geq 2$ *and satisfying* $\|xy\| \geq K_0^{\mathbb{C}} \|x\| \|y\|$ *and* $\|xy - yx\| \leq (1 - K_0^{\mathbb{C}}) \|x\| \|y\|$ *for all* x, y *in* B.
ii) *If* A *is any normed complex algebra satisfying* $\|xy\| \geq \rho \|x\| \|y\|$ *and* $\|xy - yx\| \leq (1 - \rho) \|x\| \|y\|$ *for some* $\rho > K_0^{\mathbb{C}}$ *and all* x, y *in* A*, then* A *is isomorphic to* $\mathbb{C}$.

Corollary 1.7. *Let* A *be an absolute-valued complex algebra satisfying* $\|xy - yx\| \leq (1 - \rho) \|x\| \|y\|$ *for some* $\rho > K_0^{\mathbb{C}}$ *and all* x, y *in* A*. Then* A *is isometrically isomorphic to* $\mathbb{C}$.

Theorem 1.8. *There exists a universal constant* $2^{-1/2} \leq K_1^{\mathbb{C}} < 1$ *uniquely determined by the following two properties:*

i) *There is a normed commutative complex algebra* B *with* $\dim(B) \geq 2$ *and satisfying* $\|xy\| \geq K_1^{\mathbb{C}} \|x\| \|y\|$ *for all* x, y *in* B.
ii) *If* A *is any normed commutative complex algebra satisfying* $\|xy\| \geq \rho \|x\| \|y\|$ *for some* $\rho > K_1^{\mathbb{C}}$ *and all* x, y *in* A*, then* A *is isomorphic to* $\mathbb{C}$.

Remark 1.9. In view of Corollaries 1.4 and 1.7, one can think about the greatest positive constant $C_{\mathbb{K}}$ such that, for every absolute-valued algebra A over $\mathbb{K}$ failing to the commutativity, we have

$$\sup\{\|xy - yx\| : x, y \in A, \|x\| = \|y\| = 1\} \geq C_{\mathbb{K}} \, .$$

With the help of the absolute-valued algebra B over $\mathbb{K}$ parameterizing Example 1.1, we can show that $C_{\mathbb{K}} \leq 2^{1/2}$. Indeed, for x, y in B, we have

$$\|xy - yx\|^2 = \|xy\|^2 + \|yx\|^2 - 2Re(xy|yx) = 2(\|x\|^2 \|y\|^2 - |(x|y)|^2) \, ,$$

and hence $\|xy - yx\| \leq 2^{1/2} \|x\| \|y\|$.

2. Nearly absolute valued algebras with a left unit

A consequence of the structure theory developed in [5] for absolute-valued real algebras with a left unit is that the norms of such algebras derive from inner products. Consequently, the normed spaces of absolute-valued real algebras with a left unit are uniformly non-square. In this section we prove that "most" nearly absolute-valued real algebras with a left unit are uniformly non-square. The main relevance of this result relies on the fact that the completion of a uniformly non-square normed space is a superreflexive Banach space [13, Theorem VII.4.4]. We recall that a normed space X is said to be *uniformly non-square* if there exists $0 < \sigma < 1$ such that the inequality

$$\min\{\|x+y\|, \|x-y\|\} < 2\sigma$$

holds for all x, y in the closed unit ball of X, and we note that there are absolute-valued algebras whose normed spaces are not uniformly non-square (indeed, the real or complex Banach space ℓ_1, endowed with a suitable product, becomes an absolute-valued algebra [9]).

As the next example shows, not all nearly absolute-valued real algebras with a left unit are uniformly non-square.

Example 2.1. Take A equal to $\mathbb{C}$ (regarded as a real algebra) endowed with the algebra norm $\|.\|$ defined by $\|\alpha + i\beta\| := |\alpha| + |\beta|$. Since $\|1\| = \|i\| = 1$ and $\|1+i\| = \|1-i\| = 2$, certainly A is not uniformly non-square. On the other hand, denoting by $|.|$ the usual module function on $\mathbb{C}$, for x in A we have $|x| \leq \|x\| \leq 2^{1/2}|x|$. It follows

$$\|xy\| \geq |xy| = |x||y| \geq 2^{-1}\|x\|\|y\|$$

for all x, y in A.

Let A be an algebra over $\mathbb{K}$. For x in A, we denote by L_x the operator of left multiplication by x on A. An element x of A is said to be *left-invertible* in A if the operator L_x is bijective. We denote by $L - Inv(A)$ the set of all left-invertible elements of A. We say that A is a *left-division algebra* whenever every non-zero element of A is left-invertible. Now assume that A is normed. An element x of A is said to be a *left topological divisor of zero* in A if there exists a sequence $\{x_n\}$ of norm-one elements of A such that $\lim\{xx_n\} = 0$ (equivalently, if the operator L_x is not bounded below). Clearly, nearly absolute-valued algebras have no non-zero left topological divisors of zero.

By the Banach isomorphism theorem, left-invertible elements in a complete normed algebra cannot be left topological divisors of zero. Therefore, complete normed left-division algebras have no non-zero left topological divisors of zero. The next lemma provides us with a partial converse of this fact.

Lemma 2.2. *Let A be a complete normed algebra with no non-zero left topological divisors of zero and such that $L - Inv(A)$ is non empty. Then A is a left division algebra.*

Proof. Clearly, we can assume that the dimension of A is greater than one. We argue by contradiction, so that we assume that there are no non-zero left topological divisors of zero in A and that $L - Inv(A)$ is a non-empty proper subset of $A\backslash\{0\}$. Then, since $A\backslash\{0\}$ is connected, there must exist some x_0 in the boundary of $L - Inv(A)$ relative to $A\backslash\{0\}$. For such an x_0, L_{x_0} lies in the boundary of the set of all invertible elements in the Banach algebra $BL(A)$ of all bounded linear operators on A. By [14, Lemma 56.3 and Theorem 57.4], the operator L_{x_0} is not bounded below. In this way, x_0 is a non-zero left topological divisor of zero in A, a contradiction. ∎

Lemma 2.3. *Let A be a normed real algebra with a left unit e and such that there exists $\rho > 0$ satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for all x, y in A. Then, for every two-dimensional subspace M of A with $e \in M$, there is a linear mapping φ from M to the two-dimensional euclidean real space satisfying*

$$\rho\|m\| \leq \|\varphi(m)\| \leq \|m\|$$

for all m in M.

Proof. By passing to the completion of A if necessary, we may assume that A is complete. Then, since $e \in L - Inv(A)$, it follows from Lemma 2.2 that A is a left-division algebra. Let us take a complex Banach algebra B containing isometrically $BL(A)$ as a closed real subalgebra and whose unit is the same as that of $BL(A)$ (namely, the identity mapping $\mathbf{1}$ on A) [15, Proposition 13.3]. Now, for x in $A\backslash\{0\}$, L_x is an invertible element of B satisfying $\|(L_x)^{-1}\| \leq (\rho\|x\|)^{-1}$. Therefore, for every x in A and every complex number z in $Sp(B, L_x)$ (the spectrum of L_x relative to B), we have

$$\rho\|x\| \leq |z| \leq \|x\| .$$

Finally, for x in $A\backslash\mathbb{R}e$ and α, β in $\mathbb{R}$, we may choose z in $Sp(B, L_x)$, so that $\alpha + \beta z$ belongs to $Sp(B, L_{\alpha e+\beta x})$, and we have

$$\rho\|\alpha e + \beta x\| \leq |\alpha + \beta z| \leq \|\alpha e + \beta x\| .$$

∎

Proposition 2.4. *Let A be a normed real algebra satisfying the following conditions:*

i) *There is an element a in A such that aA is dense in A.*
ii) *There exists $\rho > 0$ such that the inequality $\|xy\| \geq \rho\|x\|\|y\|$ holds for all x, y in A.*

Then, for every two-dimensional subspace M of A, there is a linear mapping φ from M to the two-dimensional euclidean real space satisfying

$$\rho^2\|m\| \leq \|\varphi(m)\| \leq \|m\|$$

for all m in M.

Proof. We may assume that A is complete. Then, since the mapping L_a is a topological embedding, it must have a complete (hence closed) range, and so, since it has dense range (by the assumption), it is in fact bijective.

Therefore, by Lemma 2.2, A is a left-division algebra. Let M be a two-dimensional subspace of A. Take a basis $\{u,v\}$ of M such that $\|u\| = \rho^{-1}$. Then, for every x in A, we have

$$\rho\|x\| \leq \|L_u^{-1}(x)\| \leq \|x\| .$$

Now, consider the real algebra B consisting of the vector space of A and the product $x\Box y := L_u^{-1}(xy)$. Then, for x,y in B, we have

$$\|x\Box y\| = \|L_u^{-1}(xy)\| \leq \|xy\| \leq \|x\|\|y\| ,$$

so that $\|.\|$ becomes an algebra norm on B. On the other hand, again for x,y in B, we have

$$\|x\Box y\| = \|L_u^{-1}(xy)\| \geq \rho\|xy\| \geq \rho^2\|x\|\|y\| .$$

Since u is a left unit for B, the existence of a linear mapping φ from M to the two-dimensional euclidean real space satisfying $\rho^2\|m\| \leq \|\varphi(m)\| \leq \|m\|$ for every m in M follows from Lemma 2.3. ∎

Taking $\rho = 1$ in the above proposition, we obtain that, if A is an absolute valued real algebra, and if there exists a in A such that aA is dense in A, then A is a prehilbert space ([5], [16]). For ρ near 1, we get the following corollary.

Corollary 2.5. *Let A be a normed real algebra satisfying the following conditions:*

i) *There is an element a in A such that aA is dense in A.*
ii) *There exists $\rho > 2^{-1/4}$ such that the inequality $\|xy\| \geq \rho\|x\|\|y\|$ holds for all x,y in A.*

Then A is uniformly non-square.

Proof. Choose σ with $2^{-1/2}\rho^{-2} < \sigma < 1$, and let x,y be in the closed unit ball of A. By Proposition 2.4, there exists a linear mapping φ from the linear hull of $\{x,y\}$ (say M) to an euclidean real space space satisfying

$$\rho^2\|m\| \leq \|\varphi(m)\| \leq \|m\|$$

for every m in M. It follows

$$[\min\{\|x+y\|,\|x-y\|\}]^2 \leq 2^{-1}(\|x+y\|^2 + \|x-y\|^2)$$
$$\leq 2^{-1}\rho^{-4}(\|\varphi(x+y)\|^2 + \|\varphi(x-y)\|^2) = \rho^{-4}(\|\varphi(x)\|^2 + \|\varphi(y)\|^2)$$
$$\leq \rho^{-4}(\|x\|^2 + \|y\|^2) \leq 2\rho^{-4} < 4\sigma^2 .$$

∎

Lemma 2.6. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ refining the Fréchet filter, and, for n in $\mathbb{N}$, let X_n be a normed space over $\mathbb{K}$ which is not uniformly non-square. Then the ultraproduct $(X_n)_\mathcal{U}$ is not uniformly non-square.*

Proof. For n in $\mathbb{N}$, there exist x_n, y_n in the closed unit ball of X_n such that

$$\min\{\|x_n + y_n\|, \|x_n - y_n\|\} \geq 2(1 - \frac{1}{n}).$$

Then (x_n) and (y_n) are elements in the closed unit ball of $(X_n)_{\mathcal{U}}$ satisfying

$$\|(x_n) + (y_n)\| = \|(x_n) - (y_n)\| = 2.$$

■

Theorem 2.7. *There exists a universal constant* $2^{-1} \leq K \leq 2^{-1/4}$ *uniquely determined by the following two properties:*

i) *There is a normed real algebra* B *which is not uniformly non-square, has a left unit, and satisfies* $\|xy\| \geq K\|x\|\|y\|$ *for all* x, y *in* B.

ii) *If* A *is any normed real algebra with a left unit and satisfying* $\|xy\| \geq \rho\|x\|\|y\|$ *for some* $\rho > K$ *and all* x, y *in* A*, then* A *is uniformly non-square.*

Proof. Let S denote the set of those elements ρ in $[0,1]$ such that there exists a normed real algebra B which is not uniformly non-square, has a left unit, and satisfies $\|xy\| \geq \rho\|x\|\|y\|$ for all x, y in B. Let us prove that S is closed in $\mathbb{R}$. To this end, since certainly 0 belongs to S, it is enough to show that, if $\{\rho_n\}$ is a sequence in S convergent to some $\rho \in [0,1]\backslash\{0\}$, then ρ lies in S. Let $\{\rho_n\}$ be such a sequence, and let ρ be its limit, so that we can assume that $\rho_n \geq \frac{\rho}{2}$ for all n in $\mathbb{N}$. Then, for every n in $\mathbb{N}$, there exists a normed real algebra B_n which is not uniformly non-square, has a left unit e_n, and satisfies $\|xy\| \geq \rho_n\|x\|\|y\|$ for all x, y in B_n. Taking an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ refining the Fréchet filter, for all x, y in the normed real algebra $(B_n)_{\mathcal{U}}$ we have $\|xy\| \geq \rho\|x\|\|y\|$, and from Lemma 2.6 we deduce that $(B_n)_{\mathcal{U}}$ is not uniformly non-square. Moreover, for n in $\mathbb{N}$ we have

$$\|e_n\| = \|(e_n)^2\| \geq \rho_n\|e_n\|^2,$$

and hence $\|e_n\| \leq \frac{1}{\rho_n} \leq \frac{2}{\rho}$. This fact allows us to consider the element (e_n) in $(B_n)_{\mathcal{U}}$, which is a left unit for $(B_n)_{\mathcal{U}}$. It follows that ρ lies in S. Now that we know that S is closed in $\mathbb{R}$, we define K as the maximum element of S, so that properties i) and ii) in the theorem hold. Finally, the inequalities $2^{-1} \leq K$ and $K \leq 2^{-1/4}$ follow from Example 2.1 and Corollary 2.5, respectively. ■

Remark 2.8. It is folklore that complete normed left-division complex algebras are isomorphic to $\mathbb{C}$. Indeed, for non-zero elements x, y in a complete normed left-division complex algebra A, we may choose a complex number λ in $Sp(BL(A), L_x \circ (L_y)^{-1})$, so that $L_{x-\lambda y} = L_x - \lambda L_y = [L_x \circ (L_y)^{-1} - \lambda \mathbf{1}] \circ L_y$ is a noninvertible element of $BL(A)$, and therefore we have $x = \lambda y$. Now, let A be a nearly absolute-valued complex algebra such that, for some element a in A, aA is dense in A. By the beginning of the proof of Proposition 2.4, we know that the completion of A is a left-division algebra. If follows that A is isomorphic to $\mathbb{C}$.

3. Nearly absolute-valued algebras with a unit

According to the noncommutative rbanik-Wright theorem [2, Theorem 1], absolute-valued real algebras with a unit are isometrically isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ (the algebra of Cayley numbers). As the next theorem shows, the consequence that absolute-valued real algebras with a unit are finite-dimensional remains true for "many" nearly absolute valued real algebras.

Theorem 3.1. *There exists a universal constant $0 \leq K_2 < 1$ uniquely determined by the following two properties:*

i) *There is a normed infinite-dimensional real algebra B with a unit and satisfying $\|xy\| \geq K_2\|x\|\|y\|$ for all x, y in B.*
ii) *If A is any normed real algebra with a unit and satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for some $\rho > K_2$ and all x, y in A, then A is finite-dimensional.*

Proof. Let S denote the set of those elements ρ in $[0, 1]$ such that there exists a normed infinite-dimensional real algebra B with a unit and satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for all x, y in B. With Lemma 1.2 instead of Lemma 2.6, we can argue as in the proof of Theorem 2.7 to obtain that S is closed in $\mathbb{R}$. Then we define K_2 as the maximum element of S, so that properties i) and ii) in the theorem hold. Since 1 does not belong to S (thanks to the noncommutative rbanik-Wright theorem), we obtain $K_2 < 1$. ∎

For an element x in an algebra A, we denote by R_x the operator of right multiplication by x on A.

Corollary 3.2. *Let A be a normed real algebra satisfying the following two conditions:*

i) *There exists $\rho > (K_2)^{1/3}$ such that the inequality $\|xy\| \geq \rho\|x\|\|y\|$ holds for all x, y in A.*
ii) *There exist a, b in A such that aA and Ab are dense in A.*

Then A is finite-dimensional.

Proof. Without loss of generality, we can assume that $\|a\| = \|b\| = \rho^{-1}$ and, by passing to the completion of A if necessary, we can also assume that A is complete. Then, arguing as in the proof of Proposition 2.4, we obtain that the operators L_a and R_b are bijective. Moreover, for every x in A, we have

$$\rho\|x\| \leq \|L_a^{-1}(x)\| \leq \|x\| \text{ and } \rho\|x\| \leq \|R_b^{-1}(x)\| \leq \|x\| .$$

Now, consider the real algebra B consisting of the vector space of A and the product $x \Box y := R_b^{-1}(x)L_a^{-1}(y)$. Then, for x, y in B, we have

$$\|x \Box y\| = \|R_b^{-1}(x)L_a^{-1}(y)\| \leq \|R_b^{-1}(x)\|\|L_a^{-1}(y)\| \leq \|x\|\|y\| ,$$

so that $\|.\|$ becomes an algebra norm on B. On the other hand, again for x, y in B, we have

$$\|x\Box y\| = \|R_b^{-1}(x)L_a^{-1}(y)\| \geq \rho\|R_b^{-1}(x)\|\|L_a^{-1}(y)\| \geq \rho^3\|x\|\|y\|\,.$$

Since $\rho^3 > K_2$, and B has a unit (namely, ab), it follows from Theorem 3.1 that B (and hence A) is finite-dimensional. ■

Corollary 3.2 above allows us to prove a general theorem on automatic continuity of homomorphisms into nearly absolute-valued algebras. Such a theorem extends [5, Theorem 4].

Theorem 3.3. *Let A be a complete normed real algebra, B a normed real algebra satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for some $\rho > (K_2)^{1/3}$ and all x, y in B, and Φ a homomorphism from A into B. Then Φ is continuous with $\|\Phi\| \leq \rho^{-1}$.*

Proof. We follow with minor changes the argument in the proof of [5, Theorem 4]. Regarding Φ as a mapping from A into the completion of its range, we can assume that B is complete and that Φ has dense range. First assume additionally that Φ is actually surjective. Then, from the obvious equality $\Phi L_x = L_{\Phi(x)}\Phi$ (for x in A) and [17, Lemma 3.1] we obtain

$$r(L_{\Phi(x)}) \leq r(L_x) \leq \|L_x\| \leq \|x\|$$

where $r(.)$ denotes spectral radius. But, for y, z in B and n in $\mathbb{N}$, we have

$$\|(L_y)^n(z)\| \geq \rho^n\|y\|^n\|z\|\,,$$

so $\|(L_y)^n\| \geq \rho^n\|y\|^n$, and so $r(L_y) = \lim\{\|(L_y)^n\|^{1/n}\} \geq \rho\|y\|$. Therefore, in the first considered case we have $\|\Phi(x)\| \leq \rho^{-1}r(L_{\Phi(x)}) \leq \rho^{-1}\|x\|$ for every x in A. Now, consider the remaining case that Φ is not surjective. Then, since Φ has dense range, B must be infinite-dimensional. Now, replacing A and B by their corresponding opposite algebras if necessary, it follows from Corollary 3.2 that all left multiplication operators on B are not surjective. To conclude the proof it is enough to show that $\rho \leq \|x\|$ whenever x belongs to A and satisfies $\|\Phi(x)\| = 1$. Let x be such an element in A, and $0 < \lambda < \rho$. Then $L_{\Phi(x)}$ is a nonsurjective continuous linear operator on B bounded below by ρ, and $\|L_{\Phi(x)} - (L_{\Phi(x)} - \lambda I_B)\| < \rho$ (where I_B denotes the identity operator on B). It follows from the proof of [5, Lemma 1] that the operator $L_{\Phi(x)} - \lambda I_B$ has closed range in B and is not surjective. Since

$$\Phi(L_x - \lambda I_A) = (L_{\Phi(x)} - \lambda I_B)\Phi$$

and Φ has dense range, we deduce that $L_x - \lambda I_A$ is not surjective, and hence $|\lambda| \leq \|L_x\|(\leq \|x\|)$ (by standard Banach algebra theory). By letting $\lambda \to \rho$, we obtain $\rho \leq \|x\|$, as required. ■

A *semi-H^*-algebra* over $\mathbb{K}$ is an algebra A over $\mathbb{K}$ endowed with a conjugate-linear vector space involution $*$ and a complete inner product $(.|.)$ satisfying

$$(xy|z) = (x|zy^*) \ = \ (y|x^*z)$$

for all x, y, z in A. Since the product of any semi-H^*-algebra is automatically continuous [18, Proposition 2.i)], up to the multiplication of the inner product by a suitable positive number, every semi-H^*-algebra becomes a complete normed algebra. The next result is a variant of [16, Theorem 1.6].

Corollary 3.4. *Let A be a real semi-H^*-algebra such that there exists an algebra norm $|||.|||$ on A satisfying $|||xy||| \geq \rho |||x||| |||y|||$ for some $\rho > (K_2)^{1/3}$ and all x, y in A. Then A is finite-dimensional.*

Proof. Let $\|.\|$ denote the natural hilbertian norm of A. Applying Theorem 3.3 to the identity mapping $(A, \|.\|) \to (A, |||.|||)$, we obtain that such a mapping is continuous, i.e., the $\|.\|$-topology is stronger than the $|||.|||$-topology. Assume that A is infinite-dimensional. Then, by Corollary 3.2, for all a, b in A we have that either aA or Ab are not $|||.|||$-dense in A. This allows us to suppose that there exists a non zero element a in A such that aA is not $|||.|||$-dense in A, say. A fortiori, for such an a, aA is not $\|.\|$-dense in A, an therefore there must exist a non zero element c in A satisfying $(aA|c) = 0$. This implies $(A|a^*c) = 0$, and hence $a^*c = 0$. Finally, we have the contradiction

$$0 = |||a^*c||| \geq (K_2)^{1/3} |||a^*||| |||c||| > \ 0 \, .$$

■

Remark 3.5. According to Remark 2.8, every nearly absolute-valued complex algebra with a unit is isomorphic to $\mathbb{C}$. We do not know if every nearly absolute-valued real algebra with a unit is finite-dimensional (equivalently, if the universal constant K_2 in Theorem 3.1 is equal to zero). If there were some infinite-dimensional nearly absolute-valued real algebra A with a unit, then, by Lemma 2.2, the completion of A would become an infinite-dimensional complete normed (two-sided) division real algebra with a unit. The existence of such a monster would answer by the negative a natural old problem, first raised in [19].

4. Nearly absolute-valued algebraic algebras

We recall that an algebra A is called *algebraic* if every single-generated subalgebra of A is finite-dimensional. According to the main theorem in [3] (see also [20]), absolute-valued algebraic real algebras are finite-dimensional. In this section we prove an extension of this result to the setting of nearly absolute-valued algebras.

Lemma 4.1. *Let $\{X_i\}_{i \in I}$ be a family of normed spaces over $\mathbb{K}, \mathcal{U}$ an ultrafilter on I, and m a natural number. Assume that* $\dim(X_i) \leq m$ *for every i in I. Then* $\dim((X_i)_{\mathcal{U}}) \leq m$.

Proof. Let $u_1 = (x_{i,1})_{i\in I}, ..., u_{m+1} = (x_{i,m+1})_{i\in I}$ be in $(X_i)_{\mathcal{U}}$. For each i in I, there exist $\lambda_{i,1}, ..., \lambda_{i,m+1}$ in $\mathbb{K}$ satisfying

$$\sum_{k=1}^{m+1} \lambda_{i,k} x_{i,k} = 0 \text{ and } \sum_{k=1}^{m+1} |\lambda_{i,k}| = 1\,.$$

For $k = 1, ..., m+1$, put $\lambda_k := \lim_{\mathcal{U}}\{\lambda_{i,k}\}_{i\in I}$. Then we easily obtain

$$\sum_{k=1}^{m+1} \lambda_k u_k = 0 \text{ and } \sum_{k=1}^{m+1} |\lambda_k| = 1\,.$$

■

For an element x in an algebra A, we denote by $A(x)$ the subalgebra of A generated by x.

Proposition 4.2. *Let $\{A_i\}_{i\in I}$ be a family of normed algebraic algebras over $\mathbb{K}$ with no non-zero divisors of zero, and $\mathcal{U}$ an ultrafilter on I. Then $(A_i)_{\mathcal{U}}$ is an algebraic algebra.*

Proof. Let $u = (x_i)$ be in $(A_i)_{\mathcal{U}}$. By [1, Chapter 11], for i in I we have $\dim(A_i(x_i)) \leq 8$. Now, $(A_i(x_i))_{\mathcal{U}}$ is a subalgebra of $(A_i)_{\mathcal{U}}$ containing u and satisfying $\dim((A_i(x_i))_{\mathcal{U}}) \leq 8$ (by Lemma 4.1). It follows $\dim((A_i)_{\mathcal{U}}(u)) \leq 8$. ■

Theorem 4.3. *There exists a universal constant $0 \leq K_3 < 1$ uniquely determined by the following two properties:*

i) *There is a normed infinite-dimensional algebraic real algebra B satisfying $\|xy\| \geq K_3\|x\|\|y\|$ for all x, y in B.*
ii) *If A is any normed algebraic real algebra satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for some $\rho > K_3$ and all x, y in A, then A is finite-dimensional.*

Proof. Let S denote the set of those elements ρ in $[0,1]$ such that there exists a normed infinite-dimensional algebraic real algebra B satisfying $\|xy\| \geq \rho\|x\|\|y\|$ for all x, y in B. Let $\{\rho_n\}$ be a sequence in S convergent to some $\rho \in [0,1]\backslash\{0\}$, so that we can assume that $\rho_n \neq 0$ for all n in $\mathbb{N}$. Then, for every n in $\mathbb{N}$, there exists a normed infinite-dimensional algebraic real algebra B_n satisfying $\|xy\| \geq \rho_n\|x\|\|y\|$ for all x, y in B_n. Taking an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ refining the Fréchet filter, for all x, y in the normed real algebra $(B_n)_{\mathcal{U}}$ we have $\|xy\| \geq \rho\|x\|\|y\|$, and from Lemma 1.2 we deduce $\dim((B_n)_{\mathcal{U}}) = \infty$. Moreover, by Proposition 4.2, $(B_n)_{\mathcal{U}}$ is algebraic. Therefore ρ lies in S. Since clearly zero belongs to S, the above argument shows that S is closed in $\mathbb{R}$. Defining K_3 as the maximum element of S, properties i) and ii) in the theorem hold. Since 1 does not belong to S (by [3, Theorem 5.1]), we obtain $K_3 < 1$. ■

We say that an algebra A is *power-commutative* if every single-generated subalgebra of A is commutative. For power-commutative absolute-valued algebras, the reader is referred to [21]. The next corollary follows directly from Theorems 1.5 and 4.3.

Corollary 4.4. *Put* $K_4 := \max\{K_1, K_3\}$*, where* K_1 *and* K_3 *are the universal constants given by Theorems 1.5 and 4.3, respectively, and let* A *be any normed power-commutative real algebra satisfying* $\|xy\| \geq \rho\|x\|\|y\|$ *for some* $\rho > K_4$ *and all* x, y *in* A*. Then* A *is finite-dimensional.*

We note that power-associative algebras, as well as flexible algebras, are power-commutative. We recall that an algebra A is said to be *power-associative* if every single-generated subalgebra of A is associative, whereas A is called *flexible* if the identity $x(yx) = (xy)x$ holds for all x, y in A. For power-associative (respectively, flexible) absolute-valued algebras, the reader is referred to [22] (respectively, [23]).

Remark 4.5. It is folklore that every algebraic complex algebra with no non-zero divisors of zero is isomorphic to $\mathbb{C}$ (see [3, pp. 296-297] for a proof). It follows that nearly absolute-valued algebraic complex algebras are isomorphic to $\mathbb{C}$.

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References

[1] Ebbinghaus, H.-D.; Hermes, H.; Hirzebruc, F.; Koecher, M.; Mainzer, K.; Neukirch, J.; Prestel, A.; Remmert, R. *Numbers;* Graduate texts in Mathematics, Vol. 123, Springer-Verlag: New York, 1990.

[2] Urbanic, K.; Wright, F. B. Absolute valued algebras. Proc. Amer. Math. Soc. **1960**, *11*, 861-866.

[3] Kaidi, A. M.; Ramírez, M. I.; Rodríguez, A. Absolute-valued algebraic algebras are finite-dimensional. J. Algebra **1997**, *195*, 295-307.

[4] Cuenca, J. A. On one-sided division infinite-dimensional normed real algebras. Publ. Mat. **1992**, *36*, 485-488.

[5] Rodríguez A. One-sided division absolute valued algebras. Publ. Mat. **1992**, *36*, 925-954.

[6] Kaplansky I. Normed algebras. Duke Math. J. **1949**, *16*, 399-418.

[7] Cabrera M.; Rodríguez A. A new simple proof of the Gelfand-Mazur-Kaplansky theorem. Proc. Amer. Math. Soc. **1995**, *123*, 2663-2666.

[8] Albert, A. A. Absolute valued real algebras. Ann. Math. **1947**, *48*, 495-501.

[9] Rodríguez, A. *Números hipercomplejos en dimensión infinita;* Academia de Ciencias de Granada: Granada, 1993.

[10] Heinrich, S. Ultraproducts in Banach space theory. J. Reine Angew. Math. **1980**, *313*, 72-104.

[11] Mathieu, M. Rings of quotients of ultraprime Banach algebras with applications to elementary operators. Proc. Centre Math. Anal. Austral. Nat. Univ. **1989**, *21*, 297-317.

[12] Albert, A. A. Absolute valued algebraic algebras. Bull. Amer. Math. Soc. **1949**, *55*, 763-768. A note of correction. Ibid. **1949**, *55*, 1191.

[13] Day, M. M. *Normed linear spaces;* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 21; Springer-Verlag: Berlin, 1973.

[14] Berberian, S. K. *Lectures in functional analysis and operator theory;* Springer-Verlag: Berlin-Heidelberg-New York, 1973.

[15] Bonsall F. F.; Duncan, J. *Complete normed algebras;* Springer-Verlag: Berlin, 1973.

[16] Cuenca J. A.; Rodríguez A. Absolute values on H^*-algebras. Commun. Algebra **1995**, *23*, 1709-1740.

[17] Rodríguez A. The uniqueness of the complete algebra norm topology in complete normed nonassociative algebras. J. Functional Analysis **1985**, *60*, 1-15.

[18] Cuenca, J. A.; Rodríguez, A. Structure theory for noncommutative Jordan H^*-algebras. J. Algebra **1987**, *106*, 1-14.

[19] Wright, F. B. Absolute valued algebras. Proc. Nat. Acad. Sci. U.S.A. **1953**, *39*, 330-332.

[20] Kaidi, A. M.; Ramírez, M. I.; Rodríguez A. Absolute-valued algebraic algebras. In *Proceedings of the International Conference on Jordan Structures (Málaga, June 1997)* (Ed. A. Castellón, J. A. Cuenca, A. Fernández, and C. Martín), 103-109, Málaga, 1999.

[21] Rodríguez A. Absolute-valued algebras of degree two. In *Non-associative algebra and its applications*, 350-356, Kluwer Academic Publishers: Dordrecht-Boston-London, 1994.

[22] El-Mallah, M. L.; Micali, A. Sur les algèbres normées sans diviseurs topologiques de zéro. Bol. Soc. Mat. Mexicana **1980**, *25*, 23-28.

[23] El-Mallah, M. L.; Micali, A. Sur les dimensions des algèbres absolument valuées. J. Algebra **1981**, *68*, 237-246.

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