FINITE-DIMENSIONAL BANACH SPACES WITH NUMERICAL INDEX ZERO

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The numerical index of a Banach space X is a constant of the space relating the behaviour of the numerical radius with that of the usual norm on $L(X)$, the Banach algebra of all bounded linear operators on the space.

The *numerical range* of an operator $T \in L(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx) \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\},$$

where X^* stands for the dual space of X and S_X is its unit sphere. This definition of numerical range was introduced by F. Bauer [1] and, concerning applications, it is equivalent to Lumer's numerical range [5]. The *numerical radius* of T is given by

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in V(T)\}.$$

It is clear that v is a seminorm on $L(X)$ satisfying $v(T) \leqslant \|T\|$ for every $T \in L(X)$. The *numerical index* of the space X is defined as

$$n(X) = \inf\{v(T) \ : \ T \in S_{L(X)}\}$$

or, equivalently, as the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in L(X)$. Note that $0 \leqslant n(X) \leqslant 1$ and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms on $L(X)$. In the complex case, it is a celebrated result due to H. Bohnenblust and S. Karlin [2] that $n(X) \geqslant 1/\mathrm{e}$, so the numerical radius is always an equivalent norm. The situation is very different in the real case, since every real Hilbert space of dimension greater than one has numerical index zero. Classical references on this topics are the monographs by F. Bonsall and J. Duncan [3, 4]. More recent results can be found in the survey paper [6] and references therein.

We deal with real Banach spaces with numerical index zero. As we already said, this class of Banach spaces contains all real Hilbert spaces of dimension greater than one. It also contains all real spaces underlying complex Banach spaces (the operator $x \longmapsto ix$ on a complex Banach space has real numerical radius 0). One may think that Banach spaces with numerical index 0 have always any kind of "complex structure", but this is not the case. Indeed, there exists an infinite-dimensional Banach space with numerical index 0, containing no isometric copy of $\mathbb{C}$ [7, Example 2.2]. Nevertheless, as the main

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result in [7] states, such an example cannot happen in the finite-dimensional context. Thus, we have the following.

Theorem 1. *Let $(X, \|\cdot\|)$ be a finite-dimensional real Banach space. Then, the following are equivalent:*

(i) *The numerical index of X is zero.*
(ii) *There are nonzero* complex *vector spaces $X_1, \dots, X_n$, a* real *vector space X_0, and positive integer numbers $q_1, \dots, q_n$ such that $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ and*

$$\left\| x_0 + \mathrm{e}^{i q_1 \rho} x_1 + \cdots + \mathrm{e}^{i q_n \rho} x_n \right\| = \| x_0 + x_1 + \cdots + x_n \|$$

for all $\rho \in \mathbb{R}$, $x_j \in X_j$ $(j = 0, 1, \dots, n)$.

In dimension two or three, the above result can be written in the more suitable form given by the following corollary.

Corollary 2. [7, Corollary 2.5] *Let X be a real Banach space with numerical index* 0*.*

(a) *If* $\dim(X) = 2$*, then X is isometrically isomorphic to the two-dimensional real Hilbert space.*
(b) *If* $\dim(X) = 3$*, then X is an absolute sum of* $\mathbb{R}$ *and the two-dimensional real Hilbert space.*

For every real Banach space X, let us denote by $Z(X)$ the subspace of $L(X)$ consisting of those operators T on X with $v(T) = 0$.

Corollary 3. [7, Corollary 2.7] *Let X be a real Banach space of dimension $n \in \mathbb{N}$. Then we have* $\dim\big(Z(X)\big) \leqslant \frac{n(n-1)}{2}$*. Moreover, X is a Hilbert space if and only if* $\dim\big(Z(X)\big) = \frac{n(n-1)}{2}$*.*

In view of Corollary 2 it might be thought that the number of complex spaces in Assertion (*ii*) of Theorem 1 can be always reduced to one. As a matter of fact, this is not true, as the following example shows.

Example 4. [7, Example 2.8] *There exists a four-dimensional real space X with $n(X) = 0$ and such that the number of complex spaces in Theorem 1.(ii) cannot be reduced to one.* Indeed, this is the case for $X = \mathbb{R}^4$ with norm

$$\|(a,b,c,d)\| = \frac{1}{4} \int_0^{2\pi} \left| \mathrm{Re}\, \left(\mathrm{e}^{2it}(a+ib) + \mathrm{e}^{it}(c+id) \right) \right| \, dt \qquad (a,b,c,d \in \mathbb{R}).$$

References

[1] F. L. Bauer, *On the field of values subordinate to a norm*, Numer. Math., **4** (1962), 103–111.

[2] H. F. Bohnenblust and S. Karlin, *Geometrical properties of the unit sphere in Banach algebras*, Ann. of Math., **62** (1955), 217–229.

[3] F. F. Bonsall and J. Duncan, *Numerical Ranges of operators on normed spaces and of elements of normed algebras*, London Math. Soc. Lecture Note Ser. **2**, Cambridge, 1971.

[4] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10**, Cambridge, 1973.

[5] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. **109** (1961), 29–43.
[6] M. Martín, *A survey on the numerical index of a Banach space*, Extracta Math. **15** (2000), 265–275.
[7] M. Martín, J. Merí and A. Rodríguez-Palacios, *Finite-dimensional Banach spaces with numerical index cero*, Indiana Univ. Math. J. (to appear).

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