Finite-Dimensional Banach Spaces with Numerical Index Zero

Miguel Martín, Javier Méri, and Ángel Rodríguez-Palacios

The numerical index of a Banach space $X$ is a constant of the space relating the behaviour of the numerical radius with that of the usual norm on $L(X)$, the Banach algebra of all bounded linear operators on the space.

The numerical range of an operator $T \in L(X)$ is the subset $V(T)$ of the scalar field defined by $V(T) = \{ x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1 \}$, where $X^*$ stands for the dual space of $X$ and $S_X$ is its unit sphere. This definition of numerical range was introduced by F. Bauer [1] and, concerning applications, it is equivalent to Lumer’s numerical range [5]. The numerical radius of $T$ is given by $v(T) = \sup \{|\lambda| : \lambda \in V(T)\}$.

It is clear that $v$ is a seminorm on $L(X)$ satisfying $v(T) \leq \|T\|$ for every $T \in L(X)$. The numerical index of the space $X$ is defined as $n(X) = \inf \{ v(T) : T \in S_{L(X)} \}$ or, equivalently, as the greatest constant $k \geq 0$ such that $k \|T\| \leq v(T)$ for every $T \in L(X)$. Note that $0 \leq n(X) \leq 1$ and $n(X) > 0$ if and only if $v$ and $\| \cdot \|$ are equivalent norms on $L(X)$. In the complex case, it is a celebrated result due to H. Bohnenblust and S. Karlin [2] that $n(X) \geq 1/e$, so the numerical radius is always an equivalent norm. The situation is very different in the real case, since every real Hilbert space of dimension greater than one has numerical index zero. Classical references on this topics are the monographs by F. Bonsall and J. Duncan [3, 4]. More recent results can be found in the survey paper [6] and references therein.

We deal with real Banach spaces with numerical index zero. As we already said, this class of Banach spaces contains all real Hilbert spaces of dimension greater than one. It also contains all real spaces underlying complex Banach spaces (the operator $x \mapsto ix$ on a complex Banach space has real numerical radius 0). One may think that Banach spaces with numerical index 0 have always any kind of “complex structure”, but this is not the case. Indeed, there exists an infinite-dimensional Banach space with numerical index 0, containing no isometric copy of $\mathbb{C}$ [7, Example 2.2]. Nevertheless, as the main
Theorem 1. Let \((X, \| \cdot \|)\) be a finite-dimensional real Banach space. Then, the following are equivalent:

(i) The numerical index of \(X\) is zero.

(ii) There are nonzero complex vector spaces \(X_1, \ldots, X_n\), a real vector space \(X_0\), and positive integer numbers \(q_1, \ldots, q_n\) such that \(X = X_0 \oplus X_1 \oplus \cdots \oplus X_n\) and

\[
\left\| x_0 + e^{i q_1 \rho} x_1 + \cdots + e^{i q_n \rho} x_n \right\| = \| x_0 + x_1 + \cdots + x_n \|
\]

for all \(\rho \in \mathbb{R}\), \(x_j \in X_j\) (\(j = 0, 1, \ldots, n\)).

Corollary 2. [7, Corollary 2.5] Let \(X\) be a real Banach space with numerical index 0.

(a) If \(\dim(X) = 2\), then \(X\) is isometrically isomorphic to the two-dimensional real Hilbert space.

(b) If \(\dim(X) = 3\), then \(X\) is an absolute sum of \(\mathbb{R}\) and the two-dimensional real Hilbert space.

Corollary 3. [7, Corollary 2.7] Let \(X\) be a real Banach space of dimension \(n \in \mathbb{N}\). Then we have \(\dim(Z(X)) \leq \frac{n(n-1)}{2}\). Moreover, \(X\) is a Hilbert space if and only if \(\dim(Z(X)) = \frac{n(n-1)}{2}\).

In view of Corollary 2 it might be thought that the number of complex spaces in Assertion (ii) of Theorem 1 can be always reduced to one. As a matter of fact, this is not true, as the following example shows.

Example 4. [7, Example 2.8] There exists a four-dimensional real space \(X\) with \(n(X) = 0\) and such that the number of complex spaces in Theorem 1.(ii) cannot be reduced to one. Indeed, this is the case for \(X = \mathbb{R}^4\) with norm

\[
\|(a, b, c, d)\| = \frac{1}{4} \int_0^{2\pi} \left| \text{Re} \left( e^{2it}(a + ib) + e^{it}(c + id) \right) \right| \, dt \quad (a, b, c, d \in \mathbb{R}).
\]

References


**Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain, e-mail:** mmartins@ugr.es

**Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain, e-mail:** jmeri@ugr.es

**Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain, e-mail:** apalacio@ugr.es