Grothendieck’s inequalities for real and complex JBW*-triples

Antonio M. Peralta* and Angel Rodríguez Palacios†

Abstract

We prove that, if $M > 4(1 + 2\sqrt{3})$ and $\varepsilon > 0$, if $V$ and $W$ are complex JBW*-triples (with preduals $V_*$ and $W_*$, respectively), and if $U$ is a separately weak*-continuous bilinear form on $V \times W$, then there exist norm-one functionals $\varphi_1, \varphi_2 \in V_*$ and $\psi_1, \psi_2 \in W_*$ satisfying

$$|U(x, y)| \leq M \|U\| \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\psi_1}^2\right)^{\frac{1}{2}} \left(\|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\varphi_1}^2\right)^{\frac{1}{2}}$$

for all $(x, y) \in V \times W$. Here, for a norm-one functional $\varphi$ on a complex JB*-triple $V$, $\|\cdot\|_{\varphi}$ stands for the prehilbertian seminorm on $V$ associated to $\varphi$ in [BF1]. We arrive in this “Grothendieck’s inequality” through results of C-H. Chu, B. Iochum, and G. Loupias [CIL], and a corrected version of the “Little Grothendieck’s inequality” for complex JB*-triples due to T. Barton and Y. Friedman [BF1]. We also obtain extensions of these results to the setting of real JB*-triples.

2000 Mathematics Subject Classification: 17C65, 46K70, 46L05, 46L10, and 46L70.

Introduction

In this paper we pay tribute to the important works of T. Barton and Y. Friedman [BF1] and C-H. Chu, B. Iochum, and G. Loupias [CIL] on the

*Supported by Programa Nacional F.P.I. Ministry of Education and Science grant, D.G.I.C.Y.T. project no. PB 98-1371, and Junta de Andalucía grant FQM 0199
†Partially supported by Junta de Andalucía grant FQM 0199
generalization of “Grothendieck’s inequalities” to complex JB*-triples. Of course, the Barton-Friedman-Chu-Iochum-Loupias techniques are strongly related to those of A. Grothendieck [Gro], G. Pisier (see [P1], [P2], and [P3]), and U. Haagerup [H], leading to the classical “Grothendieck’s inequalities” for C*-algebras. One of the most important facts contained in the Barton-Friedman paper is the construction of “natural” prehilbertian seminorms \( \| \varphi \|_p \), associated to norm-one continuous linear functionals \( \varphi \) on complex JB*-triples, in order to play, in Grothendieck’s inequalities, the same role as that of the prehilbertian seminorms derived from states in the case of C*-algebras. This is very relevant because JB*-triples need not have a natural order structure.

A part of Section 1 of the present paper is devoted to review the main results in [BF1], and the gaps in their proofs (some of which are also subsumed in [CIL]). We note that those gaps consist in assuming that separately weak*-continuous bilinear forms on dual Banach spaces, as well as weak*-continuous linear operators between dual Banach spaces, attain their norms. Section 1 also contains quick partial solutions of the gaps just mentioned. These solutions are obtained by applying theorems of J. Lindenstrauss [L] and V. Zizler [Z] on the abundance of weak*-continuous linear operators attaining their norms (see Theorems 1.4 and 1.6, respectively).

We begin Section 2 by proving a deeper correct version of the Barton-Friedman “Little Grothendieck’s Theorem” for complex JB*-triples [BF1, Theorem 1.3] (see Theorem 2.1). Roughly speaking, our result assures that the assertion in [BF1, Theorem 1.3] is true whenever we replace the prehilbertian seminorm \( \| \varphi \|_p \) arising in that assertion with \( \| \cdot \|_{\varphi_1, \varphi_2} := \sqrt{\| \cdot \|_{\varphi_1}^2 + \| \cdot \|_{\varphi_2}^2} \), where \( \varphi_1, \varphi_2 \) are suitable norm-one continuous linear functionals. It is worth mentioning that in fact our Theorem 2.1 deals with complex JBW*-triples and weak*-continuous operators, and that, in such a case, the functionals \( \varphi_1, \varphi_2 \) above can be chosen weak*-continuous. Among the consequences of Theorem 2.1 we emphasize appropriate “Little Grothendieck’s inequalities” for JBW-algebras and von Neumann algebras (see Corollary 2.5 and Remark 2.7, respectively). Corollary 2.5 allows us to adapt an argument in [P] in order to extend Theorem 2.1 to the real setting (Theorem 2.9).

Section 3 contains the main results of the paper, namely the “Big Grothendieck’s inequalities” for complex and real JBW*-triples (Theorems 3.1 and 3.4, respectively). Indeed, given \( M > 4(1 + 2\sqrt{3}) \) (respectively, \( M > 4(1 + 2\sqrt{3})(1 + 3\sqrt{2})^2 \), \( \varepsilon > 0 \), \( V, W \) complex (respectively, real) JBW*-
triples, and a separately weak*-continuous bilinear form $U$ on $V \times W$, there exist norm-one functionals $\varphi_1, \varphi_2 \in V^*$ and $\psi_1, \psi_2 \in W^*$ satisfying

$$|U(x, y)| \leq M \|U\| \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{\frac{1}{2}} \left( \|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2 \right)^{\frac{1}{2}}$$

for all $(x, y) \in V \times W$.

The concluding section of the paper (Section 4) deals with some applications of the results previously obtained. We give a complete solution to a gap in the proof of the results of [R1] on the strong* topology of complex JBW*-triples, and extend those results to the real setting. We also extend to the real setting the fact proved in [R2] that the strong* topology of a complex JBW*-triple $\mathcal{W}$ and the Mackey topology $m(\mathcal{W}, \mathcal{W}_*)$ coincide on bounded subsets of $\mathcal{W}$. From this last result we derive a Jarchow-type characterization of weakly compact operators from (real or complex) JB*-triples to arbitrary Banach spaces.

1 \hspace{1em} Discussing previous results

We recall that a complex JB*-triple is a complex Banach space $\mathcal{E}$ with a continuous triple product $\{., ., .\} : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and satisfies:

1. (Jordan Identity) $L(a, b)\{x, y, z\} = \{L(a, b)x, y, z\} = \{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$ for all $a, b, c, x, y, z$ in $\mathcal{E}$, where $L(a, b)x := \{a, b, x\}$;

2. The map $L(a, a)$ from $\mathcal{E}$ to $\mathcal{E}$ is an hermitian operator with nonnegative spectrum for all $a$ in $\mathcal{E}$;

3. $\|\{a, a, a\}\| = \|a\|^3$ for all $a$ in $\mathcal{E}$.

Complex JB*-triples have been introduced by W. Kaup in order to provide an algebraic setting for the study of bounded symmetric domains in complex Banach spaces (see [K1], [K2] and [U]).

If $\mathcal{E}$ is a complex JB*-triple and $e \in \mathcal{E}$ is a tripotent ($\{e, e, e\} = e$) it is well known that there exists a decomposition of $\mathcal{E}$ into the eigenspaces of $L(e, e)$, the Peirce decomposition,

$$\mathcal{E} = \mathcal{E}_0(e) \oplus \mathcal{E}_1(e) \oplus \mathcal{E}_2(e),$$

3
where $\mathcal{E}_k := \{ x \in \mathcal{E} : L(e,e)x = \frac{k}{2}x \}$. The natural projection $P_k(e) : \mathcal{E} \to \mathcal{E}_k(e)$ is called the Peirce k-projection. A tripotent $e \in \mathcal{E}$ is called complete if $\mathcal{E}_0(e) = 0$. By [KU, Proposition 3.5] we know that the complete tripotents in $\mathcal{E}$ are exactly the extreme points of its closed unit ball.

By a complex JBW*-triple we mean a complex JB*-triple which is a dual Banach space. We recall that the triple product of every complex JBW*-triple is separately weak*-continuous [BT], and that the bidual $\mathcal{E}^{**}$ of a complex JB*-triple $\mathcal{E}$ is a JBW*-triple whose triple product extends the one of $\mathcal{E}$ [Di].

Given a complex JBW*-triple $\mathcal{W}$ and a norm-one element $\varphi$ in the predual $\mathcal{W}_*$ of $\mathcal{W}$, we can construct a prehilbert seminorm $\| \cdot \|_\varphi$ as follows (see [BF1, Proposition 1.2]). By the Hahn-Banach theorem there exists $z \in \mathcal{W}$ such that $\varphi(z) = \|z\| = 1$. Then $(x,y) \mapsto \varphi \{x,y,z\}$ becomes a positive sesquilinear form on $\mathcal{W}$ which does not depend on the point of support $z$ for $\varphi$. The prehilbert seminorm $\| \cdot \|_\varphi$ is then defined by $\|x\|_\varphi^2 := \varphi \{x,x,z\}$ for all $x \in \mathcal{W}$.

If $\mathcal{E}$ is a complex JB*-triple and $\varphi$ is a norm-one element in $\mathcal{E}^*$, then $\| \cdot \|_\varphi$ acts on $\mathcal{E}^{**}$, hence in particular it acts on $\mathcal{E}$.

In [BF1, Theorem 1.4], J. T. Barton and Y. Friedman claim that for every pair of complex JB*-triples $\mathcal{E}, \mathcal{F}$, and every bounded bilinear form $V$ on $\mathcal{E} \times \mathcal{F}$, there exist norm-one functionals $\varphi \in \mathcal{E}^*$ and $\psi \in \mathcal{F}^*$ such that the inequality

$$|V(x,y)| \leq (3 + 2 \sqrt{3}) \|V\| \|x\|_\varphi \|y\|_\psi \quad (1.1)$$

holds for every $(x,y) \in \mathcal{E} \times \mathcal{F}$. This result is called “Grothendieck’s inequality for JB*-triples”. However, the beginning of the Barton-Friedman proof assumes that the two following assertions are true.

1. For $\mathcal{E}, \mathcal{F}$ and $V$ as above, there exists a separately weak*-continuous extension of $V$ to $\mathcal{E}^{**} \times \mathcal{F}^{**}$.

2. Again for $\mathcal{E}, \mathcal{F}$ and $V$ as above, every separately weak*-continuous extension of $V$ to $\mathcal{E}^{**} \times \mathcal{F}^{**}$ attains its norm (at a couple of complete tripotents).

We have been able to verify Assertion 1, but only by applying the fact, later proved by C-H. Chu, B. Iochum and G. Loupias [CIL, Lemma 5], that every bounded linear operator from a complex JB*-triple to the dual of another complex JB*-triple factors through a complex Hilbert space. Actually, this fact is also claimed in the Barton-Friedman paper (see [BF1, Corollary 3.2]), but their proof relies on their alleged [BF1, Theorem 1.4].
Lemma 1.1 Let $\mathcal{E}$ and $\mathcal{F}$ be complex JB*-triples. Then every bounded bilinear form $V$ on $\mathcal{E} \times \mathcal{F}$ has a separately weak*-continuous extension to $\mathcal{E}^{**} \times \mathcal{F}^{**}$.

Proof. Let $V$ be a bounded bilinear form on $\mathcal{E} \times \mathcal{F}$. Let $F$ denote the unique bounded linear operator from $\mathcal{E}$ to $\mathcal{F}^*$ which satisfies

$$V(x,y) = \langle F(x), y \rangle$$

for every $(x,y) \in \mathcal{E} \times \mathcal{F}$. By [CIL, Lemma 5], $F$ factors through a Hilbert space, and hence is weakly compact. By [HP, Lemma 2.13.1], we have $F^{**}(\mathcal{E}^{**}) \subset \mathcal{F}^*$. Then the bilinear form $\tilde{V}$ on $\mathcal{E}^{**} \times \mathcal{F}^{**}$ given by

$$\tilde{V}(\alpha,\beta) = \langle F^{**}(\alpha), \beta \rangle$$

extends $V$ and is weak*-continuous in the second variable. But $\tilde{V}$ is also weak*-continuous in the first variable because, for $(\alpha,\beta) \in \mathcal{E}^{**} \times \mathcal{F}^{**}$, the equality

$$\langle F^{**}(\alpha), \beta \rangle = \langle \alpha, F^*(\beta) \rangle$$

holds. \(\square\)

Unfortunately, as the next example shows, Assertion 2 above is not true.

Example 1.2 Take $\mathcal{E}$ and $\mathcal{F}$ equal to the complex $\ell_2$ space, and consider the bounded bilinear form on $\mathcal{E} \times \mathcal{F}$ defined by $V(x,y) := (S(x) | \sigma(y))$ where $S$ is the bounded linear operator on $\ell_2$ whose associated matrix is

$$\begin{pmatrix}
1 & 0 & \ldots & 0 & \ldots \\
2 & 2 & \ldots & 0 & \ldots \\
0 & 3 & \ldots & 0 & \ldots \\
\vdots & \vdots & \cdots & \vdots & \cdots \\
0 & 0 & \ldots & n & \ldots \\
\vdots & \vdots & \cdots & n + 1 & \cdots
\end{pmatrix},$$

and $\sigma$ is the conjugation on $\ell_2$ fixing the elements of the canonical basis. Then $V$ does not attain its norm.

It is worth mentioning that, although the bilinear form $V$ above does not attain its norm, it satisfies inequality 1.1 for every $x, y \in \ell_2$ and every norm-one elements $\varphi, \psi \in \ell_2^*$. Therefore it does not become a counterexample to
the Barton-Friedman claim. In fact we do not know if Theorem 1.4 of [BF1] is true.

Now that we know that Assertion 2 is not true, we prove that it is “almost” true.

**Lemma 1.3** Let \( \mathcal{E}, \mathcal{F} \) be complex JB*-triples. Then the set of bounded bilinear forms on \( \mathcal{E} \times \mathcal{F} \) whose separately weak*-continuous extensions to \( \mathcal{E}^{**} \times \mathcal{F}^{**} \) attain their norms is norm-dense in the space \( \mathcal{L}(^2(\mathcal{E} \times \mathcal{F})) \) of all bounded bilinear forms on \( \mathcal{E} \times \mathcal{F} \).

**Proof.** Let \( V \) be in \( \mathcal{L}(^2(\mathcal{E} \times \mathcal{F})) \). Denote by \( \tilde{V} \) the (unique) separately weak*-continuous extension of \( V \) to \( \mathcal{E}^{**} \times \mathcal{F}^{**} \). By the proof of Lemma 1.1, we can assure the existence of a bounded linear operator \( F_V : \mathcal{E} \to \mathcal{F}^* \) satisfying \( F_V^{**}(\mathcal{E}^{**}) \subset \mathcal{F}^* \) and

\[
\tilde{V}(\alpha, \beta) = <F_V^{**}(\alpha), \beta>
\]

for every \( (\alpha, \beta) \in \mathcal{E}^{**} \times \mathcal{F}^{**} \). It follows that \( \tilde{V} \) attains its norm whenever \( F_V^{**} \) does. Since the mapping \( V \mapsto F_V \), from \( \mathcal{L}(^2(\mathcal{E} \times \mathcal{F})) \) into the Banach space of all bounded linear operators from \( \mathcal{E} \) to \( \mathcal{F}^* \), is a surjective isometry, the result follows from [L, Theorem 1]. \( \square \)

An alternative proof of the above Lemma can be given taking as a key tool [A, Theorem 1].

Now note that, if \( X \) and \( Y \) are dual Banach spaces, and if \( U \) is a separately weak*-continuous bilinear form on \( X \times Y \) which attains its norm, then \( U \) actually attains its norm at a couple of extreme points of the closed unit balls of \( X \) and \( Y \) (hence at a couple of complete tripotents in the case that \( X \) and \( Y \) are complex JB*-triples). Since the Barton-Friedman proof of their claim actually shows that the inequality (1.1) holds (for suitable norm-one functionals \( \varphi \in \mathcal{E}^* \) and \( \psi \in \mathcal{F}^* \)) whenever the separately weak*-continuous extension of \( V \) given by Lemma 1.1 attains its norm at a couple of complete tripotents, the next theorem follows from Lemma 1.3.

**Theorem 1.4** Let \( \mathcal{E}, \mathcal{F} \) be complex JB*-triples. Then the set of all bounded bilinear forms \( V \) on \( \mathcal{E} \times \mathcal{F} \) such that there exist norm-one functionals \( \varphi \in \mathcal{E}^* \) and \( \psi \in \mathcal{F}^* \) satisfying

\[
|V(x,y)| \leq (3 + 2 \sqrt{3}) \|V\| \|x\|_\varphi \|y\|_\psi
\]

for every \( (x,y) \in \mathcal{E} \times \mathcal{F} \), is norm dense in \( \mathcal{L}(^2(\mathcal{E} \times \mathcal{F})) \).
Another alleged proof of the Barton-Friedman claim [BF1, Theorem 1.4] (with constant $3+2\sqrt{3}$ replaced with $4(1+2\sqrt{3})$) appears in the Chu-Iochum-Loupia paper already quoted (see [CIL, Theorem 6]). Such a proof relies on the Barton-Friedman version of the so called “Little Grothendieck’s Theorem” for complex JB*-triples [BF1, Theorem 1.3]. However, the Barton-Friedman argument for this “Little Grothendieck’s Theorem” also has a gap (see [P]).

Several authors (the second author of the present paper among others) subsumed the gap in the proof of Theorem 1.3 of [BF1] just commented, and formulated daring claims like the following (see [R1, Proposition 1] and the proof of Lemma 4 of [CM]). For every complex JBW*-triple $W$, every complex Hilbert space $H$, and every weak*-continuous linear operator $T : W \to H$, there exists a norm-one functional $\varphi \in W^*$ such that the inequality

$$\lVert T(x) \rVert \leq \sqrt{2} \lVert T \rVert \lVert x \rVert_{\varphi}$$

holds for all $x \in W$. As in the case of the Barton-Friedman big Grothendieck’s inequality, we do not know if the above claim is true. In any case, the next lemma is implicitly shown in the proof of Theorem 1.3 of [BF1].

**Lemma 1.5** Let $W$ be a complex JBW*-triple, $H$ a complex Hilbert space, and $T$ a weak*-continuous linear operator from $W$ to $H$ which attains its norm. Then $T$ satisfies inequality (1.2) for a suitable norm-one functional $\varphi \in W^*$.

We note that, for $W$ and $H$ as in the above lemma, weak*-continuous linear operators from $W$ to $H$ need not attain their norms (see the introduction of [P]). Now, from Lemma 1.5 and [Z] we obtain the following result.

**Theorem 1.6** Let $W$ be a complex JBW*-triple and $H$ a complex Hilbert space. Then the set of weak*-continuous linear operators $T$ from $W$ to $H$ such that there exists a norm-one functional $\varphi \in W^*$ satisfying

$$\lVert T(x) \rVert \leq \sqrt{2} \lVert T \rVert \lVert x \rVert_{\varphi}$$

for all $x \in W$, is norm dense in the space of all weak*-continuous linear operators from $W$ to $H$. 
2 Little Grothendieck’s Theorem for JBW*-triples

In this section we prove appropriate versions of “Little Grothendieck’s inequality” for real and complex JBW*-triples. We begin by considering the complex case, where the key tools are the Barton-Friedman result collected in Lemma 1.5, and a fine principle on approximation of operators by operators attaining their norms, due to R. A. Poliquin and V. E. Zizler [PZ].

**Theorem 2.1** Let $K > \sqrt{2}$ and $\varepsilon > 0$. Then, for every complex JBW*-triple $\mathcal{W}$, every complex Hilbert space $\mathcal{H}$, and every weak*-continuous linear operator $T : \mathcal{W} \rightarrow \mathcal{H}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}^*$ such that the inequality

$$\|T(x)\| \leq K \|T\| \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{\frac{1}{2}}$$

holds for all $x \in \mathcal{W}$.

**Proof.** Without loss of generality we can suppose $\|T\| = 1$. Take $\delta > 0$ such that $\delta \leq \varepsilon^2$ and $\sqrt{2((1 + \delta)^2 + \delta)} \leq K$. By [PZ, Corollary 2] there is a rank one weak*-continuous linear operator $T_1 : \mathcal{W} \rightarrow \mathcal{H}$ such that $\|T_1\| \leq \delta$ and $T - T_1$ attains its norm. Since $T_1$ is of rank one and weak*-continuous, it also attains its norm. By Lemma 1.5, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}^*$ such that

$$\|T_1(x)\| \leq \sqrt{2} \|T_1\| \|x\|_{\varphi_1},$$

$$\|(T - T_1)(x)\| \leq \sqrt{2} \|T - T_1\| \|x\|_{\varphi_2}$$

for all $x \in \mathcal{W}$. Therefore for $x \in \mathcal{W}$ we have

$$\|T(x)\| \leq \|(T - T_1)(x)\| + \|T_1(x)\|$$

$$\leq \sqrt{2} \|T - T_1\| \|x\|_{\varphi_2} + \sqrt{2} \|T_1\| \|x\|_{\varphi_1}$$

$$\leq \sqrt{2} (1 + \delta) \|x\|_{\varphi_2} + \sqrt{2\delta} \sqrt{\delta} \|x\|_{\varphi_1}$$

$$\leq \sqrt{2((1 + \delta)^2 + \delta)} \left( \|x\|_{\varphi_2}^2 + \delta \|x\|_{\varphi_1}^2 \right)^{\frac{1}{2}}$$

$$\leq K \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{\frac{1}{2}}.$$
Given a complex JBW*-triple $\mathcal{W}$ and norm-one elements $\varphi_1, \varphi_2 \in \mathcal{W}^*$ we denote by $\| \cdot \|_{\varphi_1, \varphi_2}$ the prehilbert seminorm on $\mathcal{W}$ given by $\| x \|_{\varphi_1, \varphi_2}^2 := \| x \|_{\varphi_1}^2 + \| x \|_{\varphi_2}^2$. The next result follows straightforwardly from Theorem 2.1.

**Corollary 2.2** Let $\mathcal{W}$ be a complex JBW*-triple and $T$ a weak*-continuous linear operator from $\mathcal{W}$ to a complex Hilbert space. Then there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}^*$ such that, for every $x \in \mathcal{W}$, we have

$$\| T(x) \| \leq 2 \| T \| \| x \|_{\varphi_1, \varphi_2}.$$ 

We recall that a JB*-algebra is a complete normed Jordan complex algebra (say $\mathcal{A}$) endowed with a conjugate-linear algebra involution $*$ satisfying $\| U_x(x^*) \| = \| x \|^3$ for every $x \in \mathcal{A}$. Here, for every Jordan algebra $\mathcal{A}$, and every $x \in \mathcal{A}$, $U_x$ denotes the operator on $\mathcal{A}$ defined by $U_x(y) := 2x \circ (xy) - x^2 \circ y$, for all $y \in \mathcal{A}$. We note that every JB*-algebra can be regarded as a complex JB*-triple under the triple product given by

$$\{ x, y, z \} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$$

(see [BKU] and [Y]). By a JBW*-algebra we mean a JB*-algebra which is a dual Banach space. Every JBW*-algebra $\mathcal{A}$ has a unit 1 [Y], so that the binary product of $\mathcal{A}$ can be rediscovered from the triple product by means of the equality $x \circ y = \{ x, 1, y \}$.

**Theorem 2.3** Let $M > 2$. Then, for every JBW*-algebra $\mathcal{A}$, every complex Hilbert space $\mathcal{H}$, and every weak*-continuous linear operator $T : \mathcal{A} \rightarrow \mathcal{H}$, there exists a norm-one positive functional $\xi \in \mathcal{A}$, such that the inequality

$$\| T(x) \| \leq M \| T \| \left( \| x \|_{\varphi_2}^3 + \varepsilon^2 \| x \|_{\varphi_1}^3 \right)^{\frac{1}{2}}$$

holds for all $x \in \mathcal{A}$.

**Proof.** Taking $K := \sqrt{M}$ and $\varepsilon := \sqrt{\frac{M-2}{2}}$ in Theorem 2.1, we find norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{A}^*$ such that

$$\| T(x) \| \leq K \| T \| \left( \| x \|_{\varphi_2}^3 + \varepsilon^2 \| x \|_{\varphi_1}^3 \right)^{\frac{1}{2}}$$

for all $x \in \mathcal{A}$. Let $i = 1, 2$. We choose $e_i \in \mathcal{A}$ with $\varphi_i(e_i) = \| e_i \| = 1$, and denote by $\xi_i$ the mapping $x \mapsto \varphi_i(x \circ e_i)$ from $\mathcal{A}$ to $\mathbb{C}$. Clearly $\xi_i$ is
a norm-one weak*-continuous linear functional on \(A\). Moreover, from the identity
\[
\{x, x, e_i\} + \{x^*, x^*, e_i\} = 2e_i \circ (x \circ x^*)
\]
we obtain that \(\xi_i\) is positive and that the equality \(\|x\|_{\varphi_i}^2 + \|x^*\|_{\varphi_i}^2 = 2\xi_i(x \circ x^*)\) holds. Therefore we have \(\|x\|_{\varphi_i}^2 \leq 2\xi_i(x \circ x^*)\) and hence
\[
\|T(x)\| \leq \sqrt{2K}\|T\| \left(\xi_2(x \circ x^*) + \varepsilon^2 \xi_1(x \circ x^*)\right)^{\frac{1}{2}}.
\]
Finally, putting \(\xi := \frac{1}{1 + \varepsilon^2}(\xi_2 + \varepsilon^2 \xi_1)\), \(\xi\) becomes a norm-one positive functional in \(A^*\) and for \(x \in A\) we have
\[
\|T(x)\| \leq \sqrt{2(1 + \varepsilon^2)}K\|T\| \left(\xi(x \circ x^*)\right)^{\frac{1}{2}} = M\|T\| \left(\xi(x \circ x^*)\right)^{\frac{1}{2}}.
\]
\[\Box\]

We recall that the bidual of every JB*-algebra \(A\) is a JBW*-algebra containing \(A\) as a JB*-subalgebra.

**Corollary 2.4** Let \(A\) be a JB*-algebra and \(T\) a bounded linear operator from \(A\) to a complex Hilbert space. Then there exists a norm-one positive functional \(\xi \in A^*\) satisfying
\[
\|T(x)\| \leq 2\|T\| \left(\xi(x \circ x^*)\right)^{\frac{1}{2}}
\]
for all \(x \in A\).

**Proof.** By Theorem 2.3, for \(n \in \mathbb{N}\) there is a norm-one positive functional \(\xi_n \in A^*\) satisfying
\[
\|T(x)\| \leq (2 + \frac{1}{n})\|T\| \left(\xi_n(x \circ x^*)\right)^{\frac{1}{2}}
\]
for all \(x \in A\). Take in \(A^*\) a weak* cluster point \(\eta\) of the sequence \(\xi_n\). Then \(\eta\) is a positive functional with \(\|\eta\| \leq 1\), and the inequality
\[
\|T(x)\| \leq 2\|T\| \left(\eta(x \circ x^*)\right)^{\frac{1}{2}}
\]
holds for all \(x \in A\). If \(\eta = 0\), then \(T = 0\) and nothing has to be proved. Otherwise take \(\xi := \frac{1}{\|\eta\|}\eta\). \[\Box\]

For background about JB- and JBW-algebras the reader is referred to [HS]. We recall that JB-algebras (respectively, JBW-algebras) are nothing but the self-adjoint parts of JB*-algebras (respectively, JBW*-algebras) [W] (respectively, [E]).
Corollary 2.5 Let $K > 2\sqrt{2}$. Then, for every JBW-algebra $A$, every real Hilbert space $H$, and every weak*-continuous linear operator $T : A \to H$, there exists a norm-one positive functional $\xi \in A_*$ such that

$$\|T(x)\| \leq K \|T\| (\xi(x^2))^{\frac{1}{2}}$$

for all $x \in A$.

Proof. Let $\hat{A}$ denote the JBW*-algebra whose self-adjoint part is equal to $A$, and $\hat{H}$ be the Hilbert space complexification of $H$. Consider the complex-linear operator $\hat{T} : \hat{A} \to \hat{H}$, which extends $T$. Clearly we have $\|\hat{T}\| \leq \sqrt{2}\|T\|$. By Theorem 2.3 there exists a norm-one positive functional $\xi \in A_*$ such that

$$\|T(x)\| = \|\hat{T}(x)\| \leq \frac{K}{\sqrt{2}} \|\hat{T}\| (\xi(x^2))^{\frac{1}{2}} \leq K \|T\| (\xi(x^2))^{\frac{1}{2}}$$

for all $x \in A$. Since $\xi$ is positive, $\xi|_A$ is in fact a norm-one positive functional in $A_*$. $\Box$

The next result follows from the above corollary in the same way that Corollary 2.4 was derived from Theorem 2.3.

Corollary 2.6 [P, Theorem 3.2]

Let $A$ be a JB-algebra, $H$ a real Hilbert space, and $T : A \to H$ a bounded linear operator. Then there is a norm-one positive linear functional $\varphi \in A^*$ such that

$$\|T(x)\| \leq 2\sqrt{2}\|T\| (\varphi(x^2))^{\frac{1}{2}}$$

for all $x \in A$.

Remark 2.7 1. Since every C*-algebra becomes a JB*-algebra under the Jordan product $x \circ y := \frac{1}{2}(xy + yx)$, it follows from Theorem 2.3 that, given $M > 2$, a von Neumann algebra $\mathcal{A}$, and a weak*-continuous linear operator $T$ from $\mathcal{A}$ to a complex Hilbert space, there exists a norm-one positive functional $\varphi \in \mathcal{A}_*$ satisfying

$$\|T(x)\| \leq M \|T\| (\varphi(\frac{1}{2}(xx^* + x^*x)))^{\frac{1}{2}}$$

for all $x \in \mathcal{A}$. A lightly better result can be derived from [H, Proposition 2.3].
As is asserted in [CIL], Corollary 2.4 can be proved by translating verbatim Pisier’s arguments for the case of C*-algebras [P2, Theorem 9.4]. We note that actually Corollary 2.4 contains Pisier’s result. Moreover, it is worth mentioning that our proof of Corollary 2.4 avoids any use of ultraproducts techniques.

Following [IKR], we define real JB*-triples as norm-closed real subtriples of complex JB*-triples. In [IKR] it is shown that every real JB*-triple \( E \) can be regarded as a real form of a complex JB*-triple. Indeed, given a real JB*-triple \( E \) there exists a unique complex JB*-triple structure on the complexification \( \hat{E} = E \oplus i E \), and a unique conjugation (i.e., conjugate-linear isometry of period 2) \( \tau \) on \( \hat{E} \) such that \( E = \hat{E}^\tau := \{ x \in \hat{E} : \tau(x) = x \} \).

The class of real JB*-triples includes all JB-algebras [HS], all real C*-algebras [G], and all J*B-algebras [Al].

By a real JBW*-triple we mean a real JB*-triple whose underlying Banach space is a dual Banach space. As in the complex case, the triple product of every real JBW*-triple is separately weak*-continuous [MP], and the bidual \( \mathcal{E}^{**} \) of a real JB*-triple \( \mathcal{E} \) is a real JBW*-triple whose triple product extends the one of \( \mathcal{E} \) [IKR]. Noticing that every real JBW*-triple is a real form of a complex JBW*-triple [IKR], it follows easily that, if \( W \) is a real JBW*-triple and if \( \varphi \) is a norm-one element in \( W^* \), then, for \( z \in W \) such that \( \varphi(z) = \|z\| = 1 \), the mapping \( x \mapsto (\varphi \{ x, x, z \})^{\frac{1}{2}} \) is a prehilbert seminorm on \( W \) (not depending on \( z \)). Such a seminorm will be denoted by \( \|x\|_{\varphi} \).

Now we proceed to deal with “Little Grothendieck’s inequality” for real JBW*-triples. We begin by showing the appropriate version of Lemma 1.5 for real JBW*-triples. Such a version is obtained by adapting the proof of a recent result of the first author for real JB*-triples (see [P]) to the setting of real JBW*-triples.

**Lemma 2.8** Let \( M > 1 + 3\sqrt{2} \). Then, for every real JBW*-triple \( W \), every real Hilbert space \( H \), and every weak*-continuous linear operator \( T : W \to H \) which attains its norm, there exists a norm one functional \( \varphi \in W^* \), such that

\[
\|T(x)\| \leq M \|T\| \|x\|_{\varphi}
\]

for all \( x \in W \).

**Proof.** We follow with minors changes the line of proof of [P, Theorem 4.3]. Without loss of generality we can suppose \( \|T\| = 1 \). Write

\[
K = \left[ 2\sqrt{2}\left( \frac{M^2}{1 + 3\sqrt{2}} - (1 + \sqrt{2}) \right) \right]^{\frac{1}{2}} > 2\sqrt{2}
\]
\[ \rho = \frac{2\sqrt{2}}{1 + \sqrt{2}}. \] By [IKR, Lemma 3.3], there exists a complete tripotent \( e \in W \) with 1 = \( \|T(e)\| \). Then denoting by \( \xi \) the linear functional on \( W \) given by \( \xi(x) := (T(x)|T(e)) \) for every \( x \in W \), \( \xi \) belongs to \( W^* \) and satisfies \( \|\xi\| = \xi(e) = 1 \). Moreover, when in the proof of [P, Theorem 4.3] Corollary 2.5 replaces [P, Theorem 3.2], we obtain the existence of a norm-one functional \( \psi \in W^* \) with \( \psi(e) = 1 \) such that

\[ \|T(x)\| \leq K\|x\|_\psi + (1 + \sqrt{2}) \|x\|_\xi \]

for all \( x \in W \). Setting \( \varphi := \frac{1}{1 + \rho}(\xi + \rho \psi) \), \( \varphi \) is a norm-one functional in \( W^* \) with \( \varphi(e) = 1 \), and we have

\[
\|T(x)\| \leq \sqrt{(1 + \sqrt{2})^2 + \frac{K^2}{\rho} \|x\|_\xi^2 + \rho \|x\|_\psi^2}
= \left( (1 + \sqrt{2})^2 + \frac{K^2}{\rho} (1 + \rho) \right)^\frac{1}{2} \|x\|_\varphi = M \|x\|_\varphi
\]

for all \( x \in W \). \( \Box \)

When in the proof of Theorem 2.1 Lemma 2.8 replaces Lemma 1.5, we arrive in the following result.

**Theorem 2.9** Let \( K > 1 + 3\sqrt{2} \) and \( \varepsilon > 0 \). Then, for every real JBW*-triple \( W \), every real Hilbert space \( H \), and every weak*-continuous linear operator \( T : W \to H \), there exist norm-one functionals \( \varphi_1, \varphi_2 \in W^* \) such that the inequality

\[ \|T(x)\| \leq K \|T\| \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{\frac{1}{2}} \]

holds for all \( x \in W \).

For norm-one elements \( \varphi_1, \varphi_2 \) in the predual of a given real JBW*-triple \( W \), we define the prehilbert seminorm \( \|\cdot\|_{\varphi_1, \varphi_2} \) on \( W \) verbatim as in the complex case.

**Corollary 2.10** Let \( W \) be a real JBW*-triple and \( T \) a weak*-continuous linear operator from \( W \) to a real Hilbert space. Then there exist norm-one functionals \( \varphi_1, \varphi_2 \in W^* \) such that, for every \( x \in W \), we have

\[ \|T(x)\| \leq 6\|T\|\|x\|_{\varphi_1, \varphi_2}. \]
3 Grothendieck’s Theorem for JBW*-triples

In this section we prove “Grothendieck’s inequality” for separately weak*-continuous bilinear forms defined on the cartesian product of two JBW*-triples.

**Theorem 3.1** Let $M > 4(1 + 2\sqrt{3})$ and $\varepsilon > 0$. For every couple $(\mathcal{V}, \mathcal{W})$ of complex JBW*-triples and every separately weak*-continuous bilinear form $V$ on $\mathcal{V} \times \mathcal{W}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{V}_*$ and $\psi_1, \psi_2 \in \mathcal{W}_*$ satisfying

$$|V(x,y)| \leq M \|V\| \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}} \left(\|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{V} \times \mathcal{W}$.

**Proof.** We begin by noticing that a bilinear form $U$ on $\mathcal{V} \times \mathcal{W}$ is separately weak*-continuous if and only if there exists a weak*-to-weak-continuous linear operator $F_U : \mathcal{V} \to \mathcal{W}_*$ such that the equality

$$U(x,y) = \langle F_U(x), y \rangle$$

holds for every $(x,y) \in \mathcal{V} \times \mathcal{W}$.

Put $T := F_V : \mathcal{V} \to \mathcal{W}_*$ in the sense of the above paragraph. By [CIL, Lemma 5] there exist a Hilbert space $\mathcal{H}$ and bounded linear operators $S : \mathcal{V} \to \mathcal{H}$, $R : \mathcal{H} \to \mathcal{W}_*$ satisfying $T = R S$ and $\|R\| \|S\| \leq 2(1 + 2\sqrt{3}) \|T\|$. Notice that in fact we can enjoy such a factorization in such a way that $R$ is injective. Indeed, take $\mathcal{H}'$ equals to the orthogonal complement of $\text{Ker}(R)$ in $\mathcal{H}$, $R' := R|_{\mathcal{H}'}$ and $S' := \pi_{\mathcal{H}'} S$, where $\pi_{\mathcal{H}'}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}'$, to have $T = R' S'$ with $R'$ injective and $\|R\| \|S'\| \leq 2(1 + 2\sqrt{3}) \|T\|$.

Next we show that $S$ is weak*-continuous. By [DS, Corollary V.5.5] it is enough to prove that $S$ is weak*-continuous on bounded subsets of $\mathcal{V}$. Let $x_\lambda$ be a bounded net in $\mathcal{V}$ weak*-convergent to zero. Take a weak cluster point $h$ of $S(x_\lambda)$ in $\mathcal{H}$. Then $R(h)$ is a weak cluster point of $T(x_\lambda) = R S(x_\lambda)$ in $\mathcal{W}_*$. Moreover, since $T$ is weak*-to-weak-continuous, we have $T(x_\lambda) \to 0$ weakly. It follows $R(h) = 0$ and hence $h = 0$ by the injectivity of $R$. Now, zero is the unique weak cluster point in $\mathcal{H}$ of the bounded net $S(x_\lambda)$, and therefore we have $S(x_\lambda) \to 0$ weakly.
Now that we know that the operator $S$ is weak*-continuous, we apply Theorem 2.1 with $K = \sqrt{\frac{M}{2(1+2\sqrt{3})}} > \sqrt{2}$ to find norm-one functionals $\varphi_1, \varphi_2 \in V_*$ and $\psi_1, \psi_2 \in W_*$ satisfying

$$\|S(x)\| \leq K \|S\| \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{\frac{1}{2}}$$

and

$$\|R^*(y)\| \leq K \|R^*\| \left( \|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2 \right)^{\frac{1}{2}}$$

for all $x \in V$ and $y \in W$. Therefore

$$|V(x,y)| = |<T(x), y>| = |<S(x), R^*(y)>|$$

$$\leq \frac{M}{2(1+2\sqrt{2})} \|R\| \|S\| \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{\frac{1}{2}} \left( \|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2 \right)^{\frac{1}{2}}$$

$$\leq M \|V\| \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{\frac{1}{2}} \left( \|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2 \right)^{\frac{1}{2}},$$

for all $(x,y) \in V \times W$. Therefore

In the same way that Theorem 2.3 was derived from Theorem 2.1, we can obtain from Theorem 3.1 that, given $M > 8 \left(1 + 2\sqrt{3}\right)$, JBW*-algebras $A, B$, and a separately weak*-continuous bilinear form $V$ on $A \times B$, there exist norm-one positive functionals $\varphi \in A_*$ and $\psi \in B_*$ satisfying

$$|V(x,y)| \leq M \|V\| \left( \varphi(x \circ x^*) \right)^{\frac{1}{2}} \left( \psi(y \circ y^*) \right)^{\frac{1}{2}}$$

for all $(x,y) \in A \times B$. As a relevant particular case we obtain the following result.

**Corollary 3.2** Let $M > 8(1 + 2\sqrt{3})$. For every couple $(A,B)$ of von Neumann algebras and every separately weak*-continuous bilinear form $V$ on $A \times B$, there exist norm-one positive functionals $\varphi \in A_*$ and $\psi \in B_*$ satisfying

$$|V(x,y)| \leq M \|V\| \left( \varphi\left(\frac{1}{2}(xx^* + x^*x)\right) \right)^{\frac{1}{2}} \left( \psi\left(\frac{1}{2}(yy^* + y^*y)\right) \right)^{\frac{1}{2}}$$

for all $(x,y) \in A \times B$.

A refined version of the above corollary can be found in [H, Proposition 2.3].

Now we proceed to deal with Grothendieck’s Theorem for real JBW*-triples. The following lemma generalizes [CIL, Lemma 5] to the real case.
Lemma 3.3 Let $E$ and $F$ be real JB*-triples and $T : E \to F^*$ a bounded linear operator. Then $T$ has a factorization $T = RS$ through a real Hilbert space with $\|R\| \|S\| \leq 4(1 + 2\sqrt{3}) \|T\|$

Proof.

Let us consider the JB*-complexifications $\hat{E}$ and $\hat{F}$ of $E$ and $F$, respectively, and denote by $\hat{T} : \hat{E} \to \hat{F^*}$ the complex linear extension of $T$, so that we easily check that $\|\hat{T}\| \leq 2\|T\|$. As we have mentioned before, $\hat{T}$ has a factorization $\hat{T} = \hat{R}\hat{S}$ through a complex Hilbert space $\mathcal{H}$, with $\|\hat{R}\| \|\hat{S}\| \leq 2(1 + 2\sqrt{3}) \|\hat{T}\|$.

Since $\hat{T}$ is the complex linear extension of $T$, the inclusion $\hat{T}(E) \subseteq F^*$ holds. Put $H := \overline{\hat{S}(E)}$, the closure of $\hat{S}(E)$ in $\mathcal{H}$. Then $H$ is a real Hilbert space and we have $\hat{R}(H) \subseteq \overline{\hat{R}(\hat{S}(E))} = \overline{\hat{T}(E)} \subseteq F^*$.

Finally we define the bounded linear operators $S := \hat{S}|_E : E \to H$ and $R := \hat{R}|_H : H \to F^*$. It is easy to see that $T = RS$ and

$$\|R\| \|S\| \leq \|\hat{R}\| \|\hat{S}\| \leq 2(1 + 2\sqrt{3}) \|\hat{T}\| \leq 4(1 + 2\sqrt{3}) \|T\|.$$ 

When in the proof of Theorem 3.1 Lemma 3.3 and Theorem 2.9 replace [CIL, Lemma 5] and Theorem 2.1, respectively, we obtain the following theorem.

Theorem 3.4 Let $M > 4(1 + 2\sqrt{3})(1 + 3\sqrt{2})^2$ and $\varepsilon > 0$. For every couple $(V, W)$ of real JBW*-triples and every separately weak*-continuous bilinear form $U$ on $V \times W$, there exist norm-one functionals $\varphi_1, \varphi_2 \in V^*$, and $\psi_1, \psi_2 \in W^*$ satisfying

$$|U(x, y)| \leq M \|U\| \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \right) \left( \|y\|_{\psi_2}^2 + \varepsilon^2 \right)^{\frac{1}{2}} \left( \|x\|_{\varphi_1}^2 + \varepsilon^2 \right)^{\frac{1}{2}}$$

for all $(x, y) \in V \times W$.

Thanks to Lemma 3.3, Lemma 1.1 remains true when real JB*-triples replace complex ones. Then Theorems 3.4 and 3.1 give rise to the real and complex cases, respectively, of the result which follows.
Corollary 3.5 Let $M > 4(1 + 2\sqrt{3}) (1 + 3\sqrt{2})^2$ (respectively, $M > 4(1 + 2\sqrt{3})$) and $\varepsilon > 0$. Then for every couple $(E, F)$ of real (respectively, complex) JB*-triples and every bounded bilinear form $U$ on $E \times F$ there exist norm-one functionals $\varphi_1, \varphi_2 \in E^*$ and $\psi_1, \psi_2 \in F^*$ satisfying

$$|U(x, y)| \leq M \|U\| (\|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2)^{\frac{1}{2}} (\|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2)^{\frac{1}{2}}$$

for all $(x, y) \in E \times F$.

Remark 3.6 In the complex case of the above corollary, the interval of variation of the constant $M$ can be enlarged by arguing as follows. Let $M > 3 + 2\sqrt{3}$, $\varepsilon > 0$, $E$ and $F$ be complex JB*-triples, and $U$ a norm-one bounded bilinear form on $E \times F$. Consider the separately weak*-continuous bilinear form $\tilde{U}$ on $E^{**} \times F^{**}$ which extends $U$, and take a weak*-to-weak continuous linear operator $T : E^{**} \to F^*$ satisfying $\tilde{U}(\alpha, \beta) = \langle T(\alpha), \beta \rangle$ for all $(\alpha, \beta) \in E^{**} \times F^{**}$. Choose $\delta > 0$ such that $\delta \leq \varepsilon^2$ and $(3 + 2\sqrt{3})(1 + \delta) \leq M$. By [PZ, Corollary 2] there is a rank one weak*-to-weak continuous linear operator $T_1 : E^{**} \to F^*$ such that $\|T_1\| \leq \delta$ and $T_2 := T - T_1$ attains its norm. Since $T_1$ is of rank one and weak*-continuous, it also attains its norm. For $i = 1, 2$, consider the separately weak*-continuous bilinear form $U_i$ on $E^{**} \times F^{**}$ defined by

$$\tilde{U}_i(\alpha, \beta) = \langle T_i(\alpha), \beta \rangle,$$

and put $U_i = \tilde{U}_i|_{E \times F}$, so that $U_i$ is a bounded bilinear form on $E \times F$ whose separately weak*-continuous extension to $E^{**} \times F^{**}$ attains its norm. By the proof of [BF1, Theorem 1.4], there exist norm-one functionals $\varphi_1, \varphi_2 \in E^*$ and $\psi_1, \psi_2 \in F^*$ such that

$$|U_i(x, y)| \leq (3 + 2\sqrt{3}) \|U_i\| \|x\|_{\varphi_i} \|y\|_{\psi_i},$$

for all $(x, y) \in E \times F$ and $i = 1, 2$.

Therefore

$$|U(x, y)| \leq |U_2(x, y)| + |U_1(x, y)| \leq (3 + 2\sqrt{3})(\|U_2\| \|x\|_{\varphi_2} \|y\|_{\psi_2} + \|U_1\| \|x\|_{\varphi_1} \|y\|_{\psi_1}) \leq (3 + 2\sqrt{3})((1 + \delta) \|x\|_{\varphi_2} \|y\|_{\psi_2} + \delta \|x\|_{\varphi_1} \|y\|_{\psi_1})$$
\[
(3 + 2\sqrt{3})(1 + \delta) \left( \|x\|_{\varphi_2} \|y\|_{\varphi_2} + \delta \|x\|_{\varphi_1} \|y\|_{\varphi_1} \right) \\
\leq (3 + 2\sqrt{3})(1 + \delta) \sqrt{\|x\|_{\varphi_2}^2 + \delta \|x\|_{\varphi_1}^2 \sqrt{\|y\|_{\varphi_2}^2 + \delta \|y\|_{\varphi_1}^2}} \\
\leq M \left( \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{1/2} \left( \|y\|_{\varphi_2}^2 + \varepsilon^2 \|y\|_{\varphi_1}^2 \right)^{1/2}
\]
for all \((x, y) \in E \times F\).

We do not know if the value \(\varepsilon = 0\) is allowed in Theorems 3.1 and 3.4. In any case, as the next result shows, the value \(\varepsilon = 0\) is allowed for a “big quantity” of separately weak*-continuous bilinear forms.

**Theorem 3.7** Let \(M > 4(1 + 2\sqrt{3}) (1 + 3\sqrt{2})^2\) (respectively, \(M > 4(1 + 2\sqrt{3})\)) and \(V, W\) be real (respectively, complex) JBW*-triples. Then the set of all separately weak*-continuous bilinear forms \(U\) on \(V \times W\) such that there exist norm-one functionals \(\varphi \in V_*\) and \(\psi \in W_*\) satisfying

\[|U(x, y)| \leq M \|U\| \|x\|_{\varphi} \|y\|_{\psi}\]

for all \((x, y) \in V \times W\), is norm dense in the set of all separately weak*-continuous bilinear forms on \(V \times W\).

*Proof.* Let \(U\) a non zero separately weak*-continuous bilinear form on \(V \times W\). By the proof of Theorem 3.4 (respectively, Theorem 3.1) there exists a real (respectively, complex) Hilbert space \(H\) such that for all \((x, y) \in V \times W\) we have

\[U(x, y) := <F(x), G(y)>,\]

where \(F : V \to H\) and \(G : W \to H^*\) are weak*-continuous linear operators satisfying \(\|F\| \|G\| \leq L \|U\|\) with \(L = 4(1 + 2\sqrt{3})\) (respectively, \(L = 2(1 + 2\sqrt{3})\)).

By [Z], there are sequences \(\{F_n : V \to H\}\) and \(\{G_n : W \to H^*\}\) of weak*-continuous linear operators, converging in norm to \(F\) and \(G\), respectively, and such that \(F_n\) and \(G_n\) attain their norms for every \(n\). Then, putting

\[U_n(x, y) := <F_n(x), G_n(y)>, \quad ((n, x, y) \in \mathbb{N} \times V \times W),\]

\(\{U_n\}\) becomes a sequence of separately weak*-continuous bilinear forms on \(V \times W\), converging in norm to \(U\). Take \(\sqrt{\frac{M}{L}} > K > 1 + 3\sqrt{2}\) (respectively,
\( \sqrt{\frac{M}{L}} > K > \sqrt{2} \). Applying Lemma 2.8 (respectively, Lemma 1.5), for \( n \in \mathbb{N} \) we find norm-one functionals \( \varphi_n \in V_* \) and \( \psi_n \in W_* \) satisfying
\[
\| F_n(x) \| \leq K \| F_n \| \| x \| \varphi_n \quad \text{and} \quad \| G_n(y) \| \leq K \| G_n \| \| y \| \psi_n
\]
for all \((x, y) \in V \times W\).

Set
\[
\delta = \frac{\| F \| - L \| U \|}{1 + L} > 0,
\]
and take \( m \in \mathbb{N} \) such that the inequalities
\[
\| F_n \| \| G_n \| - \| F \| \| G \| < \delta,
\]
\[
\| U_n \| - \| U \| < \delta, \quad \text{and} \quad \| U_n \| \geq \frac{\| U \|}{2}
\]
hold for every \( n \geq m \).

Now for \( n \geq m \) and \((x, y) \in V \times W\) we have
\[
| U_n(x, y) | \leq K^2 \| F_n \| \| G_n \| \| x \| \varphi_n \| y \| \psi_n
\]
\[
\leq K^2 \left( \| F \| \| G \| + \delta \right) \| x \| \varphi_n \| y \| \psi_n
\]
\[
\leq K^2 \left( L \| U \| + \delta \right) \| x \| \varphi_n \| y \| \psi_n
\]
\[
\leq K^2 \left( L \| U_n \| + \delta (1 + L) \right) \| x \| \varphi_n \| y \| \psi_n
\]
\[
= K^2 \left( L \| U_n \| + \left( \frac{M}{K^2} - L \right) \frac{\| U \|}{2} \right) \| x \| \varphi_n \| y \| \psi_n
\]
\[
\leq M \| U_n \| \| x \| \varphi_n \| y \| \psi_n.
\]

\( \square \)

As we noticed before Corollary 3.5, Lemma 1.1 remains true in the real setting. Then, given real or complex JB*-triples \( E, F \), the mapping sending each element \( U \in \mathcal{L}(2(E \times F)) \) to its unique separately weak*-continuous bilinear extension \( \tilde{U} \) to \( E^{**} \times F^{**} \) is an isometry from \( \mathcal{L}(2(E \times F)) \) onto the Banach space of all separately weak*-continuous bilinear forms on \( E^{**} \times F^{**} \). Therefore we obtain the following corollary.
Corollary 3.8 Let \( M > 4(1 + 2\sqrt{3}) (1 + 3\sqrt{2})^2 \) (respectively, \( M > 4(1 + 2\sqrt{3}) \)) and \( E, F \) be real (respectively, complex) JB*-triples. Then the set of all bounded bilinear forms \( U \) on \( E \times F \) such that there exist norm-one functionals \( \varphi \in E^* \) and \( \psi \in F^* \) satisfying
\[
|U(x, y)| \leq M \|U\| \|x\|_{\varphi} \|y\|_{\psi}
\]
for all \((x, y) \in E \times F\), is norm dense in \( \mathcal{L}(^2(E \times F)) \).

We note that Theorem 1.4 is finer than the complex case of the above corollary. However, since Theorem 1.4 depends on the proof of [BF1, Theorem 1.4], it is much more difficult.

Remark 3.9 We do not know if the value \( \varepsilon = 0 \) is allowed in Theorems 2.1 and 2.9 (respectively, in Theorems 3.1 and 3.4) for some value of the constant \( K \) (respectively, \( M \)). Concerning this question, it is worth mentioning that the following three assertions are equivalent:

1. There is a universal constant \( G \) such that, for every real (respectively, complex) JBW*-triple \( W \) and every couple \((\varphi_1, \varphi_2)\) of norm-one functionals in \( W_1 \times W_2 \), we can find a norm-one functional \( \varphi \in W_1 \) satisfying
\[
\|x\|_{\varphi} \leq G \|x\|_{\varphi_i}
\]
for every \( x \in W \) and \( i = 1, 2 \).

2. There is a universal constant \( \hat{G} \) such that for every couple of real (respectively, complex) JBW*-triples \( (V, W) \) and every separately weak*-continuous bilinear form \( U \) on \( V \times W \), there are norm-one functionals \( \varphi \in V_1 \), and \( \psi \in W_2 \) satisfying
\[
|U(x, y)| \leq \hat{G} \|U\| \|x\|_{\varphi} \|y\|_{\psi}
\]
for all \((x, y) \in V \times W\).

3. There is a universal constant \( \tilde{G} \) such that for every real (respectively, complex) JBW*-triple \( W \) and every weak*-continuous linear operator \( T \) from \( W \) to a real (respectively, complex) Hilbert space, there exists a norm-one functional \( \varphi \in W_1 \) satisfying
\[
\|T(x)\| \leq \tilde{G} \|T\| \|x\|_{\varphi}
\]
for all \( x \in W \).
The implication 1 \Rightarrow 2 follows from Theorems 3.1 and 3.4.

Assume that Assertion 2 above is true. Let $W$ be a real (respectively, complex) JBW*-triple, $H$ a real (respectively, complex) Hilbert space, and $T : W \to H$ a weak*-continuous linear operator. Consider the separately weak*-continuous bilinear form $U$ on $W \times H$ given by $U(x, y) := (T(x)|y)$ (respectively, $U(x, y) := (T(x)|\sigma(y))$, where $\sigma$ is a conjugation on $H$). Regarding $H$ as a JBW*-triple under the triple product $\{x, y, z\} := \frac{1}{2}((x|y)z + (z|y)x)$, and applying the assumption, we find norm-one functionals $\varphi \in W_*$ and $\psi \in H_*$ satisfying

$$|U(x, y)| \leq \hat{G} \|U\| \|x\|_{\varphi} \|y\|_{\psi}$$

$$\leq \hat{G} \|T\| \|x\|_{\varphi} \|y\|$$

for all $(x, y) \in W \times H$. Taking $y = T(x)$ (respectively, $y = \sigma(T(x))$) we obtain

$$\|T(x)\| \leq \hat{G} \|T\| \|x\|_{\varphi}$$

for all $x \in W$. In this way Assertion 3 holds.

Finally let us assume that Assertion 3 is true. Let $W$ be a real (respectively, complex) JBW*-triple and $\varphi_1, \varphi_2$ norm-one functionals in $W_*$. Since $\|., .\|_{\varphi_1, \varphi_2}$ comes from a suitable separately weak*-continuous positive sesquilinear form $\langle ., . \rangle$ on $W$ by means of the equality $\|x\|_{\varphi_1, \varphi_2}^2 = \langle x, x \rangle$, it follows from the proof of [R1, Corollary] that there exists a weak*-continuous linear operator $T$ from $W$ to a real (respectively, complex) Hilbert space satisfying $\|x\|_{\varphi_1, \varphi_2} = \|T(x)\|$ for all $x \in W$ (which implies $\|T\| \leq \sqrt{2}$). Now applying the assumption we find a norm one functional $\varphi \in W_*$ such that

$$\|x\|_{\varphi_1, \varphi_2} = \|T(x)\| \leq \hat{G} \|T\| \|x\|_{\varphi} \leq \sqrt{2}\hat{G}\|x\|_{\varphi}$$

for all $x \in W$. As a consequence, for $i = 1, 2$ we have

$$\|x\|_{\varphi_i} \leq \sqrt{2}\hat{G}\|x\|_{\varphi}$$

for all $x \in W$.

4 Some Applications

We define the strong*-topology $S^*(W, W_*)$ of a given real or complex JBW*-triple $W$ as the topology on $W$ generated by the family of seminorms $\{\|., .\|_{\varphi} :
$\varphi \in W, \|\varphi\| = 1\}$. In the complex case, the above notion has been introduced by T. J. Barton and Y. Friedman in [BF2]. When a JBW*-algebra $A$ is regarded as a complex JBW*-triple, $S^*(A, A_*)$ coincides with the so-called “algebra-strong* topology” of $A$, namely the topology on $A$ generated by the family of seminorms of the form $x \mapsto \sqrt{\xi(x \circ x^*)}$ when $\xi$ is any positive functional in $A_*$. [R1, Proposition 3]. As a consequence, when a von Neumann algebra $M$ is regarded as a complex JBW*-triple, $S^*(M, M_*)$ coincides with the familiar strong*-topology of $M$ (compare [S, Definition 1.8.7]).

We note that, if $W$ is a complex JBW*-triple, then, denoting by $W_\mathbb{R}$ the realification of $W$ (i.e., the real JBW*-triple obtained from $W$ by restriction of scalar to $\mathbb{R}$), we have $S^*(W, W_*) = S^*(W_\mathbb{R}, (W_\mathbb{R})_*)$. Indeed, the mapping $\varphi \mapsto \Re \varphi$ identifies $W_*$ with $(W_\mathbb{R})_*$, and, when $\varphi$ has norm one, the equality $\|x\|_\varphi = \|x\|_{\Re \varphi}$ holds for every $x \in W$.

**Proposition 4.1** Let $W$ be a real (respectively, complex) JBW*-triple. The following topologies coincide in $W$:

1. The strong*-topology of $W$.

2. The topology on $W$ generated by the family of seminorms of the form $x \mapsto \sqrt{\langle x, x \rangle}$, where $\langle ., . \rangle$ is any separately weak*-continuous positive sesquilinear form on $W$.

3. The topology on $W$ generated by the family of seminorms $x \mapsto \|T(x)\|$, when $T$ runs over all weak*-continuous linear operators from $W$ to arbitrary real (respectively, complex) Hilbert spaces.

**Proof.** Let us denote by $\tau_1, \tau_2$, and $\tau_3$ the topologies arising in paragraphs 1, 2, and 3, respectively. The inequality $\tau_1 \geq \tau_3$ follows from Corollary 2.10 (respectively, Corollary 2.2). Since the proof of [R1, Corollary 1] shows that for every separately weak*-continuous positive sesquilinear form $\langle ., . \rangle$ on $W$ there exists a weak*-continuous linear operator $T$ from $W$ to a real (respectively, complex) Hilbert space satisfying $\sqrt{\langle x, x \rangle} = \|T(x)\|$ for all $x \in W$, we have $\tau_3 \geq \tau_2$. Finally, since for every norm-one functional $\varphi \in W_*$ there is a separately weak*-continuous positive sesquilinear form $\langle ., . \rangle$ satisfying $\|x\|_\varphi = \sqrt{\langle x, x \rangle}$ for all $x \in W$, the inequality $\tau_2 \geq \tau_1$ follows. $\square$

For every Banach space $X$, $B_X$ will stand for the closed unit ball of $X$. For every dual Banach space $X$ (with a fixed predual denoted by $X_*$), we
denote by $m(X, X_*)$ the Mackey topology on $X$ relative to its duality with $X_*$.

**Corollary 4.2** Let $W$ be a real or complex JBW*-triple. Then the strong*-topology of $W$ is compatible with the duality $(W, W_*)$.

**Proof.** We apply the characterization of $S^*(W, W_*)$ given by paragraph 3 in Proposition 4.1. Clearly $S^*(W, W_*)$ is stronger than the weak*-topology $\sigma(W, W_*)$ of $W$. On the other hand, if $T$ is a weak*-continuous linear operator from $W$ to a Hilbert space $H$, and if we put $T = S^*$ for a suitable bounded linear operator $S : H_* \to W_*$, then $S(B_{H_*})$ is an absolutely convex and weakly compact subset of $W_*$ and we have $\|T(x)\| = \sup |< x, S(B_{H_*}) > |$. This shows that $S^*(W, W_*)$ is weaker than $m(W, W_*)$. □

The complex case of the above corollary is due to T. J. Barton and Y. Friedman [BF2]. The complex case of Proposition 4.1 is claimed in [R1, Corollary 2] (see also [R2, Proposition D.17]), but the proof relies on [R1, Proposition 1], which subsumes a gap from [BF1] (see the comments before Lemma 1.5). Now that we have saved [R1, Corollary 2], all subsequent results in [R1] concerning the strong*-topology of complex JBW*-triples are valid. Moreover, keeping in mind Proposition 4.1 and Corollary 4.2, some of those results remain true for real JBW*-triples with verbatim proof. For instance, the following assertions hold:

1. Linear mappings between real JBW*-triples are strong*-continuous if and only if they are weak*-continuous (compare [R1, Corollary 3]).

2. If $W$ is a real JBW*-triple, and if $V$ is a weak*-closed subtriple, then the inequality $S^*(W, W_*)|_V \leq S^*(V, V_*)$ holds, and in fact $S^*(W, W_*)|_V$ and $S^*(V, V_*)$ coincide on bounded subsets of $V$ (compare [R1, Proposition 2]).

It follows from the first part of Assertion 2 above and a new application of Proposition 4.1 that, if $W$ is a real JBW*-triple, and if $V$ is a weak*-complemented subtriple of $W$, then we have $S^*(W, W_*)|_V = S^*(V, V_*)$. Since every real JBW*-triple $V$ is weak*-complemented in the realification of a complex JBW*-triple $W$ (see $V$ as a real form of its JB*-complexification), and $S^*(W, W_*) = S^*(W_R, (W_R)_*)$, the results [R1, Theorem] and [R2, Theorem D.21] for complex JBW*-triples can be transferred to the real setting, providing the following result.
Theorem 4.3 Let \( W \) be a real JBW*-triple. Then the triple product of \( W \) is jointly \( S^*(W,W_*) \)-continuous on bounded subsets of \( W \), and the topologies \( m(W,W_*) \) and \( S^*(W,W_*) \) coincide on bounded subsets of \( W \).

Our concluding goal in this paper is to establish, in the setting of real JB*-triples, a result on weakly compact operators originally due to H. Jarchow [J] in the context of C*-algebras, and later extended to complex JB*-triples by C-H. Chu and B. Iochum [CI]. This could be made by transferring the complex results to the real setting by a complexification method. However, we prefer to do it in a more intrinsic way, by deriving the result from the second assertion in Theorem 4.3 according to some ideas outlined in [R2, pp. 142-143].

Proposition 4.4 Let \( X \) be a dual Banach space (with a fixed predual \( X_* \)). Then the Mackey topology \( m(X,X_*) \) coincides with the topology on \( X \) generated by the family of semi-norms \( x \mapsto \|T(x)\| \), where \( T \) is any weak*-continuous linear operator from \( X \) to a reflexive Banach space.

Proof. Let us denote by \( \tau \) the second topology arising in the statement. As in the proof of Corollary 4.2, if \( T \) is a weak*-continuous linear operator from \( X \) to a reflexive Banach space, then there exists an absolutely convex and weakly compact subset \( D \) of \( X_* \) such that the equality

\[
\|T(x)\| = \sup |<x,D>|
\]

holds for every \( x \in X \). This shows that \( \tau \leq m(X,X_*) \).

Let \( D \) be an absolutely convex and weakly compact subset of \( X_* \). Consider the Banach space \( \ell_1(D) \) and the bounded linear operator

\[
F : \ell_1(D) \to X_*
\]

given by

\[
F(\{\lambda_\varphi\}_{\varphi \in D}) := \sum_{\varphi \in D} \lambda_\varphi \varphi.
\]

Then we have \( F(\ell_1(D)) = D \), and hence \( F \) is weakly compact. By [DFJP] there exists a reflexive Banach space \( Y \) together with bounded linear operators \( S : \ell_1(D) \to Y \), \( R : Y \to X_* \) such that \( F = R S \). Then, for \( x \in X \), we have

\[
\sup |<x,D>| = \sup |<x,F(\ell_1(D))>|
\]

24
\[ = \sup |x, R(S(B_{\ell_1}(D)))| \leq \|S\| \sup |x, R(Y) > | \]

Since \(D\) is an arbitrary absolutely convex and weakly compact subset of \(X_*\), and \(R^*\) is a weak*-continuous linear operator from \(X\) to the reflexive Banach space \(Y^*\), the inequality \(m(X, X_*) \leq \tau\) follows. \(\square\)

Let \(X\) be a dual Banach space (with a fixed predual \(X_*\)). In agreement with Proposition 4.1, we define the strong*-topology of \(X\), denoted by \(S^*(X, X_*)\), as the topology on \(X\) generated by the family of semi-norms \(x \mapsto \|T(x)\|\), where \(T\) is any weak*-continuous linear operator from \(X\) to a Hilbert space.

**Proposition 4.5** Let \(X\) be a dual Banach space (with a fixed predual \(X_*\)). Then the following assertions are equivalent:

1. The topologies \(m(X, X_*)\) and \(S^*(X, X_*)\) coincide on bounded subsets of \(X\).

2. For every weak*-continuous linear operator \(F\) from \(X\) to a reflexive Banach space, there exists a weak*-continuous linear operator \(G\) from \(X\) to a Hilbert space satisfying \(\|F(x)\| \leq \|G(x)\| + \|x\|\) for all \(x \in X\).

3. For every weak*-continuous linear operator \(F\) from \(X\) to a reflexive Banach space, there exist a weak*-continuous linear operator \(G\) from \(X\) to a Hilbert space and a mapping \(N : (0, \infty) \to (0, \infty)\) satisfying

\[
\|F(x)\| \leq N(\varepsilon) \|G(x)\| + \varepsilon \|x\|
\]

for all \(x \in X\) and \(\varepsilon > 0\).

**Proof.** 1 \(\Rightarrow\) 2.– Let \(F\) be a weak*-continuous linear operator from \(X\) to a reflexive Banach space. Then, by Proposition 4.4

\[ O := \{y \in B_X : \|F(y)\| \leq 1\} \]

is a \(m(X, X_*)\)\(_{B_X}\)-neighborhood of zero in \(B_X\). By assumption, there exist Hilbert spaces \(H_1, \ldots, H_n\) and weak*-continuous linear operators \(G_i : X \to H_i\) \((i : 1, \ldots, n)\) such that

\[ O \supseteq \bigcap_{i=1}^n \{y \in B_X : \|G_i(y)\| \leq 1\}. \]
Now set $H := (\bigoplus_{i=1}^{n} H_{i})_{\ell_{2}}$, and consider the weak*-continuous linear operator $G : X \to H$ defined by $G(x) := (G_{1}(x), \ldots, G_{n}(x))$. Notice that

$$\{y \in B_{X} : \|G(y)\| \leq 1\} \subseteq \bigcap_{i=1}^{n} \{y \in B_{X} : \|G_{i}(y)\| \leq 1\} \subseteq \mathcal{O}.$$ 

Finally, if $x \in X \setminus \{0\}$, then $\frac{1}{\|y\|+\|G(x)\|} x$ lies in $\{y \in B_{X} : \|G(y)\| \leq 1\} \subseteq \mathcal{O}$, and hence $\|F(y)\| \leq 1$.

2 $\Rightarrow$ 3. Let $F$ be a weak*-continuous linear operator from $X$ to a reflexive Banach space. By assumption, for every $n \in \mathbb{N}$ there exists a Hilbert space $H_{n}$ and a weak*-continuous linear operator $G_{n}$ from $X$ to $H_{n}$ such that $\|nF(x)\| \leq \|G_{n}(x)\| + \|x\|$ for all $x \in X$. Now set $H := (\bigoplus_{n \in \mathbb{N}} H_{n})_{\ell_{2}}$, and consider the bounded linear operator $G : X \to H$ defined by $G(x) := \left\{ \frac{1}{n}G_{n}(x) \right\}$ and the mapping $N : \varepsilon \to \|G_{n}(\varepsilon)\|$ (where $n(\varepsilon)$ denotes the smallest natural number satisfying $n > \frac{1}{\varepsilon}$). Then $G$ is weak*-continuous. Indeed, given $y = \{h_{n}\} \in H$, we can take for $n \in \mathbb{N}$ $\alpha_{n}$ in $X_{\ast}$ satisfying $(G_{n}(x)|h_{n}) = \langle x, \alpha \rangle$ for every $x \in X$, so that we have

$$\sum_{n \in \mathbb{N}} \frac{\alpha_{n}}{n}\|G_{n}\| \leq \sum_{n \in \mathbb{N}} \frac{\|h_{n}\|}{n} \leq \sqrt{\sum_{n \in \mathbb{N}} \|h_{n}\|^{2}} \sqrt{\sum_{n \in \mathbb{N}} \frac{1}{n^{2}}} < \infty,$$

and hence $\alpha := \sum_{n \in \mathbb{N}} \frac{\alpha_{n}}{n}\|G_{n}\|$ is an element of $X_{\ast}$ satisfying $(G(x)|h) = \langle x, \alpha \rangle$ for all $x \in X$. Moreover, for all $\varepsilon > 0$ and $x \in X$ we have

$$\|F(x)\| \leq \frac{1}{n(\varepsilon)}\|G_{n}(\varepsilon)(x)\| + \frac{1}{n(\varepsilon)}\|x\|$$

$$\leq \|G_{n}(\varepsilon)\|\|G(x)\| + \frac{1}{n(\varepsilon)}\|x\| \leq N(\varepsilon)\|G(x)\| + \varepsilon\|x\|.$$ 

3 $\Rightarrow$ 1. Let $x_{\lambda}$ be a net in $B_{X}$ converging to zero in the topology $S^{\ast}(X, X_{\ast})$. Let $F$ be a weak*-continuous linear operator from $X$ to a reflexive Banach space, and $\varepsilon > 0$. By assumption, there exist a weak*-continuous linear operator $G$ from $X$ to a Hilbert space and a mapping $N : (0, \infty) \to (0, \infty)$ satisfying

$$\|F(x)\| \leq N\left(\frac{\varepsilon}{2}\right)\|G(x)\| + \frac{\varepsilon}{2}\|x\|$$

for all $x \in X$. Take $\lambda_{0}$ such that $\|G(x_{\lambda})\| \leq \frac{\varepsilon}{2N(\frac{\varepsilon}{2})}$ whenever $\lambda \geq \lambda_{0}$. Then we have $\|F(x_{\lambda})\| \leq \varepsilon$ for all $\lambda \geq \lambda_{0}$. By Proposition 4.4, $x_{\lambda} m(X, X_{\ast})$-converges to zero. □
We can now state the following characterization of weakly compact operators on JB*-triples.

**Theorem 4.6** Let $E$ be a real (respectively, complex) JB*-triple, $X$ a real (respectively, complex) Banach space, and $T : E \to X$ a bounded linear operator. The following assertions are equivalent:

1. $T$ is weakly compact.

2. There exist a bounded linear operator $G$ from $E$ to a real (respectively, complex) Hilbert space and a function $N : (0, +\infty) \to (0, +\infty)$ such that
   \[
   \|T(x)\| \leq N(\varepsilon)\|G(x)\| + \varepsilon\|x\|
   \]
   for all $x \in E$ and $\varepsilon > 0$.

3. There exist norm one functionals $\varphi_1, \varphi_2 \in E^*$ and a function $N : (0, +\infty) \to (0, +\infty)$ such that
   \[
   \|T(x)\| \leq N(\varepsilon) \|x\|_{\varphi_1, \varphi_2} + \varepsilon\|x\|
   \]
   for all $x \in E$ and $\varepsilon > 0$.

**Proof.** The implication 2 $\Rightarrow$ 3 follows from Corollary 2.10 (respectively, Corollary 2.2). The implication 3 $\Rightarrow$ 2 holds because, for norm-one functionals $\varphi_1, \varphi_2 \in E^*$, $\|\cdot\|_{\varphi_1, \varphi_2}$ is a prehilbert seminorm on $E$, and hence there exists a bounded linear operator $G$ from $E$ to a Hilbert space satisfying $\|G(x)\| = \|x\|_{\varphi_1, \varphi_2}$ for all $x \in E$. On the other hand, the implication 2 $\Rightarrow$ 1 is known to be true, even if $E$ is an arbitrary Banach space (see for instance [J, Theorem 20.7.3]). To conclude the proof, let us show that 1 implies 2. Assume that Assertion 1 holds. Then, by [DFJP], there exist a reflexive Banach space $Y$ and bounded linear operators $F : E \to Y$ and $S : Y \to X$ such that $T = SF$ and $\|S\| \leq 1$. By Theorem 4.3 and Proposition 4.5, there exist a weak*-continuous linear operator $\tilde{G}$ from $E^{**}$ to a Hilbert space and a mapping $N : (0, \infty) \to (0, \infty)$ satisfying
   \[
   \|F^{**}(\alpha)\| \leq N(\varepsilon) \|\tilde{G}(\alpha)\| + \varepsilon \|\alpha\|
   \]
   for all $\alpha \in E^{**}$ and $\varepsilon > 0$. By putting $G := \tilde{G}|_E$, the inequality in Assertion 2 follows. \(\Box\)
The complex case of the above theorem is established in [CI, Theorem 11], with \( \|\cdot\|_{\varphi_1,\varphi_2} \) in Assertion 3 replaced with \( \|\cdot\|_{\varphi} \) for a single norm-one functional \( \varphi \in E^* \). As we have noticed in similar occasions, we do not know if such a replacement is correct.

References


A. M. Peralta and A. Rodríguez Palacios
Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada
18071 Granada, Spain
aperalta@goliat.ugr.es and apalacio@goliat.ugr.es