CONVEX-TRANSITIVE BANACH SPACES, BIG POINTS, AND THE DUALITY MAPPING

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1. Introduction

Throughout this paper $\mathbb{K}$ will mean the field of real or complex numbers. Given a normed space X over $\mathbb{K}$, S_X, B_X, and X^* will denote the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X, and $\mathcal{G} := \mathcal{G}(X)$ will stand for the group of all surjective linear isometries from X to X. We say that an element u in a normed space X is a **big point** of X if $u \in S_X$ and $\overline{co}(\mathcal{G}(u)) = B_X$, where $\overline{co}$ means closed convex hull. The space X is said to be **convex-transitive** if all elements in S_X are big points of X. The space X is said to be **transitive** (respectively, **almost transitive**) if, for every (equivalently, some) element u in S_X, we have that $\mathcal{G}(u) = S_X$ (respectively, the closure of $\mathcal{G}(u)$ in X is equal to S_X). The notions just defined provide us with a chain of implications

$$\text{transitivity} \Rightarrow \text{almost transitivity} \Rightarrow \text{convex transitivity}\,,$$

none of which is reversible.

The literature dealing with transitivity conditions on normed spaces is linked to the Banach-Mazur "rotation" problem [1] if every transitive separable Banach space is isometric to ℓ_2. The reader is referred to the book of S. Rolewicz [21] and the survey papers of F. Cabello [9] and the authors [6] for a comprehensive view of known results and fundamental questions in relation to this matter.

In the present paper we deal with convex-transitive Banach spaces. The interest of these spaces starts with the pioneering theorem of N. J. Kalton and G. V. Wood [17] that convex-transitive complex Banach spaces having a one-dimensional hermitian projection are Hilbert spaces (see also [3]), and increases with the recent result that convex-transitive Banach spaces which either are Asplund or fulfill the Radon-Nikodym property are almost transitive and superreflexive [2]. We note that superreflexive almost transitive Banach spaces have been previously considered by C. Finet [15] (see also [13, Corollary IV.5.7]) and F. Cabello [10]. The "leit motiv" of this paper is the study of convex transitivity through the behaviour of the duality mapping of an arbitrary Banach space X at the big points of X. Either in their

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formulations or in their proofs, all main results in the paper involve the "leit motiv" just mentioned. We pass to review the results.

Recall that a subset R of a topological space E is said to be **rare** in E if the interior of the closure of R in E is empty. In the setting of our previous investigation on convex-transitive Banach spaces, the apparently weaker condition that the set of all big points of a given Banach space X is non rare in S_X often appear. If X is either Asplund or Radon-Nikodym, then such a condition is actually equivalent to the convex transitivity of X, and hence implies that X is almost transitive and superreflexive (see [2] and [5]). However, the question if the convex transitivity of an arbitrary Banach space X can be characterized by the non-rarity of the set of big points of X in S_X remained open. In Section 2 we answer affirmatively such a question (Theorem 2.5).

In Section 3 we prove that, if X is a convex-transitive complex normed space, then the complex multiples of the identity on X are the unique bounded linear operators on X which commute with all elements of $\mathcal{G}$ (Proposition 3.6). The same conclusion holds if X is a complex normed space having a big point u such that the diameter of the value of the duality mapping of X at u is "small" (Theorem 3.2). The appropriate versions for real spaces of the results just reviewed are also obtained, but they have slightly more complicated formulations (see Proposition 3.6 and Theorem 3.3, respectively, for details).

2. A characterization of convex-transitive Banach spaces

As said in the introduction, in this section we characterize the convex transitivity of a Banach space X by the existence of some non rare subset of S_X consisting only of big points of X. Our argument begins with the following lemma.

Lemma 2.1. *Let X be a normed space over $\mathbb{K}$, and let $|||.|||$ be a lower semicontinuous norm on X such that $\mathcal{G} \subseteq \mathcal{G}(X, |||.|||)$. Then $|||.|||$ is constant on the set of all big points of X.*

Proof. We can assume that $|||u||| = 1$ for some big point u of X. Then $B_{(X,|||.|||)}$ is a closed, convex, and $\mathcal{G}$-invariant subset of X containing u, so that, by the bigness of u, we have $B_X \subseteq B_{(X,|||.|||)}$, or equivalently $|||.||| \leq \|.\|$.

Now let v be an arbitrary big point of X. From the above inequality, we obtain $|||v||| \leq 1$. On the other hand, $|.| := |||v|||^{-1}|||.|||$ is a lower semicontinuous norm on X satisfying $\mathcal{G} \subseteq \mathcal{G}(X, |.|)$ and $|v| = 1$. It follows from the preceding paragraph that $|.| \leq \|.\|$, or equivalently $|||.||| \leq |||v|||\|.\|$. As a consequence, $1 = |||u||| \leq |||v|||\|u\| = |||v|||$. Therefore $|||v||| = 1$. ∎

Let X be a normed space over $\mathbb{K}$, and u an element in S_X. We consider the set $D(X,u)$ of all **states** of X relative to u, that is

$$D(X,u) := \{f \in B_{X^*} : f(u) = 1\}.$$

In some mathematical formulations of quantum mechanics, a "system of observables" is identified with the Banach space (say Y) of all self-adjoint operators on a suitable Hilbert space H and, denoting by v the identity mapping on H, the different "states of a given observable y" are the numbers of the form $g(y)$ when g runs over $D(Y,v)$; thus $D(Y,v)$ is the set of "states" of the system. We note that, for arbitrary (X,u), $D(X,u)$ is a non-empty, convex, and w^*-compact subset of X^*. The set-valued function $v \to D(X,v)$ on S_X is called the **duality mapping** of X. For x in X, the **numerical range** of x relative to (X,u), denoted by $V(X,u,x)$, is given by the equality

$$V(X,u,x) := \{f(x) : f \in D(X,u)\}.$$

Lemma 2.2. *Let X be a normed space over $\mathbb{K}$, u an element of S_X, and f a linear functional on X. Then f belongs to $D(X,u)$ if (and only if) $f(x)$ lies in $V(X,u,x)$ for every x in X.*

Proof. Assume that $f(x)$ is in $V(X,u,x)$ for every x in X. Since $V(X,u,u) = \{1\}$, we have $f(u) = 1$. On the other hand, for x in X there exist g in $D(X,u)$ such that $f(x) = g(x)$, and hence $|f(x)| = |g(x)| \leq \|x\|$. Therefore f is continuous with $\|f\| \leq 1$. ∎

Let X be a normed space, u a norm-one element of X, and x an arbitrary element in X. Then the mapping $\alpha \to \|u + \alpha x\|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex, and hence there exists $\lim_{\alpha \to 0^+} \frac{\|u+\alpha x\|-1}{\alpha}$. It is well-known that the above limit coincides with $\max\{\Re e(\lambda) : \lambda \in V(X,u,x)\}$ (see for instance [14, Theorem V.9.5]).

Lemma 2.3. *Let X be a normed space over $\mathbb{K}$, let $|||.|||$ be a norm on X which coincides with $\|.\|$ on an open subset Ω of S_X, and let u be in Ω. Then we have $D(X,u) = D((X, |||.|||), u)$.*

Proof. Let x be in X. Take $r > 0$ such that $u + \alpha x \neq 0$ and $\frac{u+\alpha x}{\|u+\alpha x\|} \in \Omega$ whenever α is in $\mathbb{R}$ with $0 < \alpha \leq r$. Then, for $0 < \alpha \leq r$, we have $|||\frac{u+\alpha x}{\|u+\alpha x\|}||| = 1$, and hence

$$\max\{\Re e(\lambda) : \lambda \in V(X,u,x)\} = \lim_{\alpha \to 0^+} \frac{\|u+\alpha x\| - 1}{\alpha}$$

$$= \lim_{\alpha \to 0^+} \frac{|||u+\alpha x||| - 1}{\alpha} = \max\{\Re e(\lambda) : \lambda \in V((X, |||.|||), u, x)\}.$$

After replacing x with μx when μ runs over $S_{\mathbb{K}}$, we obtain

$$V(X,u,x) = V((X, |||.|||), u, x).$$

Now, since x is arbitrary in X, the result follows from Lemma 2.2 ∎

The next lemma is taken from [6]. For the sake of completeness, we include here a proof.

Lemma 2.4. *Let X be a Banach space over $\mathbb{K}$, u a big point of X, and δ a positive number. Then the set*

$$\Delta_\delta(u) := \{T^*(f) : T \in \mathcal{G}, f \in X^* \text{ such that } \exists x \in S_X \text{ with } \|x-u\| \leq \delta \text{ and } f \in D(X,x)\}$$

is dense in S_{X^*}.

Proof. Let g be in S_{X^*}, and $0 < \varepsilon < 1$. Since u is a big point of X, the convex hull of $g(\mathcal{G}(u))$ is dense in $B_{\mathbb{K}}$, and therefore there exists T in $\mathcal{G}$ such that $|g(T^{-1}(u)) - 1| < \frac{\varepsilon'^2}{4}$, where $\varepsilon' := \min\{\varepsilon, \delta\}$. By the Bishop-Phelps-Bollobás theorem [8, Theorem 16.1], there are x in S_X and f in $D(X,x)$ satisfying $\|u - x\| < \varepsilon' \leq \delta$ and $\|g \circ T^{-1} - f\| < \varepsilon' \leq \varepsilon$. Since

$$\|g \circ T^{-1} - f\| = \|(T^*)^{-1}(g) - f\| = \|g - T^*(f)\|,$$

this shows that $g \in \overline{\Delta_\delta(u)}$. ∎

In fact the above lemma characterizes big points of Banach spaces. Indeed, if X is a (possibly non complete) normed space, if u is in S_X, and if $\Delta_\delta(u)$ (defined as in the lemma) is dense in S_{X^*} for every positive number δ, then u is a big point of X (see the proof of [6, Lemma 5.7] for details).

Theorem 2.5. *Let X be a Banach space over $\mathbb{K}$. Assume that there exists some non rare subset of S_X consisting only of big points of X. Then X is convex-transitive.*

Proof. Let us denote by U the set of all big points of X. Since U is closed in X (a consequence of [4, Lemma 3.7]), the assumption on X actually implies that the interior of U (say Ω) relative to S_X is non-empty. On the other hand, by [12, Theorem 5], to prove that X is convex-transitive it is enough to show that every equivalent norm $|||.|||$ on X such that $\mathcal{G} \subseteq \mathcal{G}(X, |||.|||)$ is a positive multiple of the natural norm $\|.\|$ of X. Let $|||.|||$ be an equivalent norm on X satisfying $\mathcal{G} \subseteq \mathcal{G}(X, |||.|||)$. By Lemma 2.1, we can assume that $|||.|||$ coincides with $\|.\|$ on Ω. By Lemma 2.3, we have $D(X,v) = D((X, |||.|||), v)$ for every v in Ω. Therefore we obtain $|||f||| = 1$ whenever $f \in D(X,v)$ and $v \in \Omega$. Now we fix u in Ω, and take $\delta > 0$ such that x belongs to Ω whenever x is in S_X and $\|x - u\| \leq \delta$. Since T^* is an isometry on $(X^*, |||.|||)$ whenever T is in $\mathcal{G}$, it follows from Lemma 2.4 that $|||.|||$ is equal to 1 on a dense subset of S_{X^*}. Therefore $|||.|||$ coincides with $\|.\|$ on X, as desired. ∎

3. The commutant of the group of isometries

Let X be a normed space over $\mathbb{K}$. We denote by $\mathrm{Com}(\mathcal{G})$ the set of those bounded linear operators on X which commute with all elements of $\mathcal{G}$. If there exists some big point in X, then $\mathcal{G}$ must have "many" elements, so that one could expect $\mathrm{Com}(\mathcal{G})$ to be "very small". As we show in the following example, this is not always the case.

Example 3.1. Let A and B be unital C^*-algebras without nonzero proper direct summands. Assume that A and B are not Jordan-*-isomorphic. Denote by X the complex Banach space underlying the C^*-algebra $A \oplus^\infty B$. By the Russo-Dye theorem [8, Theorem 30.2], the unit of $A \oplus^\infty B$ is a big point of X. We claim that $\mathrm{Com}(\mathcal{G})$ contains an isometric copy of the complex space ℓ_∞^2. To see this, it is enough to show that A and B are $\mathcal{G}$-invariant subsets of X. Let T be in $\mathcal{G}$. By [16, Theorem 7], there exists a unitary element v in $A \oplus^\infty B$ and a Jordan-$*$-automorphism Φ of $A \oplus^\infty B$ satisfying $T(x) = v\Phi(x)$ for every x in X. It follows from [11, Theorem 5.3] that $T(A)$ and $T(B)$ are ideals of $A \oplus^\infty B$. Since $A \oplus^\infty B = T(A) \oplus T(B)$, and $A \oplus^\infty B$ has no nonzero proper direct summands other than A and B, we deduce that either A and B are T-invariant or $T(A) = B$. But, if the equality $T(A) = B$ happened, then we would have $\Phi(A) = v^*T(A) = v^*B = B$, which is not possible because A and B are not Jordan-$*$-isomorphic.

The complex Banach space X above can be chosen either finite-dimensional (by taking $A = \mathbb{C}$ and $B = M_2(\mathbb{C})$) or a $C(K)$-space for a suitable Hausdorff compact topological space K (by taking $A = \mathbb{C}$ and $B = C([0, 1])$).

By an easy final touch of the construction above, for an arbitrary set Γ one can built a complex Banach space X having big points and such that $\mathrm{Com}(\mathcal{G})$ contains an isometric copy of the complex space $\ell_\infty(\Gamma)$.

Let X be a normed space over $\mathbb{K}$, and u a norm-one element in X. We denote by $\delta(X, u)$ the diameter of $D(X, u)$, and we put

$$n(X, u) := \inf\{\sup\{|\lambda| : \lambda \in V(X, u, x)\} : x \in S_X\}.$$

We note that, if Y is a subspace of X containing u, then the inequality $\delta(Y, u) \leq \delta(X, u)$ holds. We also note that, if $\mathbb{K} = \mathbb{C}$ and if the dimension of X is ≥ 2, then we have $\sqrt{3}\, n(X, u) \leq \delta(X, u)$ [20, Lemma 5.14].

Theorem 3.2. *Let X be a complex normed space such that there exists a big point u of X satisfying $\delta(X, u) < \frac{\sqrt{3}}{e}$. Then $\mathrm{Com}(\mathcal{G})$ is equal to $\mathbb{C}I_X$, where I_X denotes the identity mapping on X.*

Proof. If F is in $\mathrm{Com}(\mathcal{G})$, then the set $\{x \in X : \|F(x)\| \leq \|F(u)\|\}$ is closed, convex, and $\mathcal{G}$-invariant, so that the bigness of u gives $\|F\| = \|F(u)\|$. Now the mapping $F \to F(u)$ from $\mathrm{Com}(\mathcal{G})$ to X is a linear isometry sending I_X into u. On the other hand, since $\mathrm{Com}(\mathcal{G})$ is a norm-unital complex normed algebra, the Bohnenblust-Karlin theorem applies (see [7, Theorem 4.1]) to get $n(\mathrm{Com}(\mathcal{G}), I_X) \geq \frac{1}{e}$. It follows that, if $\mathrm{Com}(\mathcal{G})$ does not reduce to $\mathbb{C}I_X$, then we have

$$\frac{\sqrt{3}}{e} > \delta(X, u) \geq \delta(\mathrm{Com}(\mathcal{G})(u), u) = \delta(\mathrm{Com}(\mathcal{G}), I_X)$$

$$\geq \sqrt{3}\, n(\mathrm{Com}(\mathcal{G}), I_X) \geq \frac{\sqrt{3}}{e},$$

a contradiction. ∎

The real variant of Theorem 3.2, we are immediately proving, has a more complicated formulation.

Theorem 3.3. *There exists a universal constant $k > 0$ such that, for every real normed space X having a big point u with $\delta(X, u) < k$, the real normed algebra* $\mathrm{Com}(\mathcal{G})$ *is algebraically isomorphic to either* $\mathbb{R}$, $\mathbb{C}$, *or* $\mathbb{H}$ *(the algebra of Hamilton's quaternions).*

Proof. According to a theorem of G. Lumer [18], there exists a universal constant $k > 0$ such that every real normed algebra A having a norm-one unit $\mathbf{1}$ with $\delta(A, \mathbf{1}) < k$ is algebraically isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Now let X be a real normed space X having a big point u with $\delta(X, u) < k$. We know that the mapping $F \to F(u)$ from $\mathrm{Com}(\mathcal{G})$ to X is a linear isometry sending I_X into u. Therefore we have

$$\delta(\mathrm{Com}(\mathcal{G}), I_X) = \delta(\mathrm{Com}(\mathcal{G})(u), u) \leq \delta(X, u) < k,$$

and Lumer's theorem applies. ∎

Unfortunately, we do not know any estimate on the constant k in Theorem 3.3. We do not know in addition if the value $k = 2$ works in that theorem. In any case, the conclusion in Theorem 3.3 can attain a more illuminating form in some special settings. We recall that the norm of a normed space X is said to be **maximal** if, for every equivalent norm $|||.|||$ on X such that $\mathcal{G} \subseteq \mathcal{G}(X, |||.|||)$, we actually have $\mathcal{G} = \mathcal{G}(X, |||.|||)$. As a consequence of Lemma 2.1, we rediscover the well-known fact that convex-transitivity implies maximality of the norm. However the converse implication is not true. Indeed, the real or complex Banach space ℓ_1 has big points and maximal norm, but is not convex-transitive.

Lemma 3.4. *Let X be a normed space over $\mathbb{K}$ whose norm is maximal, and let $\mathcal{S}$ be a bounded subgroup of the group of all invertible elements of* $\mathrm{Com}(\mathcal{G})$. *Then $\mathcal{S}$ is contained in $\mathcal{G}$.*

Proof. Put $\mathcal{R} := \mathcal{S}\mathcal{G}$. Then $\mathcal{R}$ is a bounded subgroup of the group of all automorphisms of X, so that the norm $|||.|||$ on X defined by

$$|||x||| := \sup\{\|R(x)\| : R \in \mathcal{R}\}$$

is equivalent to the natural norm of X and satisfies $\mathcal{G} \subseteq \mathcal{R} \subseteq \mathcal{G}(X, |||.|||)$. Since the norm of X is maximal, and $\mathcal{S}$ is contained in $\mathcal{R}$, the result follows. ∎

Proposition 3.5. *Let X be a real normed space having a big point u. Assume that either X is smooth at u (i.e., $\delta(X, u) = 0$), or X has maximal norm and $\delta(X, u) < k$ (where k is the universal constant given by Theorem 3.3). Then either* $\mathrm{Com}(\mathcal{G})$ *reduces to* $\mathbb{R}I_X$, *or there exists a complex normed space Y such that X is the real normed space underlying Y and the equality* $\mathrm{Com}(\mathcal{G}) = \mathbb{C}I_Y$ *holds. In the last case we have $\mathcal{G} = \mathcal{G}(Y)$, and hence u becomes a big point of Y.*

Proof. Assume that X is smooth at u. Then $\mathrm{Com}(\mathcal{G})(u)$ is also smooth at u, and hence, since the mapping $F \to F(u)$ from $\mathrm{Com}(\mathcal{G})$ to X is a linear isometry, $\mathrm{Com}(\mathcal{G})$ becomes a "smooth normed" real algebra (i.e. a norm-unital real normed algebra which is smooth at its unit). By [22], $\mathrm{Com}(\mathcal{G})$ is algebraically and isometrically isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Therefore the unit sphere of $\mathrm{Com}(\mathcal{G})$ is contained in $\mathcal{G}$, so the algebra $\mathrm{Com}(\mathcal{G})$ is commutative, and so the possibility $\mathrm{Com}(\mathcal{G}) \cong \mathbb{H}$ cannot really happen. Suppose that $\mathrm{Com}(\mathcal{G})$ does not reduce to $\mathbb{R}I_X$. Then we are provided with an isometric algebra-isomorphism $\phi : \mathbb{C} \to \mathrm{Com}(\mathcal{G})$, and hence X becomes a complex normed space (say Y) under its given norm and the product of complex numbers λ by elements x of X defined by $\lambda x := (\phi(\lambda))(x)$. Now, clearly, X is the real normed space underlying Y, and the equality $\mathrm{Com}(\mathcal{G}) = \mathbb{C}I_Y$ holds. This equality implies that elements of $\mathcal{G}$ commute with all multiplications on Y by complex numbers, and hence are complex-linear operators on Y. Thus we have the inclusion $\mathcal{G} \subseteq \mathcal{G}(Y)$, and the reverse inclusion is trivially true.

Now assume that the norm of X is maximal and that $\delta(X, u) < k$. By Theorem 3.3, we are provided with an algebra-isomorphism

$$\phi : \mathbb{A} \to \mathrm{Com}(\mathcal{G}) ,$$

where $\mathbb{A}$ stands for either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. In any case, $\phi(S_{\mathbb{A}})$ is a bounded subgroup of the group of all invertible elements of $\mathrm{Com}(\mathcal{G})$, so that Lemma 3.4 applies giving that the ϕ is an isometry. Now $\mathrm{Com}(\mathcal{G})$ is algebraically and isometrically isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and the proof is concluded as in the preceding paragraph. ∎

In relation to Proposition 3.5 above, the following facts are worth mentioning. Let Y be a complex normed space, and let X denote the real normed space underlying X. Then, for u in S_X we have $\delta(X, u) = \delta(Y, u)$. Moreover, if the norm of X is maximal, and if the equality $\mathcal{G} = \mathcal{G}(Y)$ holds, then the norm of Y is maximal.

We conclude this section by describing the commutant of the group of all surjective linear isometries on a convex-transitive normed space.

Proposition 3.6. *Let X be a convex-transitive normed space over $\mathbb{K}$. If $\mathbb{K} = \mathbb{C}$, then $\mathrm{Com}(\mathcal{G}) = \mathbb{C}I_X$. If $\mathbb{K} = \mathbb{R}$, then either $\mathrm{Com}(\mathcal{G}) = \mathbb{R}I_X$, or there exists a complex normed space Y such that X is the real normed space underlying Y and the equality $\mathrm{Com}(\mathcal{G}) = \mathbb{C}I_Y$ holds. In this last case we have $\mathcal{G} = \mathcal{G}(Y)$, and hence the complex normed space Y is convex-transitive.*

Proof. We know that $\|F(u)\| = \|F\|$ whenever F belongs to $\mathrm{Com}(\mathcal{G})$ and u is a big point of X. Therefore, since X is convex transitive, we have in fact $\|F(x)\| = \|F\|\|x\|$ for all $F \in \mathrm{Com}(\mathcal{G})$ and $x \in X$. If follows $\|FG\| = \|F\|\|G\|$ for all $F, G \in \mathrm{Com}(\mathcal{G})$, i.e., $\mathrm{Com}(\mathcal{G})$ is an "absolute-valued" algebra. But unital absolute-valued algebras are smooth normed algebras (see for instance the implication $(b) \Rightarrow (a)$ in [19, Corollary 29]). Then the proof of the present proposition in the case $\mathbb{K} = \mathbb{R}$ is concluded as in the first paragraph of the proof of Proposition 3.5. For the case $\mathbb{K} = \mathbb{C}$, note that, as a biproduct of the Bohnenblust-Karlin theorem, smooth normed complex algebras are one-dimensional. ∎

Let Y be a convex-transitive complex normed space, and let X stand for the real normed space underlying Y. Then X is convex-transitive, and we have $\mathrm{Com}(\mathcal{G}) \subseteq \mathrm{Com}(\mathcal{G}(Y)) = \mathbb{C}I_Y$. For most choices of Y, there exist surjective real-linear isometries on Y which are not complex-linear, and hence we have in fact $\mathrm{Com}(\mathcal{G}) = \mathbb{R}I_X$. For instance, this is the case if Y is any complex pre-Hilbert space, or a convex-transitive complex $C(K)$- or $L_p(\mu)$-space. The possibility $\mathrm{Com}(\mathcal{G}) = \mathbb{C}I_Y$, theoretically allowed by Theorem 3.6, becomes more problematic. Actually, due to the relative scarcity of examples of convex-transitive normed spaces, we are unable to provide the reader with a choice of Y in such a way that the equality $\mathrm{Com}(\mathcal{G}) = \mathbb{C}I_Y$ holds. In other words, we do not know if there exists a convex-transitive complex normed space without surjective real-linear isometries other than the complex-linear ones.

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