CONTINUITY OF HOMOMORPHISMS INTO NORMED ALGEBRAS WITHOUT TOPOLOGICAL DIVISORS OF ZERO

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Abstract

We prove that homomorphisms from complete normed complex algebras into complete normed complex algebras with no nonzero two-sided topological divisors of zero are automatically continuous.

0.- Introduction

Although we are dealing with automatic continuity of homomorphisms between nonassociative normed algebras, a reference to the theorem of B. E. Johnson [J] on continuity of homomorphisms onto semisimple (associative) Banach algebras seems to be obliged. In fact, as is pointed out by A. M. Sinclair in the introduction of [S], Johnson's theorem germinally contains a nonassociative result, namely that Jordan-homomorphisms from Banach algebras onto semisimple Banach algebras are continuous. Concerning the automatic continuity of homomorphisms between nonassociative normed algebras, the pre-history consists of the pioneering work of V. K. Balachandran and P. S. Rema [BaRe] ([R5; pp. 109-110]), which allowed us to show that dense range homomorphisms from complete normed power-associative algebras into complete normed power-associative strongly semisimple algebras are continuous [R2] ([R5; B.1]), and the papers [PeR] and [PuY] ([R5; p. 111]). The history actually begins with the work of B. Aupetit [A] ([R5; B.2]) showing the automatic continuity of homomorphisms onto semisimple complete normed Jordan algebras, and continues with the paper of the author [R1] ([R5; pp. 119-120]), where the ultraweak radical of an arbitrary nonassociative algebra is introduced, and the continuity of homomorphisms onto complete normed nonassociative algebras with zero ultraweak radical is proved. To close this short review, let us refer to the recent papers of Berenger-Villena ([BeVi1], [BeVi2]) and Aupetit-Mathieu [AM], where the continuity of Lie-homomorphisms onto Banach algebras and other related topics are deeply discussed. For other results on automatic continuity of homomorphisms between normed nonassociative algebras the reader is referred to [R3; Theorem 4], [R6] ([R5; B.21 and E.18]), [PRiRVi; Corollary 3.1] and [PRiR; Theorem 8] (jointly collected in [R5; B.12]), [MoR; Theorem 5.3], [NRV], [Ce], [KRaR; Theorem 3.3], and [RV].

As in the associative case, the automatic continuity of nonsurjective homomorphisms between complete normed nonassociative algebras can be successfully attacked only when either the domain algebra or the range algebra is extremely " good ". For instance, we can consider the " goodness " of the range algebra consisting in the absence of topological divisors of zero, and ask whether homomorphisms into such an algebra are continuous. Particular cases of this problem have been affirmatively answered in [R3] and [KRaR]. In the present paper we are more ambitious, and raise the problem in its most general reasonable formulation. Precisely, we deal with the following

QUESTION 0.1.- *Let A be a complete normed algebra over $\mathbb{K}$, B a complete normed algebra over $\mathbb{K}$ with no nonzero two-sided topological divisors of zero, and $\Phi : A \to B$ an algebra homomorphism. Is Φ continuous?*

Here $\mathbb{K}$ denote the field of real or complex numbers, and the algebras A and B are not assumed to be associative.

If the algebra B above is associative, then it is well known (see [Ka] and [CR]) that $A = \mathbb{C}$ if $\mathbb{K} = \mathbb{C}$, and $A = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the algebra of Hamilton's quaternions) if $\mathbb{K} = \mathbb{R}$. Therefore in such a case the range of Φ is a complete normed simple algebra, and the question answers affirmatively by applying [R1; Remark 3.4.(ii)].

From now on, let us dispense the requirement of associativeness for the algebra B in Question 0.1. Then, even if B has no nonzero one-sided topological divisors of zero, B need not be finite-dimensional ([UW], [U], [Cu], [R3], and [KRaR]). Even more, B need not be simple [R4; pp. 33-34], and Φ need not have closed range [R4; p. 30]. If Φ has closed range, then, either from [R1; Remark 3.4.(ii)] and some techniques in this paper (see Corollary 2.7), or almost directly from [R1; Theorem 3.3] (see Proposition 4.8), we obtain that the answer to Question 0.1 is also affirmative.

Now Question 0.1 becomes actually interesting only in the case that B is not associative and Φ has not closed range. Our main result asserts that Question 0.1 has always an affirmative answer if $\mathbb{K} = \mathbb{C}$ (Theorem 3.5). Most techniques developed in proving this fact are also valid in the case $\mathbb{K} = \mathbb{R}$. In this line we emphasize that Question 0.1 has an affirmative answer whenever Φ has dense range and there exists some element in the unital multiplication algebra of B which is bounded below but non surjective (Proposition 2.3). As a consequence, Question 0.1 answers affirmatively whenever Φ has dense range and B is not a quasi-division algebra (Corollary 2.4). In this way quasi-division algebras naturally arise in our development. Actually, one of the key tools in the proof of the main result is that complete normed quasi-division complex algebras have dimension ≤ 2 (Proposition 3.4). As far as we know, quasi-division algebras have been not previously introduced in the literature. They are defined as those algebras C such that, for every nonzero element c in C, either the operator of left multiplication by c or the operator of right multiplication by c is bijective.

The paper is structured as follows. In Section 1, thecniques are developed as far as possible without going out the setting of Banach spaces. Indeed, if X, Y are Banach spaces over $\mathbb{K}$, and if $\Phi : X \to Y$ is a dense range linear mapping, then Φ canonically produces a certain subalgebra $\hat{X}$ of the Banach algebra $BL(X)$ of all bounded linear operators on X, as well as a homomorphism $\hat{\Phi}$ from $\hat{X}$ into $BL(Y)$ (see Notation 1.4). Then we obtain some non trivial information about the separating set $\mathcal{S}(\hat{\Phi})$ for $\hat{\Phi}$ (Theorem 1.5), which implies that elements of $\mathcal{S}(\hat{\Phi})$ are either bijective or non bounded below (Corollary 1.6). Section 2 collects those applications of Theorem 1.5 to the study of Question 0.1 which are valid for both real and complex algebras. That section includes Corollary 2.4, already reviewed in the preceding paragraph, as well as two interesting variants of it (see Corollaries 2.5 and 2.6). With one of such variants in mind we rediscover the result in [R3] that homomorphisms from complete normed algebras into absolute-valued algebras are automatically continuous (Corollary 2.8). Section 3 collects the remaining tools needed to conclude the proof of the main result. Among them we emphasize Proposition 3.4, already reviewed, and Lemma 3.3, which is applied in the proof of that proposition, and asserts that, whenever a_1, a_2, b_1, b_2 are elements of a complex Banach algebra with a unit, we can find $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$ such that $sp(\lambda_1 a_1 + \lambda_2 a_2) \cap sp(\lambda_1 b_1 + \lambda_2 b_2) \neq \emptyset$. Finally we devote Section 4 to discuss the results and techniques in the paper. In that section the reader can find examples of two-dimensional quasi-division complex algebras (Corollary 4.5) and other similar " monsters ". For instance, there are complete normed algebras over $\mathbb{K}$ with no nonzero left topological divisors of zero and such that all their elements are right topological divisors of zero (Example 4.3). Also there exist infinite-dimensional complete normed algebras over $\mathbb{K}$ with no nonzero two-sided topological divisors of zero but having both nonzero left topological divisors of zero and nonzero right topological divisors of zero (Example 4.6). Examples of such kinds were previously unknown in the literature. Moreover, in most cases, our results apply successfully to them (see again Examples 4.3 and 4.6).

1.- Working in Banach spaces

From now on, $\mathbb{K}$ will denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let X be a normed space over $\mathbb{K}$. We denote by $BL(X)$ the normed algebra of all bounded linear operators on X. An element F of $BL(X)$ is said to be *bounded below* if there exists a positive number k satisfying $k \parallel x \parallel \leq \parallel F(x) \parallel$ for every x in X. We recall that, by the Banach isomorphism theorem, if X is in fact a Banach space, then every bijective bounded linear operator on X is bounded below. Our argument begins with the following lemma.

LEMMA 1.1.- *Let X be a Banach space over $\mathbb{K}$, and P a connected subset of $BL(X)$ all elements of which are bounded below. Then either all elements of P are bijective or all elements of P are non bijective.*

Proof.- Assume that the assertion in the lemma is not true. Then

$$Q := \{F \in P : F \text{ is bijective}\}$$

is a non-empty proper subset of the connected set P, and therefore there must exist some F_0 in the boundary of Q relative to P. Since such a F_0 lies in the boundary of the set of all invertible elements of the Banach algebra $BL(X)$, it follows from [B; Lemma 56.3 and Theorem 57.4] that F_0 is not bounded below. This contradicts the assumption that all elements of P are bounded below. ■

Let F be a bounded linear operator on a normed space X. We denote by $k(F)$ the largest non-negative number k satisfying $k \parallel x \parallel \leq \parallel F(x) \parallel$ for every x in X. In this way F is bounded below if and only if $k(F) > 0$.

COROLLARY 1.2.- *Let X be a Banach space over $\mathbb{K}$. Then, for F, G in $BL(X)$ we have*

$$|k(F) - k(G)| \leq \parallel F - G \parallel \ .$$

Moreover, if $F \in BL(X)$ is bounded below and non bijective, then the open ball in $BL(X)$ with center F and radius $k(F)$ consists only of elements which are bounded below and non bijective.

Proof.- Let F, G be in $BL(X)$. For all x in X we have

$$(k(F) - \parallel G - F \parallel) \parallel x \parallel \leq \parallel F(x) \parallel - \parallel (G - F)(x) \parallel \leq \parallel G(x) \parallel \ ,$$

and hence

$$k(F) - \parallel G - F \parallel \leq k(G) \ ,$$

which proves the first assertion in the corollary. From this first assertion it follows that, if F is bounded below, and if B denotes the open ball in $BL(X)$ with center F and radius $k(F)$, then B consists only of elements which are bounded below. Therefore, if in addition F is non bijective, then, by Lemma 1.1, all elements of B are non bijective. ■

For every set E, we denote by I_E the identity mapping on E.

PROPOSITION 1.3.- *Let X and Y be Banach spaces over $\mathbb{K}$, Φ a (possibly discontinuous) linear mapping from X into Y whose range is dense in Y, and F and G be in $BL(X)$ and $BL(Y)$, respectively, such that G is non bijective and the equality $\Phi F = G\Phi$ holds. Then we have $k(G) \leq \parallel F \parallel$.*

Proof.- We may assume that G is bounded below (since otherwise we have $k(G)=0$ and nothing is to prove). Let $0<\delta<k(G)$. We have

$$\| (G-\delta I_Y)-G \| = \delta < k(G) \ ,$$

so that, since G is bounded below and non bijective, it follows from Corollary 1.2 that $G-\delta I_Y$ is bounded below and non bijective. This implies that the range of $G-\delta I_Y$ is a proper closed subspace of Y. Then, since the equality

$$\Phi(F-\delta I_X)=(G-\delta I_Y)\Phi$$

holds, and Φ has dense range, we deduce that $F-\delta I_X$ cannot be bijective. Therefore we have $\delta \leq \| F \|$, and the proof is concluded by letting $\delta \to k(G)$. ■

NOTATION 1.4.- Let X and Y be Banach spaces over $\mathbb{K}$, and Φ a linear mapping from X to Y, whose range is dense in Y. Then the set

$$\hat{X} := \{F \in BL(X) : \text{there is } G \in BL(Y) \text{ satisfying } \Phi F = G\Phi\}$$

is a (possibly non closed) subalgebra of $BL(X)$, for F in $\hat{X}$ there exists a unique G in $BL(Y)$ satisfying $\Phi F = G\Phi$, and the mapping $F \to G$ from $\hat{X}$ to $BL(Y)$ becomes an algebra homomorphism. Such a homomorphism will be denoted by $\hat{\Phi}$, and the symbol $\hat{Y}$ will stand for the closure in $BL(Y)$ of the range of $\hat{\Phi}$. $\hat{X}$ (respectively, $\hat{Y}$) will be considered as a normed (respectively, Banach) algebra under the restriction of the natural norm of $BL(X)$ (respectively, $BL(Y)$).

Given normed spaces E and F over $\mathbb{K}$, and a linear mapping $T: E \to F$, we denote by $\mathcal{S}(T)$ the *separating set* for T, namely the set of those elements f in F such that there exists a sequence $\{e_n\}$ in E satisfying $\lim\{e_n\}=0$ and $\lim\{T(e_n)\}=f$.

THEOREM 1.5.- *Let X and Y be Banach spaces over $\mathbb{K}$, and Φ a linear mapping from X to Y, whose range is dense in Y. Assume that there is some element in $\hat{Y}$ which is bounded below and non bijective. Then the separating set for $\hat{\Phi}$ consists only of elements which are not bounded below.*

Proof.- We argue by contradiction, so that there exists some element G in $\mathcal{S}(\hat{\Phi})$ which is bounded below.

In a first step we assume that such a G is non bijective. Write $G = \lim\{\hat{\Phi}(F_n)\}$ for some sequence $\{F_n\}$ in $\hat{X}$ with $\lim\{F_n\}=0$. Since the set of those elements in $BL(Y)$ which are bounded below and non bijective is open (by Corollary 1.2), there is no restriction in assuming that $\hat{\Phi}(F_n)$ is non bijective for all n. Then, by Proposition 1.3, for n in $\mathbb{N}$ we have

$$k(\hat{\Phi}(F_n)) \leq \| F_n \| \ ,$$

and therefore, since the function $k(.)$ is continuous on $BL(Y)$ (by Corollary 1.2), we obtain $k(G) \leq 0$, contradicting that G is bounded below.

To conclude the proof, remove now the assumption in the above paragraph that G is bijective. By the hypothesis in the theorem, there exists an element (say T) in $\hat{Y}$ which is bounded below and non bijective. Since G belongs to $\mathcal{S}(\hat{\Phi})$, and the separating set for an algebra homomorphism between normed algebras is an ideal of the closure of the range, also TG lies in $\mathcal{S}(\hat{\Phi})$. Then TG is an element of $\mathcal{S}(\hat{\Phi})$ which is bounded below and non bijective. But this is not possible in view of the first step of the proof. ■

The elemental inclusion $\mathcal{S}(\hat{\Phi}) \subset \hat{Y}$ gives the following corollary (which in fact is nothing but the first step in the proof of the theorem).

COROLLARY 1.6.- *Let X and Y be Banach spaces over $\mathbb{K}$, and Φ a linear mapping from X to Y, whose range is dense in Y. Then every element in $\mathcal{S}(\hat{\Phi})$ is either bijective or non bounded below.*

2.- Some first applications to normed algebras

For a vector space E over $\mathbb{K}$, we denote by $L(E)$ the associative algebra of all linear mappings from E to E. Let A be an algebra over $\mathbb{K}$. For a in A we denote by L_a^A (respectively, R_a^A) the operator of left (respectively, right) multiplication by a on A. The *unital multiplication algebra* of A is defined as the subalgebra of $L(A)$ generated by

$$\{I_A\} \cup \{L_x^A : x \in A\} \cup \{R_y^A : y \in A\} ,$$

and is denoted by $\mathcal{M}(A)$. When A is in fact a normed algebra, the operators of left and right multiplication on A by elements of A are continuous, and therefore the inclusion $\mathcal{M}(A) \subset BL(A)$ holds.

Now, let A and B be complete normed algebras over $\mathbb{K}$, and $\Phi : A \to B$ a dense range (algebra) homomorphism. Since for a in A the equalities

$$\Phi L_a^A = L_{\Phi(a)}^B \Phi \quad \text{and} \quad \Phi R_a^A = R_{\Phi(a)}^B \Phi$$

hold, with the convention of symbols in Notation 1.4 we have

$$L_a^A \in \hat{A} \ , \ R_a^A \in \hat{A} \ , \ \hat{\Phi}(L_a^A) = L_{\Phi(a)}^B \ , \text{ and } \hat{\Phi}(R_a^A) = R_{\Phi(a)}^B \ .$$

It follows from the denseness of the range of Φ and the continuity of the mappings $b \to L_b^B$ and $b \to R_b^B$ from B to $BL(B)$ that L_b^B and R_b^B lie in $\hat{B}$ for all b in B, and therefore $\mathcal{M}(B)$ is contained in $\hat{B}$. Then the following corollary follows straightforwardly from Theorem 1.5.

PROPOSITION 2.1.- *Let B be a complete normed algebra over $\mathbb{K}$ such that there is some element in $\mathcal{M}(B)$ which is bounded below and non bijective. Then, for every complete normed algebra A over $\mathbb{K}$ and every dense range homomorphism Φ from A to B, the separating set for $\hat{\Phi}$ consists only of elements which are not bounded below.*

Let A be a normed algebra over $\mathbb{K}$. An element a of A is said to be a *left* (respectively, *right*) *topological divisor of zero* in A whenever there exists a sequence $\{a_n\}$ of norm-one elements of A satisfying $\lim\{aa_n\} = 0$ (respectively, $\lim\{a_n a\} = 0$) . In this way, left (respectively, right) topological divisors of zero in A are nothing but those elements a of A such that the operator L_a^A (respectively R_a^A) is not bounded below. Elements of A which are either left or right (respectively, both left and right) topological divisors of zero are called *one-sided topological divisors of zero* (respectively, *two-sided topological divisors of zero*) in A.

Now, let A and B be complete normed algebras over $\mathbb{K}$, and $\Phi : A \to B$ a dense range homomorphism. We claim that the sets $L^B_{\mathcal{S}(\Phi)}$ and $R^B_{\mathcal{S}(\Phi)}$ are contained in $\mathcal{S}(\hat{\Phi})$. Indeed, if $\{a_n\} \to 0$ in A and $\{\Phi(a_n)\} \to b$ in B, then $\{L^A_{a_n}\} \to 0$ in $\hat{A}$ and $\{\hat{\Phi}(L^A_{a_n})\} = \{L^B_{\Phi(a_n)}\} \to L^B_b$ in $BL(B)$. Keeping in mind the claim just shown and Proposition 2.1 we obtain:

COROLLARY 2.2.- *Let B be a complete normed algebra over $\mathbb{K}$ such that there is some element in $\mathcal{M}(B)$ which is bounded below and non bijective. Then, for every complete normed algebra A over $\mathbb{K}$ and every dense range homomorphism Φ from A to B, the separating set for Φ consists only of two-sided topological divisors of zero in B.*

Now the next result follows from the closed graph theorem.

PROPOSITION 2.3.- *Let B be a complete normed algebra over $\mathbb{K}$. Assume that there is some element in $\mathcal{M}(B)$ which is bounded below and non bijective, and that B has no nonzero two-sided topological divisors of zero. Then every dense range homomorphism from a complete normed algebra over $\mathbb{K}$ into B is continuous.*

Let A be a nonzero algebra over $\mathbb{K}$. An element a in A is said to be *left* (respectively, *right*) *invertible* in A if the operator L_a^A (respectively, R_a^A) is bijective. A is said to be a *left* (respectively, *right*) *division algebra* if every nonzero element in A is left (respectively, right) invertible. If A is both a left and right (respectively, either a left or right) division algebra, then we say that A is a *two-sided division algebra* (respectively, *one-sided division algebra*). The algebra A is said to be a *quasi-division algebra* if every nonzero element in A is either left or right invertible. In the case that the algebra A is complete normed, the implications in the following diagram hold. In that diagram the abbreviation *t.d.z.* stands for *topological divisors of zero.*

Diagram I

$$
\begin{array}{ccccc}
\boxed{\begin{array}{c} A \text{ is a two-sided} \\ \text{division algebra} \end{array}} & \Rightarrow & \boxed{\begin{array}{c} A \text{ is a } \left\{ \begin{array}{c} \text{left} \\ \text{right} \end{array} \right\} \\ \text{division algebra} \end{array}} & \Rightarrow & \boxed{\begin{array}{c} A \text{ is a quasi-} \\ \text{division algebra} \end{array}} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\boxed{\begin{array}{c} A \text{ has no nonzero} \\ \text{one-sided t.d.z.} \end{array}} & \Rightarrow & \boxed{\begin{array}{c} A \text{ has no nonzero} \\ \left\{ \begin{array}{c} \text{left} \\ \text{right} \end{array} \right\} \text{ t.d.z.} \end{array}} & \Rightarrow & \boxed{\begin{array}{c} A \text{ has no nonzero} \\ \text{two-sided t.d.z.} \end{array}}
\end{array}
$$

When the complete normed algebra A above is associative, the weakest condition in the diagram implies the strongest one (see [Ka] and [CR]), so that anyone of the conditions in the diagram gives $A = \mathbb{C}$ if $\mathbb{K} = \mathbb{C}$, and $A = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ if $\mathbb{K} = \mathbb{R}$. However, in the general nonassociative setting we are dealing with, a similar situation is far from being true. Actually, as we will discuss in Section 4, when $\mathbb{K} = \mathbb{R}$ no implication in the diagram is reversible. Note also that, if A is finite-dimensional, then the six conditions in the diagram reduce to two. Indeed, in such a setting all vertical implications in the diagram are reversible, and one-sided division implies two-sided division. On the other hand, if $\mathbb{K} = \mathbb{C}$, then the implication on the left of the first line of the diagram is also reversible (see Corollary 3.2 below). In any case, the following three corollaries are direct consequences of Proposition 2.3.

COROLLARY 2.4.- *Let B be a complete normed algebra over $\mathbb{K}$. Assume that B is not a quasi-division algebra and has no nonzero two-sided topological divisors of zero. Then every dense range homomorphism from a complete normed algebra over $\mathbb{K}$ into B is continuous.*

COROLLARY 2.5.- *Let B be a complete normed algebra over $\mathbb{K}$. Assume that B is not a left (respectively, right) division algebra and has no nonzero left (respectively, right) topological divisors of zero. Then every dense range homomorphism from a complete normed algebra over $\mathbb{K}$ into B is continuous.*

COROLLARY 2.6.- *Let B be a complete normed algebra over $\mathbb{K}$. Assume that B is not a two-sided division algebra and has no nonzero one-sided topological divisors of zero. Then every dense range homomorphism from a complete normed algebra over $\mathbb{K}$ into B is continuous.*

An algebra is said to be *simple* if it has nonzero product and has no nonzero proper two-sided ideals. Since quasi-division algebras are simple, and homomorphisms from complete normed algebras onto complete normed simple algebras are continuous [R1; Remark 3.4.(ii)], the next result follows from Corollary 2.4.

COROLLARY 2.7.- *Let B be a complete normed algebra over $\mathbb{K}$ with no nonzero two-sided topological divisors of zero. Then every homomorphism from a complete normed algebra over $\mathbb{K}$ onto B is continuous.*

An *absolute value* on an algebra A over $\mathbb{K}$ is a norm $\| \, . \, \|$ on the vector space of A satisfying $\| xy \| = \| x \| \| y \|$ for all x, y in A. An *absolute-valued* algebra over $\mathbb{K}$ is a nonzero algebra over $\mathbb{K}$ endowed with an absolute value. Absolute-valued algebras become the nicest examples of normed algebras with no non-zero one-sided topological divisors of zero.

COROLLARY 2.8 [R3; Theorem 4].- *Let B be an absolute-valued algebra over $\mathbb{K}$. Then every homomorphism from a complete normed algebra over $\mathbb{K}$ into B is continuous.*

Proof.- Let A be a complete normed algebra over K, and $\Phi : A \to B$ a homomorphism. Replacing B by the completion of the range of Φ, we may assume that B is complete and that Φ has dense range. If B is not a division algebra, then the continuity of Φ follows from Corollary 2.6. Otherwise, by [W], B is finite-dimensional, so Φ is surjective, and so the continuity of Φ follows from Corollary 2.7. ■

In [R3; Theorem 4] it is proved that homomorphisms from complete normed algebras into absolute-valued algebras are in fact contractive. But, for such homomorphisms, contractiveness is an easy consequence of continuity (see for instance [CuR; Lemma 2.1]).

3.- The main result

As usual, by a *Banach algebra* we mean a complete normed associative algebra. For an element x in a complex Banach algebra B with a unit $\mathbf{1}$, we denote by $sp(x)$ the *spectrum* of x relative to B, namely the set of those complex numbers λ such that there is no y in B satisfying $y(x - \lambda\mathbf{1}) = (x - \lambda\mathbf{1})y = \mathbf{1}$. When F is a bounded linear operator on a complex Banach space X, $sp(F)$ will stand for the spectrum of F relative to the complex Banach algebra $BL(X)$, so that $0 \in sp(F)$ means that F is not bijective. The fact that elements of complex Banach algebras with a unit have nonempty spectra can be reformulated as follows.

LEMMA 3.1.- *If b_1, b_2 are elements of a complex Banach algebra B with a unit, then there exists (λ_1, λ_2) in $\mathbb{C}^2\backslash\{(0,0)\}$ such that $0 \in sp(\lambda_1 b_1 + \lambda_2 b_2)$.*

Proof.- If $0 \in sp(b_1)$, then the result holds with $(\lambda_1, \lambda_2) = (1, 0)$. Otherwise, we can choose μ in $sp(b_1^{-1} b_2)$, and consider the equality $b_2 - \mu b_1 = b_1(b_1^{-1} b_2 - \mu \mathbf{1})$, to obtain that the result is true with $(\lambda_1, \lambda_2) = (-\mu, 1)$. ■

The following corollary is folklore (see for instance [KRaR; Remark 2.8]).

COROLLARY 3.2.- *Every complete normed one-sided division complex algebra is isomorphic to $\mathbb{C}$.*

Proof.- Let A be a complete normed left-division complex algebra. According to the previous lemma, whenever x_1 and x_2 are in A we can find (λ_1, λ_2) in $\mathbb{C}^2\backslash\{(0,0)\}$ such that

$$0 \in sp(\lambda_1 L^A_{x_1} + \lambda_2 L^A_{x_2}) = sp(L^A_{\lambda_1 x_1 + \lambda_2 x_2}),$$

so we have $\lambda_1 x_1 + \lambda_2 x_2 = 0$, and so the system $\{x_1, x_2\}$ is linearly dependent. ■

If x, y are elements of a complex Banach algebra B, then the inclusion $sp(L^B_x - R^B_y) \subset sp(x) - sp(y)$ holds. This fact was proved first by M. Rosenblum (see [Ro; Corollary 3.3], where, because of a misprint, the oposite inclusion arises). An alternative proof is the following. Since L^B_x and R^B_y are commuting elements of the complex Banach algebra $BL(B)$, we can apply Gelfand's theory to obtain $sp(L^B_x - R^B_y) \subset sp(L^B_x) - sp(R^B_y)$ (see for instance [Ru; Theorem 11.23]), and the result follows by keeping in mind that the mapping $z \to L^B_z$ (respectively, $z \to R^B_z$) from B to $BL(B)$ is an algebra homomorphism (respectively, antihomomorphism), and consequently the inclusion $sp(L^B_x) \subset sp(x)$ (respectively, $sp(R^B_y) \subset sp(y)$) holds.

LEMMA 3.3.- *Let B be a complex Banach algebra with a unit, and a_1, a_2, b_1, b_2 be elements in B. Then there exists a couple (λ_1, λ_2) in $\mathbb{C}^2\backslash\{(0,0)\}$ such that*

$$sp(\lambda_1 a_1 + \lambda_2 a_2) \cap sp(\lambda_1 b_1 + \lambda_2 b_2) \neq \emptyset.$$

Proof.- By Lemma 3.1, there is a couple (λ_1, λ_2) in $\mathbb{C}^2\backslash\{(0,0)\}$ such that

$$0 \in sp(\lambda_1(L^B_{a_1} - R^B_{b_1}) + \lambda_2(L^B_{a_2} - R^B_{b_2})).$$

But, by the result of Rosenblum quoted above, we have

$$sp(\lambda_1(L^B_{a_1} - R^B_{b_1}) + \lambda_2(L^B_{a_2} - R^B_{b_2}))$$

$$= sp(L^B_{\lambda_1 a_1 + \lambda_2 a_2} - R^B_{\lambda_1 b_1 + \lambda_2 b_2})$$

$$\subset sp(\lambda_1 a_1 + \lambda_2 a_2) - sp(\lambda_1 b_1 + \lambda_2 b_2).$$

It follows that, for such a couple $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, we have

$$sp(\lambda_1 a_1 + \lambda_2 a_2) \cap sp(\lambda_1 b_1 + \lambda_2 b_2) \neq \emptyset. \blacksquare$$

In Section 4 we will exhibit a quasi-division complex algebra of dimension 2. Now, with the help of the above proposition, we can prove that no greater dimension is allowed for complete normed quasi-division complex algebras.

PROPOSITION 3.4.- *Every complete normed quasi-division complex algebra has dimension* ≤ 2.

Proof.- Assume that there is a complete normed quasi-division complex algebra A with $3 \leq dim(A)$. Denote by Ω_1 (respectively, Ω_2) the set of all left (respectively, right) invertible elements of A. By Corollary 3.2, Ω_1 and Ω_2 are proper subsets of $A \setminus \{0\}$. Since $A \setminus \{0\} = \Omega_1 \cup \Omega_2$, and Ω_1, Ω_2 are open, and $A \setminus \{0\}$ is connected, there must exist some x in $\Omega_1 \cap \Omega_2$. Take x_1, x_2 in A such that the system $\{x_1, x_2, x\}$ is linearly independent. Applying Lemma 3.3 to the complex Banach algebra $B := BL(A)$, we find (λ_1, λ_2) in $\mathbb{C}^2 \backslash \{(0,0)\}$ and λ in $\mathbb{C}$ such that

$$\lambda \in sp(\lambda_1 L^A_{x_1}(L^A_x)^{-1} + \lambda_2 L^A_{x_2}(L^A_x)^{-1}) \cap sp(\lambda_1 R^A_{x_1}(R^A_x)^{-1} + \lambda_2 R^A_{x_2}(R^A_x)^{-1}).$$

Then, putting $y := \lambda_1 x_1 + \lambda_2 x_2 - \lambda x$, y becomes a nonzero element of A which is neither left nor right invertible, contradicting that A is a quasi-division algebra. $\blacksquare$

Now we are ready to formulate and prove the main result of the paper

THEOREM 3.5.- *Let B be a complete normed complex algebra with no nonzero two-sided topological divisors of zero. Then every homomorphism from a complete normed complex algebra into B is continuous.*

Proof.- Let A be a complete normed complex algebra, and $\Phi : A \to B$ a homomorphism. Since the absence of non-zero two-sided topological divisors of zero is inherited by every subalgebra of B, we may replace B by the closure of the range of Φ, and assume without loss of generality that Φ has dense range. If B is not a quasi-division algebra, then the continuity of Φ follows from Corollary 2.4. Otherwise, by Proposition 3.4, B is finite-dimensional, so Φ is surjective, and so the continuity of Φ follows from Corollary 2.7. $\blacksquare$

4.- Discussion of results and methods

This concluding section of the paper is devoted to discuss the field of applicability of our results on automatic continuity, namely Theorem 3.5, and Corollaries 2.4, 2.5, 2.6, and 2.7. We note that, after Theorem 3.5, those corollaries only have interest in the case of real algebras. We include also in this section a remark on the techniques developed to obtain Corollary 2.7.

We already introduced real or complex absolute-valued algebras as examples of normed algebras with no nonzero one-sided topological divisors of zero, and applied Corollaries 2.6 and 2.7 to rediscover the result in [R3] that homomorphisms from complete normed algebras to absolute-valued algebras are automatically continuous. The remaining tools in the proof were that, for absolute-valued algebras, completeness is not a relevant assumption in relation to our results (because the completion of an absolute-valued algebra is an absolute-valued algebra too), and the theorem in [W] that absolute-valued two-sided division algebras are finite-dimensional. For examples of infinite-dimensional absolute-valued algebras the reader is referred to [UW], [U], [Cu], and [R3].

Our horizon admits a first enlargement by considering the so-called nearly absolute-valued algebras [KRaR]. *Nearly absolute-valued algebras* over $\mathbb{K}$ are defined as those normed algebras A over $\mathbb{K}$ such that there exists a positive number $\rho = \rho(A)$ satisfying $\| xy \| \geq \rho \| x \| \| y \|$ for all x, y in A. The class of nearly absolute-valued algebras contains that of equivalent algebra renormings of absolute-valued algebras, but is much larger than this last class. Indeed, there exist infinite-dimensional nearly absolute-valued commutative algebras over $\mathbb{K}$ [KRaR; Example 1.1], whereas every absolute-valued commutative algebra over $\mathbb{K}$ is finite-dimensional [UW]. As in the case of absolute-valued algebras, nearly absolute-valued algebras have no nonzero one-sided topological divisors of zero, and, for them, completeness is not a relevant assumption in relation to our results. Therefore, by Theorem 4.4, homomorphisms from complete normed complex algebras into nearly absolute-valued complex algebras are automatically continuous (a result implicitly contained in [KRaR]). The continuity of homomorphisms into nearly absolute-valued real algebras is not so well understood. This is so because we do not know if nearly absolute-valued two-sided division real algebras are finite-dimensional. Actually we only know the existence of a universal constant $0 \leq K < 1$ such that every nearly absolute-valued two-sided division real algebra A with $\rho(A) > K$ is finite-dimensional [KRaR; Corollary 3.2]. Then, arguing as in the proof of Corollary 2.8 we rediscover the result in [KRaR; Theorem 3.3] asserting the automatic continuity of homomorphisms from complete normed real algebras into nearly absolute-valued real algebras A with $\rho(A) > K$.

A first tool to obtain previously unknown applications of the main results in this paper is the following claim, whose proof is straightforward.

CLAIM 4.1.- *Let A be a normed algebra over $\mathbb{K}$ with no nonzero left topological divisors of zero, and F a norm-one bounded linear operator on A which is injective but not bounded below. Denote by $\Box$ the product of A, and by*

$B = B(A, F)$ the algebra whose vector space is that of A and whose product is defined by $xy := F(x)\Box y$. Then B (with the same norm as A) is a normed algebra over $\mathbb{K}$ with no nonzero left topological divisors of zero, but every element in B is a right topological divisor of zero in B.

The above claim not only will produce new examples of applicability of Theorem 3.5, but also will provide us with an interesting discussion about the limitations of Corollaries 2.4, 2.5, and 2.6. For such a discussion we need the following straightforward consequence of Lemma 1.1.

LEMMA 4.2 [KRaR; Lemma 2.2].- *Let B be a complete normed algebra over $\mathbb{K}$. Assume that there is some left (respectively, right) invertible element in B, and that B has no non-zero left (respectively, right) topological divisors of zero. Then B is a left (respectively, right) division algebra.*

EXAMPLE 4.3.- Let I be an infinite set, and $1 \leq p < \infty$. Then certainly we may find a norm-one bounded linear operator F, on the classical Banach space $\ell_p(I)$ over $\mathbb{K}$, which is injective but not bounded below. Moreover we can choose a product on the Banach space $\ell_p(I)$ converting it into an absolute-valued algebra [R3; Remark 3.(1)]. We denote by A the absolute-valued algebra just appeared, and consider the complete normed algebra $B = B(A, F)$ given by Claim 1, so that B has no nonzero left topological divisors of zero but is far from being a nearly absolute-valued algebra. If $\mathbb{F} = \mathbb{C}$, then homomorphisms from complete normed complex algebras into B are continuous (by Theorem 3.5). If $\mathbb{K} = \mathbb{R}$ and $p \neq 2$, then A is not a left division algebra (since norms of left division absolute-valued algebras derive from inner products [R3; Proposition 4 and Remark 4.(1)]), so B also fails to be a left division algebra (by the definition of the product of B and Lemma 4.2), and so dense range homomorphisms from complete normed real algebras into B are continuous (by Corollary 2.5). Let us finally assume that $\mathbb{K} = \mathbb{R}$ and $p = 2$. Then the product on $\ell_p(I)$, which converts it into the absolute-valued algebra A, can be chosen in such a way that A is not a left division algebra. For instance, this is the case if we apply the constructions in either [R3; Remark 3.(1)] or [U]. With such a choice, things behave as above. But we also can choose the product on $\ell_p(I)$, converting it into the absolute-valued algebra A, in such a way that A becomes a left division algebra (see [Cu] and [R3; Theorem 3]). If we do so, then B is a complete normed left division real algebra all elements of which are right topological divisors of zero, and therefore none of Corollaries 2.4, 2.5 and 2.6 can be applied to B. Actually we do not know if dense range homomorphisms from complete normed real algebras to such a monster B are continuous. In any case, homomorphisms from complete normed real algebras onto such a normed algebra B are continuous (by Corollary 2.7).

Now that we are provided with abundant examples of complete normed algebras with no nonzero left (respectively, right) topological divisors of zero but having nonzero right (respectively, left) topological divisors of zero, we attack the more complicated question of finding complete normed algebras with no nonzero two-sided topological divisors of zero but having some nonzero left topological divisor of zero as well as some nonzero right topological divisor of zero. An appropriate tool to win in this line is the following claim. An element x in an algebra A is said to be a *left* (respectively, *right*) *divisor of zero* in A if there exists $y \in A\backslash\{0\}$ such that $xy = 0$ (respectively, $yx = 0$). For an algebra A, the existence of nonzero left divisors of zero in A is equivalent to the existence of nonzero right divisors of zero in A. When this is the case, we simply say that *A has nonzero divisors of zero.*

CLAIM 4.4.- *Let A be a nonzero algebra over $\mathbb{K}$, and let $B = B(A)$ denote the algebra over $\mathbb{K}$ whose vector space is $A \times A$ and whose product is defined by*

$$(x_1, x_2)(y_1, y_2) := (x_1y_2, x_1y_1 + x_2y_2) .$$

Then B has nonzero divisors of zero. Moreover we have:

1. *B is a quasi-division algebra if and only if A is a two-sided division algebra.*
2. *If A is in fact a normed algebra, and if we consider B as a normed algebra under the norm $\| (x_1, x_2) \| := \| x_1 \| + \| x_2 \|$, then B has no nonzero two-sided topological divisors of zero if and only if A has no nonzero one-sided topological divisors of zero.*

Proof.- Since for x_2 and y_1 in A the equality $(0, x_2)(y_1, 0) = 0$ holds, the existence in B of nonzero divisors of zero is not in doubt.

Assume that B is a quasi-division algebra. Let a be in $A\backslash\{0\}$. Since $(0, a)$ is not left invertible in B, it must be right invertible in B. This means that the mapping $(x_1, x_2) \rightarrow (x_1a, x_2a)$ from B to B is bijective. Equivalently, the mapping $b \rightarrow ba$ from A to A is invertible, i.e. a is right inversible in A. Analogously, the fact that $(a, 0)$ is not right invertible in B allows us to obtain that a is left invertible in A. Since a is arbitrary in $A\backslash\{0\}$, A is a two-sided division algebra. Now assume that A is a two-sided division algebra. Let $x = (x_1, x_2)$ be in $B\backslash\{0\}$. If $x_1 \neq 0$, then the operator L_x^B is bijective with inverse mapping given by

$$(y_1, y_2) \ \rightarrow \ (\ (L_{x_1}^A)^{-1}(y_2 \ - \ L_{x_2} \circ (L_{x_1}^A)^{-1}(y_1)) \ , \ (L_{x_1}^A)^{-1}(y_1) \ ) \ .$$

If $x_1 = 0$ then the operator R_x^B is bijective with inverse mapping given by

$$(y_1, y_2) \ \rightarrow \ (\ (R_{x_2}^A)^{-1}(y_1) \ , \ (R_{x_2}^A)^{-1}(y_2) \ ) \ .$$

Since x is arbitray in $B\backslash\{0\}$, B is a quasi-division algebra.

In this last paragraph of the proof we suppose that A is actually a normed algebra, and consider B as a normed algebra under the norm

$$\| (x_1, x_2) \| := \| x_1 \| + \| x_2 \| .$$

Assume that B has no nonzero two-sided topological divisors of zero. Then, for each nonzero element a in A we have that $(0, a)$ (respectively, $(a, 0)$) is not a right (respectively, left) topological divisor of zero in B. This implies that such an a is not a one-sided topological divisor of zero in A. Therefore A has no nonzero one-sided topological divisors of zero. Now assume that A has no nonzero one-sided topological divisors of zero. Let $x = (x_1, x_2)$ be in $B \backslash \{0\}$. First suppose that $x_1 \neq 0$. If $\{y_n\} = \{(y_1^n, y_2^n)\}$ is a sequence in B with $\{xy_n\} \to 0$, then we have in A

$$\{x_1 y_2^n\} \to 0 \ \text{ and } \ \{x_1 y_1^n + x_2 y_2^n\} \to 0 \ ,$$

so $\{y_2^n\} \to 0$ and $\{y_1^n\} \to 0$ (since x_1 is not a left topological divisor of zero in A), and so $\{y_n\} \to 0$. Therefore x is not a left topological divisor of zero in B. Now suppose that $x_1 = 0$ (so that $x_2 \neq 0$). Then, for every $y = (y_1, y_2)$ in B we have $yx = (y_1 x_2, y_2 x_2)$, and we easily realize that x is not a right topological divisor of zero in B (since x_2 is not a right topological divisor of zero in A). ■

Because of its relevance in relation to Proposition 3.4, we emphasize here the following consequence of the claim just proved.

COROLLARY 4.5.- *Let B denote the complex algebra whose vector space is $\mathbb{C}^2$ and whose product is defined by*

$$(\lambda_1, \lambda_2)(\mu_1, \mu_2) := (\lambda_1 \mu_2, \lambda_1 \mu_1 + \lambda_2 \mu_2) \ .$$

Then B is a quasidivision algebra.

EXAMPLE 4.6.- Taking in Claim 4.4 A equal to any complete absolute-valued infinite-dimensional algebra over $\mathbb{K}$, we obtain an infinite-dimensional complete normed algebra $B = B(A)$ over $\mathbb{K}$ which, by Assertion 2 in the claim, has no nonzero two-sided topological divisors of zero but has nonzero divisors of zero. If $\mathbb{K} = \mathbb{C}$, then the continuity of homomorphisms from complete normed algebras into B follows from Theorem 3.5. Assume that $\mathbb{K} = \mathbb{R}$. Then A is not a two-sided division algebra (since A is an infinite-dimensional absolute-valued algebra, and the already quoted result in [W] applies), so B is not a quasi-division algebra (by Assertion 1 in the claim), and so dense range homomorphims from complete normed real algebras to B are continuous (by Corollary 2.4).

With the information given until now, the reader can find examples showing that, in the setting of complete normed real algebras A, none of the implications in Diagram I is reversible. The same can be said in the setting of complete normed complex algebras, with the unique exception that, in such a setting, division and one-sided division are equivalent notions (by Corollary 3.2). Trying to conclude the discussion of our main results, we note the following consequence of Lemma 4.2.

COROLLARY 4.7.- *Let B be a complete normed algebra over $\mathbb{K}$. Assume that B is a quasi-division algebra and that either B has no nonzero left topological divisors of zero or B has no nonzero right topological divisors of zero. Then B is a one-sided division algebra.*

Proof.- Assume that B is a quasi-division algebra and has no nonzero left topological divisors of zero. If B is a left division algebra, then there is nothing to prove. Otherwise, by the assumptions and Lemma 4.2, B is a right division algebra. ■

It follows from the above corollary that, if B is a complete normed algebra over $\mathbb{K}$ which has no nonzero left (respectively, right) topological divisors of zero but is not a left (respectively, right) division algebra, then either B is a quasi-division algebra or has no nonzero one-sided topological divisors of zero. This means that, whenever Corollary 2.5 is applicable to B, we can be sure that either Corollary 2.4 or Corollary 2.6 also applies. With Examples 4.3 and 4.6 in mind, the reader can realize that, in the actually interesting case that $\mathbb{K} = \mathbb{R}$, no more dependences between Corollaries 2.4, 2.5, and 2.6 can be found.

We conclude the paper by commenting on the techniques applied in the proof of Corollary 2.7. By the sake of convenience, we obtained Corollary 2.7 by combining Corollary 2.4 with the result in [R1] that homomorphisms from complete normed algebras onto complete normed simple algebras are continuous. In fact, the result in [R1] just quoted consists of a general theorem establishing the automatic continuity of homomrphisms from complete normed algebras onto complete normed algebras with zero " ultra-weak radical " [R1; Theorem 3.3], and the remark that simple algebras have zero ultra-weak radical [R1; Remark 3.4.(ii)]. It is worth mentioning that the automatic continuity of homomorphism from complete normed algebras onto complete normed algebras with no nonzero two-sided topological divisors of zero, assured by Corollary 2.7, is in fact a particular case of [R1; Theorem 3.3]. This follows from the next proposition. In the proof, $uw - Rad(A)$ will mean the ultraweak radical of an algebra A, and, when A is associative, $Rad(A)$ will stand for the Jacobson radical of A.

PROPOSITION 4.8.- *Every complete normed algebra over $\mathbb{K}$ with no nonzero two-sided topological divisors of zero has zero ultra-weak radical.*

Proof.- We begin by noting that, if B is an associative normed algebra over $\mathbb{K}$, and if b is an element of B, then we have:

1. $r(b) := \lim\{\| b^n \|^{1/n}\} = 0$ whenever b belongs to $Rad(B)$.
2. b is a two-sided topological divisor of zero in B whenever $r(b) = 0$.

Indeed, i) is proved in [BD; Proposition 25.1] under the unnecessary assumption that B is complete. On the other hand, ii) is clearly true whenever b is in fact nilpotent. Otherwise we put $x_n := \frac{b^n}{\|b^n\|}$, so that we have $\| x_n \| = 1$ and

$$liminf \{\| bx_n \|\} = liminf \{\| x_n b \|\}$$

$$= liminf \{\frac{\| b^{n+1} \|}{\| b^n \|}\} \leq lim \{\| b^n \|^{1/n}\} \ = \ r(b) \ ,$$

and therefore b is a two-sided topological divisor of zero in B whenever $r(b) = 0$.

Now let A be a complete normed algebra over $\mathbb{K}$. By the definition of the ultraweak radical of A [R1; Definition 3.2 and Remark 1.8], we have

$$uw - Rad(A) \ = \ \sum_{i \in I} A_i \ ,$$

where $\{A_i\}_{i \in I}$ is a family of subspaces of A satisfying the following property: for every $i \in I$ there exists a subalgebra B_i of $BL(A)$ such that $L^A_{x_i}$ and $R^A_{x_i}$ belong to $Rad(B_i)$ whenever x_i is in A_i. It follows from the first paragraph in the proof that, for i in I and x_i in A_i, $L^A_{x_i}$ and $R^A_{x_i}$ are two-sided topological divisors of zero in B_i (hence also in $BL(A)$), and therefore, by [B; Theorem 57.4], x_i is a two-sided topological divisor of zero in A. In this way we have proved that $uw - Rad(A)$ consists only of finite sums of two-sided topological divisors of zero. ■

Acknowledgements.- The author is very grateful to Antonio Moreno Galindo for his constant wiliness to discuss on the matter developed in this paper. Among other relevant contributions, Moreno Galindo discovered the first example of a two-dimensional quasi-division complex algebra, and showed that finite-dimensional quasi-division complex algebras have dimension ≤ 2. These facts allowed the author to conjecture that Proposition 3.4 could be true. The author also thanks J. Becerra and M. V. Velasco for fruitful remarks.

References

[A] B. AUPETIT, The uniqueness of the complete norm topology in Banach algebras and Banach Jordan algebras. *J. Funct. Anal.* **47** (1982), 7.25.

[AM] B. AUPETIT and M. MATHIEU, The continuity of Lie homomorphisms. *Studia Math.* **138** (2000), 193-199.

[BaRe] V. K. BALACHANDRAN and P. S. REMA, Uniqueness of the norm topology in certain Banach Jordan algebras. *Publ. Ramanujan Inst.* **1** (1969), 283-289.

[B] S. K. BERBERIAN, *Lectures in functional Analysis and operator theory.* Springer-Verlag, New York, 1974.

[BeVi1] M. I. BERENGUER and A. R. VILLENA, Continuity of Lie mappings on the skew elements of Banach algebras with involution. *Proc. Amer. Math. Soc.* **126** (1998), 2717-2720.

[BeVi2] M. I. BERENGUER and A. R. VILLENA, Continuity of Lie isomorphisms of Banach algebras. *Bull. London Math. Soc.* **31** (1999), 6-10.

[BoD] F. F. BONSALL and J. DUNCAN, *Complete normed algebras.* Springer-Verlag, New Yoork, 1973.

[CR] M. CABRERA and A. RODRIGUEZ, A new simple proof of the Gelfand-Mazur-Kaplansky theorem. *Proc. Amer. Math. Soc.* **123** (1995), 2663-2666.

[Ce] A. CEDILNIK and A. RODRIGUEZ, Automatic continuity of homomorphisms into normed quadratic algebras. *Pub. Math. Debrecen* (to appear).

[Cu] J. A. CUENCA, On one-sided division infinite-dimensional normed real algebras. *Publ. Mat.* **36** (1992), 485-488.

[CuR] J. A. CUENCA and A. RODRIGUEZ, Absolute values on H^*-algebras. *Commun. Algebra* **23** (1995), 1709-1740.

[J] B. E, JOHNSON, The uniqueness of the (complete) norm topology. *Bull. Amer. Math. Soc.* **73** (1967), 537-539.

[KRaR] A. KAIDI, M. I. RAMIREZ, and A. RODRIGUEZ, Nearly absolute-valued algebras. Preprint.

[Ka] I. KAPLANSKY, Normed algebras. *Duke Math. J.* **16** (1949), 399-418.

[MoR] A. MORENO and A. RODRIGUEZ, Algebra norms on tensor products of algebras, and the norm extension problem. *Linear Algebra Appl.* **269** (1998), 257-305.

[NVR] M. NEUMANN, A. RODRIGUEZ and M. V. VELASCO, Continuity of homomorphisms and derivations on algebras of vector-valued functions. *Quart. J. Math. Oxford* **50** (1999), 279-300.

[PRiR] J. PEREZ, L. RICO and A. RODRIGUEZ, Full subalgebras of Jordan Banach algebras and algebra norms on JB^*-algebras. *Proc. Amer. Math. Soc.* **121** (1994), 1133-1143.

[PRiRVi] J. PEREZ, L. RICO, A. RODRIGUEZ and A. R. VILLENA, Prime Jordan-Banach algebras with non-zero socle. *Commun. Algebra* **20** (1992), 17-53.

[PeR] I. PEREZ DE GUZMAN and A. RODRIGUEZ, Jordan-epimorfismos sobre un álgebra alternativa normada completa semisimple Kleinfeld. In *Actas de las VII Jornadas Matemáticas Hispano-Lusitanas, Publicaciones de la*

Sección de Matemáticas de la Universidad Autónoma de Barcelona **21** (1980), 205-209.

[PuY] P. S. PUTTER and B. YOOD, Banach Jordan $*$-algebras. *Proc. London Math. Soc.* **41** (1980), 21-44.

[R1] A. RODRIGUEZ, The uniqueness of the complete norm topology in complete normed nonassociative algebras. *J. Funct. Anal.* **60** (1985), 1-15.

[R2] A. RODRIGUEZ, An approach to Jordan-Banach algebras from the theory of nonassociative complete normed algebras. In *Algèbres de Jordan and algèbres normées non associatives, Ann. Sci. Univ. Clermont-Ferrand II Math.* **27** (1991), 1-57.

[R3] A. RODRIGUEZ, One-sided division absolute valued algebras. *Publ. Mat.* **36** (1992), 925-954.

[R4] A RODRIGUEZ, *Números hipercomplejos en dimensión infinita.* Discurso de ingreso en la Academia de Ciencias Matemáticas, Físico-Químicas y Naturales de Granada. Granada, 1993. *Math. Rev.* **95h**:46075.

[R5] A. RODRIGUEZ, Jordan structures in Analysis. In *Jordan algebras, Proceedings of the Conference held in Oberwolfach, Germany, August 9-15, 1992* (Ed. W. Kaup, K. McCrimmon, and H. P. Petersson), 97-186, Walter de Gruyter, Berlin-New York 1994.

[R6] A. RODRIGUEZ, Continuity of densely valued homomorphisms into H^*-algebras. *Quart. J. Math. Oxford* **46** (1995), 107-118.

[RV] A. RODRIGUEZ and M. V. VELASCO, Continuity of homomorphisms and derivations on normed algebras which are tensor products of algebras with involution. *Rocky Mountain J. Math.* (to appear).

[Ro] M. ROSEMBLUM, On the operator equation $BX - XA = Q$. *Duke Math. J.* **23** (1956), 263-270.

[Ru] W. RUDIN, *Functional Analysis.* Tata NcGraw-Hill Publishing Company, New Delhi, 1977.

[S] A. M. SINCLAIR, Jordan automorphisms on a semisimple Banach algebra. *Proc. Amer. Math. Soc.* **25** (1970), 526-528.

[U] K. URBANIK, Absolute valued algebras with an involution. *Fundamenta Math.* **49** (1961), 247-258.

[UW] K. URBANIK and F. B. WRIGHT, Absolute valued algebras. *Proc. Amer. Math. Soc.* **11** (1960), 861-866.

[W] F. B. WRIGHT, Absolute valued algebras. *Proc. Nat. Acad. Sci. U.S.A.* **39** (1953), 330-332.