Introducing Analysis in Zelmanov's theorems for Jordan systems

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Keywords: Prime Jordan-Banach algebras, prime JB^*-algebras, prime JB^*-triples.
AMS Subject Classification: 17C65, 46H20, 46H70, 46A30.

ABSTRACT

In this paper we survey in detail the applications of Zelmanov's prime theorems on Jordan structures to the study of normed Jordan algebras and triples. Such a study includes primitive Jordan-Banach algebras, simple normed Jordan algebras, nondegenerately ultraprime Jordan-Banach complex algebras, and prime JB^*-algebras, JB-algebras, real JB^*-triples, complex JB^*-triples, real JBW^*-triples, and complex JBW^*-triples.

INTRODUCTION

In a series of papers (see [74], [75], [76], [77], and [78]) E. I. Zelmanov provided the mathematical community with his surprising classification theorems for prime nondegenerate Jordan algebras and triples. Zelmanov's prime theorems, though received enthusiastically by algebraists since its appearance, have taken a relatively long time to be assimilated by analysts in order to obtain new structure theorems for normed prime nondegenerate Jordan algebras and triples, results that cannot be attacked by the familiar technique of the existence of a nonzero socle or by duality methods in JBW^*-theory. The reason could be that the formulation of Zelmanov's prime theorems, in order to attains a nice simple form, perhaps conceals in their formulations some crucial information that is needed in the applications. This means for the analyst the necessity of finding out the deep and very difficult proofs of Zelmanov's theorems, a fact that takes time. Fortunately this time has been already taken, and we are presenting in this paper relevant examples of the application of Zelmanov's techniques to the structure of normed Jordan algebras and triples. In the case of JB^*-algebras and JB^*-triples, the results obtained refine in a very nontrivial way the classical theory.

The material we are reviewing has been partially surveyed in other papers (see [19], [24], [29], [43], [54], [60], [61], [62], and [63]). In the present paper we tray to offer a complete panoramic view of such a material. In reviewing the results, we have not respected the chronology of their appearance. In fact we have preferred to assemble the different results according to their nature. Thus, we collect in Sections 1 and 2, those structure theorems which are of algebraic-topological type. This means that the description of the normed Jordan structures considered in such theorems is made up to bicontinuous isomorphisms. This is the case of primitive Jordan-Banach algebras (Theorem 1.8), simple normed Jordan algebras (Theorem 2.1), and nondegenerately ultraprime Jordan-Banach complex algebras (Theorem 2.2). Section 3 remains in the spirit of results of algebraic-topological nature, and deals with the so-called "norm extension problem", which in its roots is crucially related to normed versions of Zelmanov's prime theorems for Jordan structures. Sections 4 and 5 are devoted to collect structure theorems of geometric type. In these cases the description of the normed Jordan structures is achieved up to isometric isomorphisms. Actually, results of such a geometric nature are known to date only in the setting of JB- and JB^*-algebras, and JB^*-triples. The description of prime JB^*-algebras, JB-algebras, real JB^*-triples, complex JB^*-triples, real JBW^*-triples, and complex JBW^*-triples is given by Theorems 4.5, 4.6, 5.2, 5.1, 5.4 and 5.5, respectively. The concluding section of the paper (Section 6) is devoted to notes and remarks.

1. Primitive Jordan-Banach algebras

Along this paper $\mathbb{F}$ will denote a field of characteristic different from two. A *Jordan algebra* over $\mathbb{F}$ is a commutative algebra over $\mathbb{F}$ satisfying the Jordan identity:

$$(x^2.y).x = x^2.(y.x) \ .$$

If A is an associative algebra with product denoted by yuxtaposition ab, then its symmetrization A^+, which has the same vector space and the symmetric or Jordan product $a.b := (ab + ba)$, is a Jordan algebra. Subalgebras of A^+ are called *Jordan subalgebras* of A. If the associative algebra A has an involution $*$, then the set $H(A, *)$ (of all $*$-invariant elements in A) becomes a nice example of a Jordan subalgebra of A. We remark that, unless stated otherwise, by an involution on an algebra we mean a LINEAR ALGEBRA involution.

Jordan algebras which are isomorphic to Jordan subalgebras of associative algebras are called *special*. A Jordan algebra is said to be *exceptional* whenever it is not special.

Symmetric bilinear forms on vector spaces produce, in a very transparent way, relevant examples of Jordan algebras. Let X be a vector space over $\mathbb{F}$ and $f : X \times X \to \mathbb{F}$ be a symmetric bilinear form. Then the vector space $\mathbb{F}\mathbf{1} \oplus X$ with the product defined by

$$(\alpha\mathbf{1} + x).(\beta\mathbf{1} + y) := (\alpha\beta + f(x, y))\mathbf{1} + (\alpha y + \beta x)$$

is a Jordan algebra over $\mathbb{F}$ denoted by $J(X, f)$ and called the *Jordan algebra of the (symmetric) bilinear form f (on the vector space X).* Although not obvious, such an algebra is special (it can be seen as a Jordan subalgebra of the Clifford algebra of f, see [38, VII, §1]).

Let J be a Jordan algebra. For x, y in J we denote by $U_{x,y}$ the linear operator on J defined by $U_{x,y}(z) := x.(y.z) + y.(x.z) - (x.y).z$ and put $U_x := U_{x,x}$. An *inner ideal* of J is a subspace I of J such that $U_I(J^1) \subseteq I$, where J^1 denote the unital hull of J. We say that an inner ideal I in J is *e-modular* for some $e \in J$ if

$$U_{1-e}(J) \subseteq I, U_{1-e,I}(J^1) \subseteq I, \text{ and } e - e^2 \in I.$$

We will say that an inner ideal M is *e-maximal* if it is maximal among all proper e-modular inner ideals. We say that M is *maximal-modular* if it is e-maximal for some e. A modular inner ideal is proper if and only if it excludes its modulus [34, Proposition 2.10]. Therefore, via Zorn's Lemma, any proper e-modular inner ideal is contained in an e-maximal inner ideal. Following [74] and [34], we say that the Jordan algebra J is *primitive* if there exists a maximal-modular inner ideal of J containing no non-zero ideals of J.

The structure of primitive Jordan algebras is given by the next variant of the Zelmanov prime theorem [75]. Such a variant has been obtained independently by A. Anquela, F. Montaner, and T. Cortés [6] and V. G. Skosyrsky [69].

Theorem 1.1. *The primitive Jordan algebras over $\mathbb{F}$ are the following:*

i) *The finite-dimensional central simple exceptional Jordan algebras over a field extension of $\mathbb{F}$;*
ii) *Jordan algebras of a nondegenerate symmetric bilinear form on a vector space X over a field Ω extension of $\mathbb{F}$ with $\dim_\Omega(X) \geq 2$;*
iii) *Jordan subalgebras of the Martindale algebra of symmetric quotients, $Q(A)$, containing A as an ideal, where A is a primitive associative algebra over $\mathbb{F}$;*
iv) *Jordan subalgebras of $Q(A)$ contained in $H(Q(A), *)$ and containing $H(A, *)$ as an ideal, where A is a primitive associative algebra over $\mathbb{F}$ with a linear algebra involution.*

The structure of primitive Jordan-Banach algebras obtained in [19] and [20], which will be reviewed in what follows, consists of a case-by-case Banach treatment of the above theorem.

We begin by considering cases $i), ii)$, and $iii)$ in Theorem 1.1, whose normed study does not need any Zelmanovian technique.

An algebra A is said to be *alternative* if for every x, y in A the equalities

$$x^2y = x(xy) \text{ and } yx^2 = (yx)x$$

hold. Every finite-dimensional central simple alternative not associative algebra A over $\mathbb{F}$ is 8-dimensional over $\mathbb{F}$ and has a unit $\mathbf{1}$ and an involution "$^-$" satisfying

$$x + \overline{x} \in \mathbb{F}\mathbf{1} \text{ and } x\overline{x} \in \mathbb{F}\mathbf{1}$$

[68, Theorem 3.17 and Chapter III, §4]. Such algebras are called *octonion algebras* over $\mathbb{F}$ and the involution "$^-$" above is called the *standard* involution. Over any field $\mathbb{F}$, there exists a unique octonion algebra (denoted by $C(\mathbb{F})$) having divisors of zero [68, Lemma 3.16]. As a consequence, if $\mathbb{F}$ is algebraically closed, then $C(\mathbb{F})$ is the unique octonion algebra over $\mathbb{F}$. It is also well-known that there are only two non-isomorphic octonion algebras over $\mathbb{R}$, namely $C(\mathbb{R})$ and the more familiar one denoted by $\mathbb{O}$, which of course is a division algebra.

Given an octonion algebra C over $\mathbb{F}$ and $\gamma_1, \gamma_2, \gamma_3$ non-zero elements in $\mathbb{F}$, we denote by $M_3(C)$ the algebra of all $3x3$ matrices with entries in C (with the usual matrix product) and by Γ the diagonal matrix

$$\Gamma := \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\} \ .$$

If we consider the involution $*$ on $M_3(C)$ given by $X \to X^* := \Gamma^{-1}\overline{X}^t\Gamma$, where $X^t := (\overline{x_{ji}})$ when $X = (x_{ij})$, then the subspace $H_3(C, \Gamma) := H(M_3(C), *)$ of all $*$-invariant elements in $M_3(C)$, endowed with the symmetrized product, becomes a 27-dimensional central simple exceptional Jordan algebra over $\mathbb{F}$ [68, Theorem 4.8]. When Γ is equal to the identity mapping, we simply write $H_3(C) := H_3(C, \Gamma)$.

If either $\mathbb{F}$ is algebraically closed or $\mathbb{F} = \mathbb{R}$, then the construction above becomes specially relevant in view of the following theorem (see [1, Theorems 4 and 10]).

Theorem 1.2. i) *If $\mathbb{F}$ is algebraically closed, then $H_3(C(\mathbb{F}))$ is the unique finite-dimensional exceptional simple Jordan algebra over $\mathbb{F}$.*

ii) *There are exactly three non-isomorphic finite-dimensional exceptional central simple Jordan algebras over $\mathbb{R}$, namely $H_3(C(\mathbb{R}))$, $H_3(\mathbb{O})$, and $H_3(\mathbb{O}, diag\{1, -1, 1\})$.*

Now the normed treatment of case *i*) in Theorem 1.1 reduces to putting together Theorem 1.2 and the next immediate consequence of the Gelfand-Mazur theorem. From now on, $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$.

Proposition 1.3. *Let J be a unital algebra over a field extension $\mathbb{F}$ of $\mathbb{K}$. If J is a normed algebra over $\mathbb{K}$, then $\mathbb{F} = \mathbb{C}$ if $\mathbb{K} = \mathbb{C}$, and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ if $\mathbb{K} = \mathbb{R}$.*

Now we pass to deal with the normed treatment of Jordan algebras of a bilinear form. If $(X, \|.\|)$ is a normed space over $\mathbb{K}$ and if f is a continuous symmetric bilinear form on X with $\|f\| \leq 1$, then the Jordan algebra $J(X,f) = \mathbb{K}\mathbf{1} \oplus X$ with norm $\|.\|$ defined by

$$\|\alpha\mathbf{1} + x\| := |\alpha| + \|x\|$$

becomes a normed algebra called the *normed Jordan algebra of the continuous symmetric bilinear form f on the normed vector space* $(X, \|.\|)$. Clearly, such a normed algebra is complete if and only if X is a Banach space. The following result asserts that, up to a topological isomorphism, these algebras are the unique Jordan algebras of a symmetric bilinear form which are normed algebras.

Proposition 1.4. *Let $(J, \|.\|)$ be a normed Jordan algebra over $\mathbb{K}$ and assume that $J = J(X,f)$ for a symmetric bilinear form f on a vector space X over a field $\mathbb{F}$ extension of $\mathbb{K}$. Then $\mathbb{F} = \mathbb{C}$ if $\mathbb{K} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ if $\mathbb{K} = \mathbb{R}$. Moreover, there exists a norm $|\!|\!| \cdot |\!|\!|$ on X such that X is a normed space over $\mathbb{F}$, f is continuous with $|\!|\!| f |\!|\!| \leq 1$, and the norm $\|.\|$ in J is equivalent to the norm given by $\alpha\mathbf{1} + x \to |\alpha| + |\!|\!| x |\!|\!|$.*

With more or less precision in the formulation and proof, the content of the above proposition often arises in the literature, mainly in the case $\mathbb{K} = \mathbb{C}$ (see for example [10], [23], and [59]). The actual formulation of Proposition 1.4 is taken from [19], where a complete proof is given.

The Banach treatment of case iii) in Theorem 1.1 was made in [17] and [19] for $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$, respectively, given rise to the result collected in the next proposition. For a normed space X, we denoted by $BL(X)$ the normed algebra of all bounded linear operators on X.

Proposition 1.5. *Let $(J, \|.\|)$ a Jordan-Banach algebra over $\mathbb{K}$ and assume that there exists a primitive associative algebra A over $\mathbb{K}$ such that J is a Jordan subalgebra of $Q(A)$ containing A as an ideal. Then there exists a Banach space X over $\mathbb{K}$ and a one-to-one homomorphism Φ from $Q(A)$ into the Banach algebra $BL(X)$ such that $\Phi(A)$ acts irreducibly on X and the restriction of Φ to J is contiuous.*

The remaining part of this section is devoted to the Banach treatment of case *iv*) of Theorem 1.1, where the application of Zelmanovian techniques become crucial.

From now on, $\mathbf{X}$ will stand for a countably infinite set of indeterminates. We denote by $\mathcal{A}(\mathbf{X})$ the free associative algebra (over a prefixed field $\mathbb{F}$) on $\mathbf{X}$, and by $\mathcal{J}(\mathbf{X})$ the free special Jordan algebra over $\mathbb{F}$ on $\mathbf{X}$, namely the Jordan subalgebra of $\mathcal{A}(\mathbf{X})$ generated by $\mathbf{X}$. Intuitively, the elements of $\mathcal{J}(\mathbf{X})$, called *Jordan polynomials*, are those elements in $\mathcal{A}(\mathbf{X})$ which can be obtained from that of $\mathbf{X}$ by a finite process of taking sums, Jordan products, and products by elements of $\mathbb{F}$. If $*$ denotes the unique involution on $\mathcal{A}(\mathbf{X})$ fixing the elements of $\mathbf{X}$, we clearly have $\mathcal{J}(\mathbf{X}) \subseteq H(\mathcal{A}(\mathbf{X}), *)$. For every element a in any algebra with involution $*$, put $\{a\} := \frac{1}{2}(a + a^*)$. Following [49], we say that a Jordan polynomial $\mathbf{p}$ (involving m indeterminates, say $\mathbf{x}_1, ..., \mathbf{x}_m$) is an *imbedded pentad eater* if there exist a natural number k and Jordan polynomials $\mathbf{p}_j^i (1 \leq i \leq k, 1 \leq j \leq 3)$ involving $m+4$ indeterminates such that, for all natural numbers r and s and all $\mathbf{z}_1, \ldots, \mathbf{z}_r, \mathbf{y}_1, \ldots, \mathbf{y}_4, \mathbf{w}_1, \ldots, \mathbf{w}_s$ in $\mathbf{X}$, we have in $\mathcal{A}(\mathbf{X})$

$$\{\mathbf{z}_1 \ldots \mathbf{z}_r \mathbf{y}_1 \ldots \mathbf{y}_4 \mathbf{p} \mathbf{w}_1 \ldots \mathbf{w}_s\} = \sum_{i=1}^{k} \{\mathbf{z}_1 \ldots \mathbf{z}_r \mathbf{p}_1^i \mathbf{p}_2^i \mathbf{p}_3^i \mathbf{w}_1 \ldots \mathbf{w}_s\},$$

where to be brief in the second side of the equality we have written $\mathbf{p}_j^i$ instead of $\mathbf{p}_j^i(\mathbf{y}_1, \ldots, \mathbf{y}_4, \mathbf{x}_1, \ldots, \mathbf{x}_m)$. The set of all imbedded pentad eaters is a subspace of $\mathcal{J}(X)$, and, in fact, an ideal of $\mathcal{J}(X)$ [5, Theorem 2.7], which is denoted by I_5. For any special Jordan algebra J, $I_5(J)$ will mean the ideal of J of all valuations on J of the polynomials in I_5.

Let J be a prime Jordan algebra. Then every non-zero element in the centre Z of J is not a divisor of zero in J, and therefore, if $Z \neq 0$, then we can consider the "álgebra of fractions" JZ^{-1}, which is called the *central localization* of J [80, pp. 185-186]. J is called an *Albert ring* (respectively, a *central order in the Jordan algebra of a bilinear form*) if $Z \neq 0$ and its central localization is a central simple 27-dimensional exceptional Jordan algebra (respectively, a Jordan algebra of a bilinear form on a vector space) over the field of fractions ZZ^{-1}.

The celebrated Zelmanov's prime theorem for Jordan algebras [75] asserts that, if J is a nondegenerate (i.e., $x \in J$ and $U_x = 0$ implies $x = 0$) prime Jordan algebra, and if J is neither an Albert ring nor a central order in a Jordan algebra of a bilinear form, then there exists a $*$-prime associative algebra $(A, *)$ generated by $H(A, *)$ such that J can be seen as a Jordan subalgebra of $Q(A)$ contained in $H(Q(A), *)$ and containing $H(A, *)$ as an ideal. The proof of Zelmanov's theorem shows that the algebra A above can be chosen with the additional property that $I_5(J) = H(A, *)$ (see [49] for details). Precisely thanks to the above equality we were able to prove in [20] the following "germinal" normed version of the Zelmanov prime theorem.

Theorem 1.6. *Let $(J, \|.\|)$ be a prime nondegenerate normed Jordan algebra over $\mathbb{K}$, and assume that J is neither an Albert ring nor a central order in the Jordan algebra of a bilinear form. Then there exists a normed $*$-prime associative algebra $(A, *, \| \! | \cdot | \! \|)$ over $\mathbb{K}$, which is generated in a purely algebraic sense by $H(A, *)$, such that J can be seen as a Jordan subalgebra of $Q(A)$ contained in $H(Q(A), *)$ and containing $H(A, *)$ as an ideal, and the following properties are satisfied:*

i) $\|h\| \leq \| \! | h | \! \|$ *for all h in $H(A, *)$,*

ii) *If a is in A, and satisfies $aJ + Ja \subseteq A$, then the mappings $x \to ax$ and $x \to xa$ from $(J, \|.\|)$ into $(A, \| \! | \cdot | \! \|)$ are continuous,*

iii) $\| \! | a^* | \! \| = \| \! | a | \! \|$ *for all a in A, and*

iv) *If* $(\hat{A}, *)$ *denotes the completion of* $(A, *)$ *relative to the norm* $|||\,.\,|||$*, then every nonzero* $*$*-ideal of* $\hat{A}$ *meets* $H(\hat{A}, *)$.

If the norm $\|.\|$ on the Jordan algebra J above is complete, then a better result holds. Such a result is derived in [20] from Theorem 1.6 and a "very Zelmanovian" theorem on extensions of Jordan homomorphisms due to K. McCrimmon [48, Theorem 2.2], and reads as follows.

Theorem 1.7. *Let* $(J, \|.\|)$ *be a prime nondegenerate Jordan-Banach algebra over* $\mathbb{K}$*, and assume that* J *is neither an Albert ring nor a central order in the Jordan algebra of a bilinear form. Then there exists a normed* $*$*-prime associative algebra* $(A, *, |||\,.\,|||)$ *over* $\mathbb{K}$*, which is generated in a purely algebraic sense by* $H(A, *)$*, such that* J *can be seen as a Jordan subalgebra of* $Q(A)$ *contained in* $H(Q(A), *)$ *and containing* $H(A, *)$ *as an ideal, and the following properties are satisfied:*

i) *If* a *is in* A*, and satisfies* $aJ + Ja \subseteq A$*, then the mappings* $x \to ax$ *and* $x \to xa$ *from* $(J, \|.\|)$ *into* $(A, |||\,.\,|||)$ *are continuous,*
ii) $|||\, a^* \,||| = |||\, a \,|||$ *for all* a *in* A;
iii) *If* $(\hat{A}, *, |||\,.\,|||)$ *denotes the completion of* $(A, *, |||\,.\,|||)$*, then the inclusion* $A \subseteq Q(A)$ *extends in a unique way to a one-to-one* $*$*-homomorphism* $\hat{A} \hookrightarrow Q(A)$ *mapping* $H(\hat{A}, *)$ *into* J *and satisfying* $\|h\| \leq |||\, h \,|||$ *for all* h *in* $H(\hat{A}, *)$*, and*
iv) *Every nonzero* $*$*-ideal of* $\hat{A}$ *meets* $H(A, *)$.

Theorems 1.6 and 1.7 make no direct reference to primitive Jordan algebras. However, as matter of fact, primitive Jordan algebras are particular examples of prime nondegenerate Jordan algebras, and we were able to obtain the structure of primitive Jordan-Banach algebras (as we are presenting here) only by passing through the "germinal" complete normed version of Zelmanov's prime theorem provided by Theorem 1.7. All the remaining material collected in this section is also involved in the proof of the structure theorem of primitive Jordan-Banach algebras. Such a theorem was proved in [20] and [19] for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$, respectively, and reads as follows.

Theorem 1.8. *A Jordan-Banach algebra* J *over* $\mathbb{K}$ *is primitive (if and) only if one of the following assertions holds:*

i) $J = H_3(C(\mathbb{C}))$ *if* $\mathbb{K} = \mathbb{C}$*, and* $J = H_3(C(\mathbb{C}))$*,* $J = H_3(C(\mathbb{R}))$*,* $J = H_3(\mathbb{O})$*, or* $J = H_3(\mathbb{O}, diag\{1, -1, 1\})$ *if* $\mathbb{K} = \mathbb{R}$.
ii) J *is the Jordan-Banach algebra of a continuous nondegenerate symmetric bilinear form on a Banach space over* $\mathbb{F}$ *with* $\dim_{\mathbb{F}}(X) \geq 2$*, where* $\mathbb{F} = \mathbb{C}$ *if* $\mathbb{K} = \mathbb{C}$*, and* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ *if* $\mathbb{K} = \mathbb{R}$.

iii) *There exist a Banach space X over $\mathbb{K}$ and an associative subalgebra A of $BL(X)$ acting irreducibly on X such that J can be seen as a Jordan subalgebra of $BL(X)$ containing A as an ideal, and the inclusion $J \hookrightarrow BL(X)$ is continuous.*

iv) *There exist a Banach space X over $\mathbb{K}$ and an associative subalgebra A of $BL(X)$ acting irreducibly on X such that J can be seen as a Jordan subalgebra of $BL(X)$, the inclusion $J \hookrightarrow BL(X)$ is continuous, the identity mapping on J extends to an involution $*$ on the subalgebra B of $BL(X)$ generated by J, A is a $*$-invariant subset of B, $H(A,*)$ is an ideal of J, and A is generated by $H(A,*)$.*

2. Strong-versus-light normed versions of the Zelmanov prime theorem

The classification theorem for primitive Jordan-Banach algebras we have just reviewed becomes an example of the so called "light" normed versions of the Zelmanov prime theorem. This means that the topology of the norm of the normed Jordan algebra J in cases iii) and iv) of the theorem does not arise as the restriction to J of the topology of some algebra norm on its natural associative envelope (in our present case, the subalgebra of $BL(X)$ generated by J). There are in the literature examples of "strong" (i.e., free of the above pathology) versions of the Zelmanov prime theorem, like the following.

Theorem 2.1 ([22])**.** *Up to bicontinuous isomorphisms, the simple (complete) normed Jordan algebras over $\mathbb{K}$ are the following:*

i) $H_3(C(\mathbb{C}))$ *if* $\mathbb{K} = \mathbb{C}$, *and* $H_3(C(\mathbb{C}))$, $H_3(C(\mathbb{R}))$, $H_3(\mathbb{O})$, *and* $H_3(\mathbb{O}, diag\{1,-1,1\})$ *if* $\mathbb{K} = \mathbb{R}$.

ii) *The Jordan algebras $J(X,f)$ of a continuous nondegenerate symmetric bilinear form f on a (complete) normed vector space X over $\mathbb{F}$ with $\dim_{\mathbb{F}}(X) \geq 2$, where $\mathbb{F} = \mathbb{C}$ if $\mathbb{K} = \mathbb{C}$, and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ if $\mathbb{K} = \mathbb{R}$.*

iii) *The Jordan algebras of the form A^+, where A is a simple (complete) normed associative algebra over $\mathbb{K}$ with a unit.*

iv) *The Jordan algebras of the form $H(A,*)$, where A is a simple (complete) normed associative algebra over $\mathbb{K}$ with a unit and $*$ is an isometric involution on A.*

In fact Theorem 2.1 was proved in [22] only for the case $\mathbb{K} = \mathbb{C}$, but, with the help of Proposition 1.3, Theorem 1.2, and Proposition 1.4, the arguments in the proof remain valid for the case $\mathbb{K} = \mathbb{R}$.

Another strong normed version of the Zelmanov prime theorem is the one proved in [23] for nondegenerately ultraprime Jordan-Banach algebras. A normed Jordan algebra J is said to be nondegenerately ultraprime if there exists a countably incomplete ultrafilter $\mathcal{U}$ on a suitable set such that the corresponding normed ultrapower $J_{\mathcal{U}}$ is prime and nondegenerate. With the Beidar-Mikhalev-Slin'ko characterization of prime nondegenerate Jordan algebras [9], it can be proved easily that a normed Jordan algebra J is nondegenerately ultraprime if and only if there exists $k > 0$ such that $\|U_{x,y}\| \geq k\|x\|\|y\|$ for all x, y in J. As a consequence, all normed ultrapowers of a nondegenerately ultraprime normed Jordan algebra are prime and nondegenerate. Following ideas by M. Mathieu in [45] and [47], M. Cabrera and A. Rodríguez introduced in [23] *ultra-τ-prime* normed associative algebras with continuous involution τ, which can be characterized without any reference to ultrapowers as those normed associative algebras A with continuous involution τ satisfying

$$\max\{\|M_{a,b}\|, \|M_{a^\tau,b}\|\} \geq k\|a\|\,\|b\|$$

for some fixed $k > 0$ and all a, b in A, where $M_{a,b}(c) := acb$ for every c in A. For such an ultra-τ-prime normed associative algebra (A, τ), a large τ-invariant subalgebra $Q_b(A)$ of its symmetric Martindale algebra of quotients can be converted in an ultra-τ-prime normed algebra in such a way that the natural embedding $A \hookrightarrow Q_b(A)$ becomes a topological embedding. Then Jordan subalgebras of $Q_b(A)$ contained in $H(Q_b(A), \tau)$ and containing $H(A, \tau)$ are examples of nondegerately ultraprime normed Jordan algebras. Now, the main result in [23] reads as follows.

Theorem 2.2. *Up to bicontinous isomorphisms, the nondegenerately ultraprime Jordan-Banach complex algebras are the following:*

i) $H_3(C(\mathbb{C}))$.

ii) *The Jordan-Banach algebras $J(X, f)$ of a continuous nondegenerate symmetric bilinear form f on a complex Banach space X with* $\dim(X) \geq 2$ *and such that the natural embedding $x \to f(.,x)$ from X into its dual is topological.*

iii) *The closed Jordan subalgebras of $Q_b(A)$ contained in $H(Q_b(A), \tau)$ and containing $H(A, \tau)$ as an ideal, where A is an ultra-τ-prime complex Banach algebra with continuous involution τ such that $H(A, \tau)$ generates A as a Banach algebra.*

The proof of the above theorem is very long and difficult. For a summary of the main tools involved in such a proof the reader is referred to [60, p. 167].

Despite the examples of strong normed versions of Zelmanov's prime theorem provided by Theorems 2.1 and 2.2 in some particular settings, a general strong normed version of Zelmanov's theorem cannot be expected. Nor even for normed simple Jordan algebras without a unit (see [18, Remark 4] for details). This is a consequence of the next theorem. We denote by $M_\infty(\mathbb{K})$ the simple associative algebra of all countably infinite matrices over $\mathbb{K}$ with a finite number of non-zero entries.

Theorem 2.3 ([18])**.** *There exists an algebra norm $\|.\|$ on $M_\infty(\mathbb{K})^+$ which is not equivalent to any algebra norm on $M_\infty(\mathbb{K})$. More precisely, there exists an involution $*$ on $M_\infty(\mathbb{K})$ such that there is no algebra norm $\vert\!\vert\!\vert\,.\,\vert\!\vert\!\vert$ on $M_\infty(\mathbb{K})$ such that the restrictions of $\|.\|$ and $\vert\!\vert\!\vert\,.\,\vert\!\vert\!\vert$ to $H(M_\infty(\mathbb{K}), *)$ are equivalent.*

We showed in [18, Section 3] that, for a suitable choice of the pathological norm $\|.\|$ above, the completions of the normed Jordan algebras $(M_\infty(\mathbb{K})^+, \|.\|)$ and $(H(M_\infty(\mathbb{K}), *), \|.\|)$ are primitive Jordan-Banach algebras over $\mathbb{K}$ whose topologies cannot be obtained by restricting to them the topology of any algebra norm in their natural associative envelopes. In other words, a strong normed version of Theorem 1.8 cannot be expected.

Strong normed versions of the Zelmanov prime theorem depend heavily on the so called "norm extension problem", which will be considered in some detail in the next section. As a matter of fact, one of the main results in [64] (see Theorem 3.1) essentially links the norm extension problem with the continuity of a typical non-Jordan polynomial (namely, the "tetrad"). This invited us to apply the techniques in the proof of Theorem 1.5 to obtain analytical characterizations of Jordan polynomials, a question previously tried out in [8]. In this direction we proved the following result.

Theorem 2.4 ([21])**.** *An associative polynomial $\mathbf{p}$ over $\mathbb{K}$ is a Jordan polynomial if and only if, for every algebra norm $\|.\|$ on $M_\infty(\mathbb{K})^+$, the action of $\mathbf{p}$ on $M_\infty(\mathbb{K})$ is $\|.\|$-continuous.*

Actually a better result holds: There exists an algebra norm $\|.\|$ on $M_\infty(\mathbb{K})^+$ such that Jordan polynomial over $\mathbb{K}$ are those associative polynomials which act $\|.\|$-continuously on $M_\infty(\mathbb{K})$ [50].

3. The norm extension problem

Throughout this section C will stand for a (possibly non associative) algebra over $\mathbb{K}$ endowed with an involution $*$. As in the associative case, we denote by $H(C, *)$ the subspace of C consisting of all $*$-invariant elements in C, and we will consider $H(C, *)$ as an algebra over $\mathbb{K}$ under the *symmetrized product* $x \circ y := \frac{1}{2}(xy + yx)$. Obviously, if $\|.\|$ is an algebra norm on C, then its restriction to $H(C, *)$ is an algebra norm on $H(C,*)$. The so called *norm extension problem* (in short, NEP) is the following.

NEP. Given $(C,*)$ as above, and an algebra norm $\|.\|$ on $H(C,*)$, is there an algebra norm on C whose restriction to $H(C,*)$ is equivalent to $\|.\|$?

It is easy to see that, when we are able to answer affirmatively the above question, then actually we can choose the algebra norm on C extending the topology of the norm $\|.\|$ on $H(C,*)$ in such a way that $*$ becomes continuous. It is also not difficult to realize that, in studying the NEP, the additional assumption that the algebra C is a $*$-*tight envelope* of $H(C,*)$ (i.e., C is generated by $H(C,*)$, and every nonzero $*$-invariant ideal of C has nonzero intersection with $H(C,*)$) is not too restrictive. Indeed, in any case one can find an algebra with involution (D,τ) such that (D,τ) is a τ-tight envelope of $H(D,\tau)$, and the algebras $H(D,\tau)$ and $H(C,*)$ are isomorphic.

Since, as we commented in the previous section, strong normed versions of Zelmanov‘s prime theorem crucially depends on the norm extension problem, we are reviewing in detail in the present section the main results about that problem.

Assume that the algebra C is associative and that the NEP has an affirmative answer. Then, clearly, the *tetrad mapping*

$$xyzt \to \{xyzt\} := \frac{1}{2}(xyzt + tzyx)$$

from $H(C,*)\times H(C,*)\times H(C,*)\times H(C,*)$ to $H(C,*)$ is $\|.\|$-continuous. The following partial converse of the fact just quoted was proved in [64, Corollary 1].

Theorem 3.1. *Assume that C is associative and a $*$-tight envelope of $H(C,*)$. Then the NEP has an affirmative answer if (and only if) the tetrad mapping is $\|.\|$-continuous.*

The two following positive results on the norm extension problem involve in their proof the general criterium given by the above theorem.

Theorem 3.2 ([64, Theorem 2])**.** *Assume that C is associative and a $*$-tight envelope of $H(C,*)$, that $H(C,*)$ is semiprime, and that the norm $\|.\|$ on $H(C,*)$ is complete. Then the NEP has an affirmative answer.*

Theorem 3.3 ([18, Theorem 4])**.** *Assume that C is associative and a $*$-tight envelope of $H(C,*)$, and that $H(C,*)$ is simple and has a unit. Then the NEP has an affirmative answer.*

The last theorem is an example of a *global* affirmative answer to the NEP. This means that the NEP answers affirmatively without any condition on the algebra norm $\|.\|$ on $H(C,*)$, and hence, under suitable algebraic assumptions (for instance, if C is associative and a $*$-tight envelope of $H(C,*)$, and $H(C,*)$ is simple and has a unit), every algebra norm on $H(C,*)$ can be extended to an algebra norm on C.

According to Theorem 2.3, for $C = M_\infty(\mathbb{K})$ with a suitable involution $*$ there exists an algebra norm $\|.\|$ on $H(C,*)$ such that the NEP has a negative answer. This shows that neither the assumption of completeness of the norm $\|.\|$ in Theorem 3.2 nor that the existence of a unit for $H(C,*)$ in Theorem 3.3 can be removed.

Although, in relation to the analytic treatment of Zelmanov's prime theorem, the norm extension problem is only interesting in the case that the algebra C is associative, in some positive results on that problem the associativity of C is not needed. This is the case of the paper [55]. Let B be an algebra with a unit and an involution $*$. If $b_1, ..., b_n$ are $*$-invariant invertible elements in the nucleus of B (see [38, p. 18] for the definition of the nucleus), and if we put $d := diag\{b_1, ..., b_n\}$, then the operator $*$ on $M_n(B)$ given by $(b_{ij})^* := d^{-1}(b_{ji}^*)d$ is an involution on $M_n(B)$. Involutions on $M_n(B)$ defined in this way are called *canonical involutions.* The *standard involution* on $M_n(B)$ is nothing but the canonical involution corresponding to the identity diagonal matrix. In its easiest form, the main result of [55] reads as follows.

Theorem 3.4 ([55, Theorem 3.3]). *The NEP has an affirmative answer whenever C is of the form $M_n(B)$, for some natural number $n \geq 3$ and some (possibly non associative) algebra B over $\mathbb{K}$ with a unit and an involution, and the involution $*$ on C is a canonical involution. Moreover, for $(C,*)$ as above, the following two assertions hold:*

i) Two algebra norms on C are equivalent whenever they make $*$ continuous and their restriction to $H(C,*)$ are equivalents (UNIQUENESS OF THE EXTENDED NORM TOPOLOGY).

ii) If the algebra norm $\|.\|$ on $H(C,*)$ is complete, then the essentially unique algebra norm on C making $*$ continuous and generating on $H(C,*)$ the topology of $\|.\|$ is complete too (COMPLETE-TO-COMPLETE EXTENSION PROPERTY).

Let $(A,*)$ be a finite dimensional $*$-simple associative complex algebra, and let n denote the *degree* of $H(A,*)$ (i.e., n is the smallest natural number such that every single-generated subalgebra of $H(A,*)$ has dimension less or equal to n). Then $(A,*)$ is isomorphic to $(M_n(\mathbb{D}),*)$, where $\mathbb{D}$ is a complex composition algebra and $*$ on $M_n(\mathbb{D})$ denotes the standard involution relative to the Cayley involution on $\mathbb{D}$ (see [38, pp. 208-209]). Now, let us consider in addition a complex algebra B with a unit and a involution. Then we have

$$A \otimes B = M_n(\mathbb{D}) \otimes B = M_n(\mathbb{C}) \otimes \mathbb{D} \otimes B = M_n(\mathbb{D} \otimes B),$$

and the tensor involution on $A \otimes B$ is noting but the standard involution on $M_n(\mathbb{D} \otimes B)$ relative to the tensor involution on $\mathbb{D} \otimes B$. It follows from

Theorem 3.4 (with $\mathbb{D} \otimes B$ instead of B) that, if $n \geq 3$, then the NEP has a global affirmative answer whenever C is of the form $A \otimes B$, for A and B as above, and the involution on C is the tensor involution. With some additional effort (see the proof of Theorem 3.5 in [55], for details), the result we have just shown for the case $\mathbb{K} = C$ remains essentially true when $\mathbb{K} = R$. In this way we have the following abstract version of Theorem 3.4 (note that, if $(A, *)$ is a unital $*$-simple associative algebra over $\mathbb{K}$, then $H(A, *)$ is a unital simple Jordan algebra and therefore the centre of $H(A, *)$ is a field).

Theorem 3.5 ([55, Theorem 3.5])**.** *The NEP has an affirmative answer whenever $(C, *)$ is of the form $(A, *) \otimes (B, *)$, where $(A, *)$ is a finite dimensional $*$-simple associative algebra over $\mathbb{K}$ whose hermitian part $H(A, *)$ is of degree ≥ 3 over its centre, and $(B, *)$ is a (possibly non associative) algebra with involution and a unit over $\mathbb{K}$. Moreover, for $(C, *)$ as above, we enjoy the uniqueness of the extended norm topology and the complete-to-complete extension property.*

We note that the affirmative answer to the NEP given by Theorems 3.4 and 3.5 is of global type.

The necessity of the assumptions in Theorems 3.4 and 3.5 are fully discussed in [55]. Actually, concerning the hypothesis $n \geq 3$ in Theorem 3.4 we have the next "anti-theorem" (see [55, Theorem 4.3] or [54]).

Theorem 3.6. *Let X be an arbitrary infinite-dimensional normed space over $\mathbb{K}$. Then there exists an associative algebra B over $\mathbb{K}$ with a unit, and an involution $*$ on B, satisfying:*

i) $X = H(M_2(B), *)$, as vector spaces.

ii) Up to multiplication by a suitable positive number if necessary, the norm of X becomes an algebra norm on $H(M_2(B), *)$.

iii) There is no algebra norm on $M_2(B)$ whose restriction to $H(M_2(B), *)$ is equivalent to the norm of X.

iv) $M_2(B)$ is a $*$-tight envelope of $H(M_2(B), *)$.

We note that the above "anti-theorem" shows in addition that the assumption of semiprimeness of $H(C, *)$ in Theorem 3.2 cannot be removed.

A relevant part of Theorem 3.5 remains true if we relax the assumption of finite dimensionality for the associative $*$-simple algebra A to the mere existence of a unit for A, but we assume that the algebra B is associative. This is proved in [55, Theorem 5.5], by applying Zelmanovian techniques, and precisely reads as follows.

Theorem 3.7. *The NEP has a global affirmative answer whenever $(C, *)$ is of the form $(A, *) \otimes (B, *)$, where $(A, *)$ is a unital $*$-simple associative algebra over $\mathbb{K}$ whose degree over its centre is ≥ 3 , and $(B, *)$ is a unital associative algebra with involution over $\mathbb{K}$.*

In obtaining the above theorem, the following purely algebraic fact becomes crucial. If $(A, *)$ is a $*$-simple associative algebra whose hermitian part is of degree ≥ 2 over its centroid, and if $(B, *)$ is a unital involutive algebra, then the involutive algebra $(A, *) \otimes (B, *)$ is a $*$-tight envelope of its hermitian part [55, Proposition 5.1].

To conclude this section, let us consider the NEP in the case that $H(C, *)$ is finite dimensional. If C is associative, and if $H(C, *)$ generates C, then it is well know that the equality $C = H(C, *) + H(C, *)^2 + H(C, *)^3$ holds. Therefore, if in addition $\dim H(C, *) < \infty$, then $\dim C < \infty$, and hence the NEP has an obvious and global affirmative answer. If C is not associative, a similar result cannot be expected. This is a consequence of the following "anti-theorem".

Theorem 3.8 ([53, Theorem 3]). *Let X be an arbitrary vector space over $\mathbb{K}$ with $\dim X \geq 2$. Then there exists an algebra C over $\mathbb{K}$ with involution $*$ satisfying the following conditions:*

i) *$X = H(C, *)$ as vector spaces.*
ii) *C is a $*$-tight envelope of $H(C, *)$.*
iii) *The product of $H(C, *)$ is zero (and hence every norm on X is an algebra norm on $H(C, *)$) .*
iv) *There is no algebra norm on C.*

4. Prime JB^*-algebras

A JB^*-algebra is a complete normed complex Jordan algebra J with a conjugate-linear algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in J.

Given a C^*-algebra A, the Jordan algebra A^+ becomes naturally a JB^*-algebra under the same norm and involution as those of A. If A is a C^*-algebra and τ is a $*$-involution on A (i.e., a $\mathbb{C}$-linear algebra involution commuting with $*$), then $H(A, \tau)$ is a norm-closed $*$-invariant Jordan subalgebra of A and hence a JB^*-algebra.

Applying [59, Theorems 3.2 and 3.3] with suitable simplification due to the commutativity, all JB^*-algebras which are Jordan algebras of a bilinear form arise by taking an arbitrary complex Hilbert space H, choosing a conjugation σ on H (which always exists and is essentially unique [33, Lemma 7.5.6]), and then by considering $J = \mathbb{C} \oplus H$ with Jordan product

$$(\lambda + h).(\mu + k) := [\lambda\mu + (h|\sigma(k))] + [\lambda k + \mu h]$$

involution

$$(\lambda + h)^* := \overline{\lambda} + \sigma(h)$$

and norm given for $x = \lambda + h$ by

$$\|x\|^2 := \|\!| x |\!\| + \sqrt{\|\!| x |\!\|^4 - |(x|x^{\#})|^2}$$

where $x^{\#} := \overline{\lambda} - \sigma(h)$, $(.|.)$ denotes the natural inner product on J regarded a ℓ_2-sum of $\mathbb{C}$ and H, and $\| x \| := \sqrt{(x|x)}$. We note that the algebras J just described are simple (and hence prime) whenever $\dim J \neq 2$. Moreover, since prime JB^*-algebras are central [67], when $\dim J \neq 2$ such algebras are the unique prime JB^*-algebras which are central orders in Jordan algebras of bilinear forms.

To complete the list of JB^*-algebras of classical type, let us say that the complex Jordan algebra $H_3(C(\mathbb{C}))$ can be structured as a JB^*-algebra [72], and this structure is essentially unique [58, Corollary 2.10 and Proposition 2.1]. The fact already quoted that prime JB^*-algebras are central, together with Theorem 1.2, implies that the JB^*-algebra $H_3(C(\mathbb{C}))$ is the unique prime JB^*-algebra which is an Albert ring.

According to the above comments, to get a classification of all prime JB^*-algebras it is enough to describe those prime JB^*-algebras which are neither an Albert ring nor a central order in the Jordan algebra of a bilinear form. To this end it was necessary to delve (lightly in this case) into the proof of Zelmanov's prime theorem [75], extracting some arguments that can be summarized in the following proposition (see also [49]).

Proposition 4.1. *Let B be an associative algebra with an involution τ, J a prime nondegenerate Jordan subalgebra of B contained in $H(B,\tau)$, and assume that J is not a central order in a Jordan algebra of a bilinear form. Then there exists a τ-invariant subalgebra A of B such that $H(A,\tau)$ is a nonzero ideal of J.*

Now, if the prime JB^*-algebra J is neither an Albert ring nor a central order in the Jordan algebra of a bilinear form, we may appeal to the classical theory of JB^*-algebras in order to select a specially well-behaved associative envelope B for J, to which Proposition 4.1 will be applied. The contribution of the classical JB^*-theory is the following (at this time folklore) Proposition.

Proposition 4.2 ([30, Proposition 1.2])**.** *For every special JB^*-algebra J, there exists a C^*-algebra B with $*$-involution τ such that J is a closed $*$-invariant Jordan subalgebra of B contained in $H(B,\tau)$.*

The arguments we are reviewing are nothing but the first observations in the paper by A. Fernández, E. García, and A. Rodríguez [30], where a fine JB^*-version of the Zelmanov prime theorem was provided. In the search for this result they were inspired by a recent one of P. Ara. He showed in [7] that, for a prime C^*-algebra A, the symmetric Martindale algebra of quotients $Q(A)$ coincides with the "symmetric algebra of bounded quotients" $Q_b(A)$. Since $Q_b(A)$ is a pre-C^*-algebra, its completion $Q_b(A)\hat{}$ became affectively an ideal candidate to play in the JB^*-case the role played by $Q(A)$ in the original Zelmanov's prime theorem. In fact, Fernández, García, and Rodríguez were able to replace $Q_b(A)\hat{}$ by the smaller and more familiar C^*-algebra $M(A)$ of multipliers on A. Recall that, for a semiprime associative algebra A, the symmetric Martindale algebra of quotients $Q(A)$ contains the subalgebra of multipliers $M(A)$, which in its turn contains A as an essential ideal. Recall also that, in the case that A is a C^*-algebra, $M(A)$ is in a natural way a C^*-algebra containing A as a C^*-subalgebra. For the proof of Zelmanov's prime theorem for JB^*-algebras some advances in the classical JB^*-theory were made in [30] concerning JB^*-algebras which contain closed essential ideals of classical type. We state these results in the following two propositions.

Proposition 4.3 ([30, Proposition 1.3]). *Let J be a JB^*-algebra containing a closed essential ideal that, regarded as a JB^*-algebra, is of the form A^+ for a suitable C^*-algebra A. Then J can be viewed as a closed $*$-invariant Jordan subalgebra of the C^*-algebra $M(A)$ containing A.*

Proposition 4.4 ([30, Proposition 1.4]). *Let J be a JB^*-algebra containing a closed essential ideal of the form $H(A, \tau)$ for a suitable C^*-algebra A with $*$-involution τ, and assume A is generated as a C^*-algebra by $H(A, \tau)$. Then J can be regarded as a closed $*$-invariant Jordan subalgebra of $M(A)$ contained in $H(M(A), \tau)$ and containing $H(A, \tau)$.*

When all the above results are put together, the following theorem follows easily.

Theorem 4.5 ([30, Theorem 2.3]). *The prime JB^*-algebras are the following:*

i) *The JB^*-algebra $H_3(C(\mathbb{C}))$.*
ii) *The JB^*-algebras which are Jordan algebras of bilinear forms and have dimension different from 2.*
iii) *The closed $*$-invariant Jordan subalgebras of $M(A)$ containing A, where A is a prime C^*-algebra.*
iv) *The closed $*$-invariant Jordan subalgebras of $M(A)$ contained in $H(M(A), \tau)$ and containing $H(A, \tau)$, where A is a prime C^*-algebra with $*$-involution τ.*

JB-algebras are defined as those complete normed Jordan real algebras B satisfying $\|x\|^2 \leq \|x^2 + y^2\|$ for all x, y in B. They were introduced by E. M. Alfsen, F. W. Shultz, and E. Stormer [2], and their basic theory is today nicely collected in [33]. According to the main results in [72] and [73], JB^*-algebras are in a bijective categorical correspondence with JB-algebras. The correspondence is obtained by passing from each JB^*-algebra to its selfadjoint part.

An easy consequence of the Theorem 4.5 is the next corollary (see the proof of [30, Corollary 2.4] for details).

Corollary 4.6. *The prime JB-algebras are the following:*

i) *The JB-algebra $H_3(\mathbb{O})$.*
ii) *The selfadjoint parts of those JB^*-algebras which are Jordan algebras of bilinear forms and have dimension different from 2.*
iii) *The closed Jordan subalgebras of $M(R)$ contained in the self-adjoint part of $M(R)$ and containing the self-adjoint part of R, where R is a prime REAL C^*-algebra.*

Another not difficult consequence of Theorem 4.5 is the following.

Corollary 4.7 ([30, Corollary 3.1])**.** *The topologically simple JB^*-algebras are the following:*

i) *The JB^*-algebra $H_3(C(\mathbb{C}))$.*
ii) *The JB^*-algebras which are Jordan algebras of bilinear forms and have dimension different from 2.*
iii) *The JB^*-algebras of the form A^+, where A is a topologically simple C^*-algebra.*
iv) *The JB^*-algebras of the form $H(A,\tau)$, where A is a topologically simple C^*-algebra with $*$-involution τ.*

When Theorem 4.5 is regarded under the light of the tools developed in [6] for the proof of Theorem 1.1, the next result, first formulated in [60, Theorem F.9], is obtained.

Theorem 4.8. *The primitive JB^*-algebras are the following:*

i) *The JB^*-algebra $H_3(C(\mathbb{C}))$.*
ii) *The JB^*-algebras which are Jordan algebras of bilinear forms and have dimension different from 2.*
iii) *The closed $*$-invariant Jordan subalgebras of $M(A)$ containing A, where A is a primitive C^*-algebra.*
iv) *The closed $*$-invariant Jordan subalgebras of $M(A)$ contained in $H(M(A),\tau)$ and containing $H(A,\tau)$, where A is a primitive C^*-algebra with $*$-involution τ.*

With the well-known result of Dixmier that separable prime C^*-algebras are primitive, the above theorem implies easily that *separable prime JB^**-algebras are primitive.

5. Prime JB^*-triples

A *complex JB^*-triple* is a complex Banach space $\mathcal{A}$ with a continuous triple product $\{\cdots\} : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ which is linear and symmetric in the outer variables, and conjugate linear in the middle variable, and satisfies

i) for all $x \in \mathcal{A}$, the mapping $a \mapsto \{xxa\}$ from $\mathcal{A}$ to $\mathcal{A}$ is a hermitian element (in the sense of [11, Definition §10.12]) of the complex Banach algebra $BL(\mathcal{A})$, and has nonnegative spectrum;
ii) $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$ (the main identity);
iii) $\|\{aaa\}\| = \|a\|^3$.

Complex JB^*-triples were introduced by W. Kaup in order to provide an algebraic setting for the study of bounded symmetric domains in complex Banach spaces. The open unit ball of every complex JB^*-triple is a bounded symmetric domain [40], and every bounded symmetric domain in any complex Banach space is bi-holomorphically equivalent to the open unit ball of a suitable complex JB^*-triple [41].

Fundamental examples of complex JB^*-triples are provided by JB^*-algebras, with triple product defined by

$$\{xyz\} := x.(y^*.z) + z.(y^*.x) - (x.z).y^*.$$

As a consequence, complex C^*-algebras are JB^*-triples under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x). \tag{1}$$

A larger class of complex JB^*-triples consists of the so-called ternary rings of operators [79], which are nothing but norm-closed subspaces of complex C^*-algebras, closed under the associative triple product xy^*z. Ternary rings of operators are seen as complex JB^*-triples by symmetrizing their associative triple products in the outer variables. A still larger class is that of complex JC^*-triples, i.e. JB^*-subtriples of complex C^*-algebras. The classical structure theory for complex JB^*-triples consists of a precise classification of certain prime complex JB^*-triples (the so-called "complex Cartan factors") and the fact that every complex JB^*-triple has a faithful family of Cartan factor representations. Complex Cartan factors come in six different types. Those of type **I** are prime ternary rings of operators, whereas the ones of type **II** and **III** are the hermitian parts of certain prime ternary rings of operators relative to suitable complex-linear triple involutions. Complex Cartan factors of type **IV** (called *complex spin factors*) are nothing but the JB^*-algebras which are Jordan algebras of a bilinear form and have dimension greater than 2, when they are regarded as complex JB^*-triples. Complex spin factors are JC^*-triples, but in general they are neither ternary rings of operators nor hermitian parts of ternary rings of operators. Complex Cartan factors of types **V** and **VI** are exceptional (i.e., they are not JC^*-triples). Exceptional complex Cartan factors are very scarce: exactly, there is a single member in each type. In fact, the type **VI** complex Cartan factor is nothing but the JB^*-algebra $H_3(C(\mathbb{C}))$, regarded as a complex JB^*-triple, and the type **V** complex Cartan factor is a distinguished subtriple of the type **VI** one.

Applying the techniques of E. Zelmanov in [76], [77], and [78] (see also [3] and [4]), we proved in [56] a classification theorem for general prime complex JB^*-triples, which, roughly speaking, asserts that prime complex JB^*-triples, which are neither spin factors nor exceptional Cartan factors, are "essentially" either prime ternary rings of operators or hermitian parts of prime ternary rings of operators. More precisely, our theorem establishes that, if J is a prime complex JB^*-triple, and if J is neither a spin factor nor an exceptional Cartan factor, then J contains a non-zero closed triple ideal which is either a prime ternary ring of operators or the hermitian part of a prime ternary ring of operators relative to a linear triple involution. By noticing that the multiplier complex JB^*-triple $M(R)$ (in the sense of [14]) of any ternary ring of operators R is also a ternary ring of operators, to which every linear triple involution on R extends uniquely, it follows that, for J as above, we have one of the following possibilities:

i) $R \subseteq J \subseteq M(R)$
ii) $H(R,\tau) \subseteq J \subseteq H(M(R),\tau)$,

where in both cases R is a prime ternary ring of operators, in the second case τ is a linear triple involution on R and $H(R,\tau)$ stands for the hermitian part of R relative to τ, the right inclusions much be read as "J is a JB^*-subtriple of ...", and consequently the left inclusions read as "... is a closed triple ideal of J".

The result just reviewed arises in [56, Theorem 8.2] in a lightly different formulation involving "matricially decomposed" complex C^*-algebras instead of ternary rings of operators. We preferred such a reformulation because of the scarcity of a well-developed theory for ternary rings of operators. We note that, if $A = \sum_{i,j\in\{1,2\}} A_{ij}$ is a matricially decomposed complex C^*-algebra, then A_{12} is a ternary ring of operators, and that, conversely, it follows from [79] that every ternary ring of operators is of the form A_{12} for some matricially decomposed complex C^*-algebra A. Later, in [57, Proposition 1.3] we proved that matricial decomposition of a given C^*-algebra A are in one-to-one correspondence with projections (i.e., $*$-invariant idempotents) in the multiplier C^*-algebra $M(A)$ of A. Moreover, we showed in [57, Proposition 1.1 and 1.2] that no non-prime JB^*-triples are included among those listed in [56, Theorem 8.2]. In this way the definitive classification theorem for prime JB^*-triples reads as follows.

Theorem 5.1. *A complex JB^*-triple J is prime if and only if one of the following assertions hold for J:*

i) *J is either the type* **V** *or the type* **VI** *complex Cartan factor.*
ii) *J is a complex spin factor.*

iii) *There exist a prime complex C^*-algebra A and a projection e in $M(A)$ such that J can be regarded as a JB^*-subtriple of the complex C^*-algebra $M(A)$ contained in $eM(A)(1-e)$ and containing $eA(1-e)$.*
iv) *There exist a prime complex C^*-algebra A, a projection e in $M(A)$, and a $*$-involution τ on A with $e + e^\tau = 1$ such that J can be regarded as a JB^*-subtriple of the complex C^*-algebra $M(A)$ contained in $H(eM(A)e^\tau, \tau)$ and containing $H(eAe^\tau, \tau)$.*

After some forerunners ([15], [25], [27], [28] and [70]), real JB^*-triples have recently attracted the attention of several authors. Real JB^*-triples are defined as norm-closed real subtriples of complex JB^*-triples (or, equivalently, as real forms of complex JB^*-triples). They have been introduced and studied in the paper of J.M. Isidro, W. Kaup and A. Rodríguez [37], where, as main result, it is proved that surjective linear mappings between real JB^*-triples are isometric if and only if they preserve the cube mapping $x \mapsto \{xxx\}$.

As in the complex case, real C^*-algebras are real JB^*-triples under the triple product formally defined as in (1). It is also important for our approach the fact that, if A is a real C^*-algebra, then the self-adjoint part of A is a JB^*-subtriple of A, and hence a real JB^*-triple. Other relevant examples of real JB^*-triples are obtained from real C^*-algebras A with a $*$-involution τ, by considering the set $S(A, \tau)$ of all skew elements of A relative to τ.

In [56] we also applied Zelmanovian techniques to obtain the corresponding classification theorem for prime real JB^*-triples. Let us say that a real JB^*-triple is a generalized real Cartan factor if it is either a complex Cartan factor (regarded as a real JB^*-triple) or a real form of a complex Cartan factor (compare [42, Lemma 4.5]). With these conventions and the help [77, Lemma 4], the classification theorem for prime real JB^*-triples, proved in [56, Theorem 8.4], reads as follows. As usual, given a C^*-algebra A, A_{sa} will denote the self-adjoint part of A.

Theorem 5.2. *A real JB^*-triple J is prime if and only if one of the following assertions hold for J:*

i) *J is an exceptional generalized real Cartan factor.*
ii) *J is a generalized real spin factor.*
iii) *There exists a prime real C^*-algebra A such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $M(A)_{sa}$ and containing A_{sa} .*
iv) *There exists a prime real C^*-algebra A with $*$-involution τ such that J can be regarded as a JB^*-subtriple of the real C^*-algebra $M(A)$ contained in $S(M(A), \tau) \cap M(A)_{sa}$ and containing $S(A, \tau) \cap A_{sa}$.*

Now, let us comment on the techniques applied in the proof of Theorems 5.1 and 5.2. In Zelmanov's work, *Jordan triples* over a field $\mathbb{F}$ of characteristic different from 2 and 3 are defined as vector spaces over $\mathbb{F}$ endowed with a triple product which is $\mathbb{F}$-linear in each of its variables, is symmetric in the outer variables, and satisfies the same main identity required for JB^*-triples. The Zelmanov classification of non-degenerate prime Jordan triples relies on an apparently ingenuous alternative by considering three mutually excluding cases, namely, non i-special, Clifford, and hermitian. A Jordan triple T is said to be special if it can be seen as a subtriple of an associative algebra A endowed with the triple product

$$\{abc\} := \frac{1}{2}(abc + cba), \tag{2}$$

and i-special if it is the homomorphic image of a special Jordan triple. An i-special Jordan triple T over $\mathbb{F}$ is said to be Clifford or hermitian depending on whether or not all the identities collected in a certain ideal of the free special Jordan triple over $\mathbb{F}$ vanish on T. We recall that the *free special Jordan triple* over $\mathbb{F}$ is the Jordan subtriple generated by $\mathbf{X}$ in the free associative algebra $\mathcal{A}(\mathbf{X})$ over $\mathbb{F}$ on $\mathbf{X}$, where $\mathbf{X}$ is a countably infinite set of indeterminates, and $\mathcal{A}(\mathbf{X})$ is regarded as a Jordan triple under the triple product given by (2).

Roughly speaking, a part of Zelmanov's prime theorem for Jordan triples establishes the scarcity, up to suitable scalar extensions, of non-degenerate prime Jordan triples which are not of hermitian type. The remaining part of Zelmanov's theorem shows that non-degenerate prime Jordan triples of hermitian type over $\mathbb{F}$ are "essentially" of the form $H(A,*) \cap S(A,\tau)$ for some associative algebra A over $\mathbb{F}$ with two commuting involutions $*$ and τ. Here $H(A,*) \cap S(A,\tau)$ is regarded as a subtriple of A with triple product defined by (2).

The conjugate-linear behaviour of the triple product of a complex JB^*-triple in its middle variable becomes a first handicap in applying Zelmanovian notions and techniques in our setting. Concerning notions, there are no problems: we see complex JB^*-triples as Jordan triples over $\mathbb{R}$, and consider separately the non i-special, hermitian, and Clifford cases. However, a verbatim application of Zelmanovian techniques to prime complex JB^*-triples would provide in the best of cases only a determination of the real structure of such JB^*-triples (see for instance [56, Theorem 5.3]). To overcome this difficulty, we designed in [56] different strategies, which are explained in what follows. Our determination of non i-special complex prime JB^*-triples (the first part of [56, Theorem 2.4]) actually avoids Zelmanov's prime theorem for Jordan triples, and only uses Zelmanov's prime theorem for Jordan algebras through its version for JB^*-algebras (Theorem 4.5). Concerning prime complex JB^*-triples of Clifford type, we start with a rather artisanal determination of complex Cartan factors of Clifford type [56, Proposition 6.1]. Such a determination leads us to realize that Banach ultraproducts of arbitrary families of complex Cartan factors of Clifford type are Hilbert spaces up to equivalent renormings [56, Corollary 6.2]. Then we replace algebraic ultraproducts with Banach ultraproducts in an argument in [78, pp. 63-64] (see also [4]) to obtain that every prime complex JB^*-triple of Clifford type is in fact a complex Cartan factor [56, Proposition 7.3]. The determination of Clifford and non i-special prime real JB^*-triples (second parts of Theorems 7.4 and 2.4, respectively, of [56]) follows easily from that of complex ones, by applying classical theory.

In studying real or complex JB^*-triples of hermitian type, a new handicap arises. Indeed, in the Zelmanovian theory, the associative envelopes for special Jordan triples are associative algebras regarded as Jordan triples under the triple product (2), whereas the natural associative envelopes for real (respectively, complex) JC^*-triples are real (respectively, complex) C^*-algebras regarded as JB^*-triples under the triple product (1). Concerning prime real JB^*-triples of hermitian type [56, Theorem 4.5], things are not too difficult because, if A is a real C^*-algebra, then the two triple products of A given by (1) and (2) coincide on the self-adjoint part A_{sa} of A, and moreover every real JC^*-triple can be represented into a real JB^*-triple of the form A_{sa} for some real C^*-algebra A [37, Corollary 2.4]. Then Zelmanovian techniques apply almost verbatim. The proof of the structure theorem for prime complex JB^*-triples of hermitian type [56, Theorem 5.9] is much difficult. Following an idea of O. Loos in [44, 2.9], when a complex JB^*-triple J is regarded as a real Jordan pair, such a real Jordan pair is in fact the realification of a Jordan pair (say V) over $\mathbb{C}$. In the case that J is prime and hermitian, the polarization of V (say T) is a Jordan triple over $\mathbb{C}$ of hermitian type, which can be represented into the secondary diagonal of a matricially decomposed complex C^*-algebra regarded as Jordan triple under the product (2). Then Zelmanovian techniques successfully apply to T, providing enough information for J. Such an information is collected in [56, Proposition 5.6], which, together with [57, Proposition 1.3 and Remark 1.4], reads as follows.

Proposition 5.3. *Let J be a complex JC^*-triple of hermitian type. Then J contains a non-zero closed triple ideal of the form*

$$H(A, \tau) \cap eA(1 - e),$$

where A is a C^*-algebra, e is a projection in $M(A)$, τ is a $*$-involution on A satisfying $e^\tau = 1 - e$, and A is generated as C^*-algebra by $H(A, \tau) \cap eA(1 - e)$.

Finally, applying the theory of multipliers of complex JB^*-triples developed in [14], suitable variants for complex JB^*-triples of Propositions 4.3 and 4.4 are obtained (see [56, Proposition 5.8]). These results, together with Proposition 5.3 above,, lead to the description of prime complex JB^*-triples of hermitian type (see [56, Theorem 5.9] and [57, Theorem 2.12] for details).

(Real or complex) *JBW^*-triples* are defined as those JB^*-triples which are Banach dual spaces. From a given JBW^*-triple J we can obtain new JBW^*-triples by considering the so-called *JBW^*-subtriples* of J, namely the w^*-closed JB^*-subtriples of J. Prime JBW^*-triples are called *JBW^*-factors*. The *complex Cartan factors* already quoted are nothing but those complex JBW^*-factor such that the closed unit balls of their preduals have extreme points (see [35] and [32]).

The Zelmanovian classification of real and complex JBW^*-factors was also attacked in [57], by slightly modifying the techniques applied in the determination of general prime real and complex JB^*-triples. The results obtained in this line are given by Theorems 5.4 and 5.5 which follow.

Theorem 5.4 ([57, Theorem 3.4])**.** *A real JBW^*-triple J is a JBW^*-factor if and only if one of the following assertions hold for J:*

i) *J is an exceptional generalized real Cartan factor.*
ii) *J is a generalized real spin factor.*
iii) *There exists a real W^*-factor A such that $J = A_{sa}$.*
iv) *There exists a real W^*-factor A with $*$-involution τ such that $J = S(A,\tau) \cap A_{sa}$.*

Theorem 5.5 ([57, Theorem 3.8])**.** *A complex JBW^*-triple J is a JBW^*-factor if and only if one of the following assertions hold for J:*

i) *J is either the type* **V** *or the type* **VI** *complex Cartan factor.*
ii) *J is a complex spin factor.*
iii) *There exist a complex W^*-factor A and a projection e in $M(A)$ such that $J = eA(1-e)$.*
iv) *There exist a complex W^*-factor A, a projection e in $M(A)$, and a $*$-involution τ on A with $e+e^\tau = 1$ such that $J = H(eAe^\tau, \tau)$.*

An apparently different classification of complex JBW^*-factors can be derived from the general structure theory of complex JBW^*-triples developed by G. Horn and E. Neher (see [35] and [36]). According to that theory, every complex JBW^*-factor J which is neither an exceptional Cartan factor nor a spin factor must satisfy one of the following three assertions:

(a) There exist a complex W^*-factor B and a projection p in B such that $J = pB$.
(b) There exists a complex W^*-factor B with $*$-involution π such that $J = H(B,\pi)$.
(c) There exists a complex W^*-factor B with $*$-involution π such that $J = S(B,\pi)$.

The concluding part of [57] is devoted to show how the classification of complex JBW^*-factors just quoted can be derived from Theorem 5.5 (see [57, Claim 3.9 and Corollary 3.16]).

6. Notes and remarks

6.1. Theorem 1.8 has been applied by A. R. Villena in [71] to extend to the setting of Jordan-Banach algebras the celebrated Johson-Sinclair theorem [39] about the automatic continuity of derivations on semisimple (associative) Banach algebras. Villena's result has been generalized recently by N. Boudi, A. Fernández, H. Marhnine, and C. Zarhouti in the setting of Jordan-Banach triples and pairs (see [31] and [12]).

6.2. In relation to the germinal normed versions of Zelmanov's prime theorem for Jordan algebras given by Theorems 1.6 and 1.7, it would be interesting to tray a description of those (complete) normed prime Jordan algebras which are either Albert rings or central orders in Jordan algebras of bilinear forms. The tensor product $\mathcal{D} \otimes H_3(C(\mathbb{C}))$, where $\mathcal{D}$ denotes the disk algebra, becomes an example of an algebra in such a (complete) situation.

6.3. Concerning Theorem 2.2, it would be interesting to know if the algebras arising in case *iii*) of that theorem fall in one of the following two cases:

(1) The closed Jordan subalgebras of $Q_b(A)$ containing A as an ideal, where A is an ultraprime complex Banach algebra. Here $Q_b(A)$ denotes the Mathieu's symmetric algebra of bounded quotients of such an algebra A [47].
(2) The closed Jordan subalgebras of $Q_b(A)$ contained in $H(Q_b(A), \tau)$ and containing $H(A, \tau)$ as an ideal, where A is an ultraprime complex Banach algebra with continuous involution τ.

This problem seems to be an essentially associative problem, namely if every ultra$-\tau$-prime prime complex Banach algebra A, with continuous involution τ, is ultraprime. Even a very particular case of this question is also open, namely, if every ultra-τ-prime simple complex Banach algebra, with continuous involution τ, is ultraprime.

6.4. The "uniqueness of the extended norm topology" and the "complete-to-complete extension property", first discovered in the setting of Theorem 3.4, have been later systematically considered in [65] and [66].

Assume that the algebra with involution $(C, *)$ is associative and a $*$-tight envelope of $H(C, *)$, that $H(C, *)$ is simple with a unit, and that the algebra norm $\|.\|$ on $H(C, *)$ is complete. Then, according to either Theorem 3.2 or Theorem 3.3, the NEP has an affirmative answer. As main result, it is shown in [65] that, under the above assumptions, the topologies of all algebra norms on C making $*$ continuous and extending the topology of $\|.\|$ on $H(C,*)$ coincide, and are complete. This result is applied in [65, Section 3] to simplify the original proof of the complete case of Theorem 2.1.

Now assume that $(C,*)$ is of the form $(A,*)\otimes(B,*)$, where $(A,*)$ is a finite dimensional $*$-simple associative algebra over $\mathbb{K}$ whose hermitian part $H(A,*)$ is of degree ≥ 2 over its centre, and $(B,*)$ is a (possibly non associative) algebra with involution and a unit over $\mathbb{K}$. Then, according to Theorem 3.6, the NEP need not have a global affirmative answer. However, if for some particular algebra norm $\|.\|$ on $H(C,*)$ the NEP has an affirmative answer, then, as in Theorem 3.5, we enjoy the uniqueness of the extended norm topology and the complete-to-complete extension property [66].

The uniqueness of the extended norm topology is applied in [65] and [66] (see also [55, Theorem 5.3]) to derive the continuity of homomorphisms and derivations from certain normed $*$-algebras into arbitrary normed algebras and modules, respectively, from the continuity of such isomorphisms and derivations on the hermitian part.

6.5. Theorem 4.5 has been applied in [30], together with results by M. Mathieu in [46], to show that prime JB^*-algebras are nondegenerately ultraprime. Non-Zelmanovian proofs and generalizations of the fact just quoted can be found in [16] and [26].

6.6. Let A be an associative algebra with two commuting involutions τ and π, and consider the Jordan triple $T := H(A,\tau)\cap S(A,\pi)$ under the triple product $\{xyz\} := \frac{1}{2}(xyz+zyx)$. As we commented in Section 5, Jordan triples as the one T above play an important role in Zelmanov's prime theorem for Jordan triples. In fact they play a role similar to that played by the Jordan algebra $H(A,*)$ in Zelmanov's prime theorem for Jordan algebras. Thus, a "triple-norm extension problem" merits consideration in relation to eventual future normed versions of Zelmanov's prime theorem for Jordan triples .

Let A and T be as above. If $\|.\|$ is an algebra norm on A, then, clearly, the restriction of $\|.\|$ to T is a *triple-norm* on T (i.e., a norm on the vector space T making the triple product of T continuous). The converse question, called the *triple-norm extension problem*, is the following: given a triple-norm $|\!|\!|\,.\,|\!|\!|$ on T, is there an algebra norm on A whose restriction to T is equivalent to $|\!|\!|\,.\,|\!|\!|$?

Assume from now on that A is a "τ-π-tight envelope" of T. Then the triple-norm extension problem has an affirmative answer if (and only if) the pentad mapping $\{...\}_5$ is $|\!|\!|\,.\,|\!|\!|$-continuous, where $\{.....\}_5$ is the function from $T\times T\times T\times T\times T$ to T defined by

$$\{t_1t_2t_3t_4t_5\}_5 := \frac{1}{2}(t_1t_2t_3t_4t_5 + t_5t_4t_3t_2t_1)$$

[51, Theorem 1.2] (compare Theorem 3.1). Moreover, if T is nondegenerate (i.e.,the conditions $x \in T$ and $\{xTx\} = 0$ imply $x = 0$), and if the triple norm $|||\,.\,|||$ on T is complete then the triple-norm extension problem has an affirmative answer [52, Theorem 2] (compare Theorem 3.2).

6.7. *Jordan-$*$-triples* are defined as complex vector spaces endowed with a triple product which is symmetric and linear in the outer variables an conjugate-linear in de middle variable, and satisfies the same main identity required for JB^*-triples. With the help of Zelmanov's prime theorem for Jordan triples [78] (see also [3]), K. Bouhya and A. Fernández classified in [13] prime Banach Jordan-$*$-triples with nonzero socle and without nilpotent elements. As a consequence, the authors of [13] rediscovered the Bunce-Chu structure theorem for compact JB^*-triples [14].

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