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Abstract

We collect in this note the results in [6] where we show that weakly compact operators on a non-reflexive normed space cannot be bijective but might be surjective, study the behaviour of surjective weakly compact operators on a non-reflexive normed space when they are perturbed by small scalar multiples of the identity, and derive the recent result of J. Spurný [8] that compact operators on an infinite-dimensional normed space cannot be surjective. As a novelty, we prove here that the linear hull of the identity and all compact or weakly compact operators on a normed space is a normed Q-algebra.

Keywords: Compact operator, weakly compact operator, normed Q-algebra.

1 Introduction

As said in the abstract, the aim of this note is to review in some detail the results in [6] about compact and weakly compact operators on non-complete normed spaces, as well as to derive a new consequence, which has its own interest in the theory of normed Q-algebras.

The existence of suitable infinite-dimensional normed spaces X and Y such that there are bijective compact operators from X to Y is well-known. It is also known that, in this situation, the space Y cannot be complete. On the other hand, the space X above can be chosen arbitrarily among the duals of infinite-dimensional separable Banach spaces (see Proposition 2.3), and, in particular, among the infinite-dimensional reflexive separable spaces. More specifically, the choice $X = \ell_2$ is allowed. In the opposite direction, the space X above can be also chosen non complete (see Proposition 2.4). This gives examples of normed spaces X and Y such that there exists a bijective weakly compact operator from X

to Y, and both X and Y are non reflexive. One of the main results in [6] asserts that this last situation cannot happen in the case that $X = Y$ (Theorem 3.2). As a consequence, if T is a weakly compact operator on a normed space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), then the set of those $\lambda \in \mathbb{K}$ such that $T - \lambda$ is not bijective becomes a compact subset of $\mathbb{K}$ (Corollary 3.3).

Other relevant result in [6] is the one asserting that the requirement of bijectivity in Theorem 3.2, mentioned above, cannot be relaxed to that of surjectivity. Indeed, we can find non-complete (hence non-reflexive) normed spaces X, of arbitrary density character, such that there are surjective weakly compact operators from X to X (Proposition 4.1). Moreover, we show in [6] that, if T is a surjective weakly compact operator on a non-reflexive normed space over $\mathbb{K}$, then there exists $\delta > 0$ such that $T - \lambda$ is surjective but not injective whenever λ is in $\mathbb{K}$ with $0 < |\lambda| < \delta$ (Theorem 4.3). Since this conclusion cannot be true if the operator T is in fact compact, we derive in [6] the recent result of J. Spurný [8] that compact operators on an infinite-dimensional normed space cannot be surjective (Corollary 4.4). Actually, a detailed inspection of the argument in [8] allows us to realize that, if X is a normed space, and if T is a surjective weakly compact operator from X to X, then $X/\ker(T)$ is reflexive (Theorem 4.5), being provided in this way with an alternative proof of Theorem 3.2.

We take the opportunity of publishing this note to prove a new result. Indeed, we apply a part of the previously reviewed material to show in Corollary 3.7 that the linear hull of the identity and all compact or weakly compact operators on a normed space is a normed Q-algebra (see Definition 3.5). This has its own interest because of the scarcity of natural examples of normed Q-algebras.

2 Some basic facts about weakly compact operators

We recall that a linear operator T, from a normed space X to a normed space Y, is called compact (respectively, weakly compact) if $T(B_X)$ is a relatively compact (respectively, weakly compact) subset of Y. Here B_X stands for the closed unit ball of X. All results in this section are taken from [6], being aware that some of them can be considered as folklore.

Proposition 2.1. *Let X and Y be normed spaces, and let T be a compact (respectively, weakly compact) linear operator from X to Y. If Y is infinite-dimensional (respectively, non reflexive), then $T(X)$ is of the first category in Y.*

It follows from Proposition 2.1 that, *if Y is a normed space of the second category in itself, and if there exists a surjective compact (respectively, weakly compact) operator from some normed space to Y, then Y is finite-dimensional (respectively, reflexive).* As a consequence, we have the following.

Corollary 2.2. *Let Y be a Banach space such that there exists a surjective compact (respectively, weakly compact) operator from some normed space to Y. Then Y is finite-dimensional (respectively, reflexive).*

The version of Corollary 2.2 for compact operators is well-known (see for example [9, Theorem V.7.4]).

Both compact and weakly compact versions of Corollary 2.2 do not remain true if the assumption that Y is a Banach space is relaxed to the one that Y is an arbitrary normed space. Actually, suitable infinite-dimensional normed spaces X and Y are built in [8] such that there exists a BIJECTIVE compact (so, weakly compact) operator from X to Y. The space Y of [8] is of course non complete (and hence, non reflexive), whereas, although strangely introduced, the space X is (isometrically isomorphic to) ℓ_2. More examples of bijective compact operators between infinite-dimensional normed spaces are given by Proposition 2.3 immediately below. Given a normed space X, we denote by X^* the (topological) dual of X.

Proposition 2.3. *Let X be a separable Banach space, and let Y be an infinite-dimensional Banach space. Then there exists a bijective compact operator from X^* to some subspace of Y.*

Now, the existence of bijective compact operators starting from non-complete normed spaces follows from the following.

Proposition 2.4. *Let $(X, \|\cdot\|)$ be a normed space, and let f be a $\|\cdot\|$-discontinuous linear funcional on X. Then the norm $|||\cdot|||$ on X defined by $|||x||| := \|x\| + |f(x)|$ is not complete. Moreover, compact (respectively, weakly compact) operators starting from X remain compact (respectively, weakly compact) when they are regarded as operators starting from $(X, |||\cdot|||)$.*

Given a linear operator T on a vector space X, and any scalar λ, we write $T - \lambda$ instead of $T - \lambda I_X$, where I_X stands for the identity mapping on X. Given normed spaces X and Y, and a bounded linear operator $T : X \to Y$, we denote by $T^* : Y^* \to X^*$ the transpose of T. By noticing that a linear operator T on a normed space X is weakly compact if and only if the inclusion $T^{**}(X^{**}) \subseteq X$ holds, the concluding result in this section is easily obtained.

Proposition 2.5. *Let X be a normed space over $\mathbb{K}$, let T be a weakly compact operator on X, and let λ be in $\mathbb{K} \setminus \{0\}$. Then $\ker(T - \lambda)$ is a reflexive Banach space, and we have*

$$\ker(T - \lambda) = \ker(T^{**} - \lambda).$$

Moreover, the following assertions are equivalent:

1. *$T - \lambda$ is surjective.*
2. *$T^{**} - \lambda$ is surjective.*
3. *$T - \lambda$ is open.*

3 Bijective weakly compact operators

As a consequence of Corollary 2.2 and Propositions 2.3 and 2.4, we are provided with examples of normed spaces X and Y such that there exists a bijective compact operator from X to Y, and both X and Y are non complete. In particular, we are provided with examples of normed spaces X and Y such that there exists a bijective weakly compact operator from X to Y, and both X and Y are non reflexive. Now, the fact that this last situation cannot happen in the case that $X = Y$ becomes one of the main results in [6]. The key tool to prove this result is Lemma 3.1 immediately below.

It is well-known and easy to realize that, if T is a linear operator on vector space X satisfying $T^2(X) = T(X)$ and $\ker(T^2) = \ker(T)$, then we have $X = \ker(T) \oplus T(X)$. On the other hand, it is also known that, if T is a bounded linear operator on a Banach space X, and if $T(X)$ is algebraically complemented in X by a closed subspace of X, then $T(X)$ is closed in X (a consequence of [9, Theorem IV.5.10]). By putting together the two facts just reviewed, we obtain the following.

Lemma 3.1. *Let X be a Banach space, and let T be a bounded linear operator on X satisfying $T^2(X) = T(X)$ and $\ker(T^2) = \ker(T)$. Then $T(X)$ is closed in X.*

When Lemma 3.1 applies with (X^{**}, T^{**}) instead of (X, T), where now X is a normed space, and T is a bijective weakly compact operator on X, we obtain the following.

Theorem 3.2. *Let X be a normed space such that there exists a bijective weakly compact operator from X to X. Then X is a reflexive Banach space.*

Let T be a linear operator on a vector space X over a field $\mathbb{F}$. The spectrum of T is defined as the subset $\sigma(T)$ of $\mathbb{F}$ given by

$$\sigma(T) := \{\lambda \in \mathbb{F} : T - \lambda \text{ is not bijective}\}.$$

As a consequence of [4, Proposition VI.1.9], if X is in fact a Banach space, and if the linear operator T is bounded, then we have $\sigma(T) = \sigma(T^*)$. Keeping in mind the fact just quoted, and invoking Proposition 2.5 and Theorem 3.2, we get in [6] the following.

Corollary 3.3. *Let X be a normed space over $\mathbb{K}$, and let T be a weakly compact operator on X. Then we have $\sigma(T) = \sigma(T^*)$. As a consequence, $\sigma(T)$ is a compact subset of $\mathbb{K}$, and is nonempty whenever $\mathbb{K} = \mathbb{C}$.*

A subalgebra B of an associative algebra A with a unit $\mathbf{1}$ is said to be full in A if $\mathbf{1}$ lies in B, and $Inv(B) = B \cap Inv(A)$, where $Inv(\cdot)$ stands for the set of all invertible elements of the algebra under consideration. Now, denote by $\mathcal{K}(X)$ (respectively, $\mathcal{W}(X)$) the algebra of all compact (respectively, weakly compact) operators on a given normed space X. Involving Proposition 2.5, Theorem 3.2, and the compact version of Corollary 2.2, we derive in [6] the following.

Corollary 3.4. *Let X be a normed space over $\mathbb{K}$. Then both $\mathcal{K}(X) + \mathbb{K}I_X$ and $\mathcal{W}(X) + \mathbb{K}I_X$ are full subalgebras of the algebra of all (possibly discontinuous) linear operators on X.*

Now, we consider the following.

Definition 3.5. *A normed algebra A is said to be a Q-algebra if it has a unit, and the set of all invertible elements of A is open in A.*

Normed Q-algebras compose a well-understood class of (possibly non-complete) normed algebras which, concerning spectral theory, behave like Banach algebras (see for example [7, Pages 208-260]). According to [7, Page 213], normed Q-algebras were introduced by I. Kaplansky [5]. Of course, every Banach algebra with a unit is a normed Q-algebra. More generally, any full subalgebra of a Banach algebra with a unit becomes a normed Q-algebra. A. Wilansky's question [10], asking if the converse of the last assertion is true, was famous for some years. Curiously enough, as pointed out in [2], this question had been answered affirmatively ten years before in [1, Theorem 2], by proving that every normed Q-algebra is a full subalgebra in its completion.

The next proposition allows us to find normed Q-algebras of bounded linear operators on a normed space.

Proposition 3.6. *Let X be a normed space, and let B be an algebra of bounded linear operators on X such that:*

1. *The identity mapping I_X lies in B.*
2. *If F belongs to B and is bijective, then F^{-1} belongs to B.*
3. *An operator $F \in B$ is bijective if and only if so is F^*.*

Then B is a normed Q-algebra.

Proof. For F in B, we have that F is invertible in B if and only if F is bijective (by the assumption 2), if and only if F^* is invertible in the Banach algebra $L(X^*)$ of all bounded linear operators on X^* (by the assumption 3 and the Banach isomorphism theorem). Now, let us consider the mapping h from B into $L(X^*)$ defined by $h(F) = F^*$. It follows that

$$Inv(B) = h^{-1}(Inv(L(X^*))).$$

Since h is continuous, and $Inv(L(X^*))$ is open in $L(X^*)$, we conclude that $Inv(B)$ is open in B. □

As a matter of fact, there are not many natural examples of non-complete normed Q-algebras. Fortunately, Proposition 3.6 above, together with some of our previous results, provides us with two of them. Indeed, we have the following.

Corollary 3.7. *Let X be a normed space over $\mathbb{K}$. Then both $\mathcal{K}(X) + \mathbb{K}I_X$ and $\mathcal{W}(X) + \mathbb{K}I_X$ are normed Q-algebras.*

Proof. Let B stand for either $\mathcal{K}(X) + \mathbb{K}I_X$ or $\mathcal{W}(X) + \mathbb{K}I_X$, so that the assumption 1 in Proposition 3.6 is fulfilled by B. On the other hand, by Corollary 3.4 (respectively, Corollary 3.3), the assumption 2 (respectively, the assumption 3) in Proposition 3.6 is also fulfilled by B. □

4 Surjective weakly compact operators

The following proposition, proved in [6], shows that Theorem 3.2 does not remain true when surjectivity replaces bijectivity.

Proposition 4.1. *Let X be a reflexive Banach space containing closed subspaces Y and Z such that Y has a Schauder basis, Z is isomorphic to X, and $X = Y \oplus Z$. Then there exists a couple (M, T), where M is a dense proper subspace of X, and T is a surjective weakly compact operator from M to M.*

We note that all requirements on the space X in the above proposition are fulfilled in the case that $X = \ell_p(I)$, where I is any infinite set, and $1 < p < \infty$. Therefore we are provided with surjective weakly compact operators on non-complete normed spaces of arbitrary density character. In what follows, we are going to realize that such operators have a rather pathological behaviour, which prohibits them to be compact. The key tool to prove this is Lemma 4.2 immediately below.

Let T be a linear operator on a vector space X. The descent $d(T)$ of T is defined by the equality

$$d(T) := \min\{n \in \mathbb{N} \cup \{0\} : T^n(X) = T^{n+1}(X)\},$$

with the convention that $\min \emptyset = \infty$. The following result is stated in [3, Proposition 1.1] for complex spaces, but its proof works verbatim in the real case.

Lemma 4.2. *Let X be a Banach space over $\mathbb{K}$, and let T be a bounded linear operator on X with finite descent $d := d(T)$. Then there exists $\delta > 0$ such that, for every $\lambda \in \mathbb{K}$ with $0 < |\lambda| < \delta$, we have:*

1. *$T - \lambda$ is surjective.*
2. *$\dim(\ker(T - \lambda)) = \dim(\ker(T) \cap T^d(X))$.*

Roughly speaking, the following theorem is proved in [6] by keeping in mind Corollary 2.2, Theorem 3.2, and Proposition 2.5, and arguing as in the proof of Theorem 3.2 with Lemma 4.2 instead of Lemma 3.1.

Theorem 4.3. *Let X be a non-reflexive normed space over $\mathbb{K}$, and let T be a surjective weakly compact operator on X. Then X is non complete, and T is non injective. Moreover, there exists $\delta > 0$ such that $T - \lambda$ is open (and hence surjective) but non injective whenever λ is in $\mathbb{K}$ with $0 < |\lambda| < \delta$.*

It is well-known that, if T is a compact operator on a normed space X over $\mathbb{K}$, and if λ is a nonzero element in $\mathbb{K}$, then $T - \lambda$ is injective if and only if it is surjective (see for example the first comment after [9, Theorem V.7.9]). Therefore, the last conclusion in Theorem 4.3 cannot be true if the surjective weakly compact operator T in that theorem is actually compact. Thus, invoking the compact version of Corollary 2.2, we derive the main result in [8], namely the following.

Corollary 4.4. *Let X be a normed space such that there exists a surjective compact operator from X to X. Then X is finite-dimensional.*

The original proof in [8] of Corollary 4.4 is much simpler than the one sketched above. Actually, as done in [6], Spurný's argument in [8] can be adapted to the case of surjective weakly compact operators, giving rise to the following.

Theorem 4.5. *Let X be a normed space, and let T be a surjective weakly compact operator from X to X. Then $X/\ker(T)$ is a reflexive Banach space.*

As pointed out in [6], Theorem 4.5 contains Theorem 3.2 in a straightforward way, and has the following consequence.

Corollary 4.6. *Let X be a normed space. Then the following assertions are equivalent:*

1. *There exists a surjective weakly compact operator from X to X.*
2. *There exists a closed subspace M of X such that X/M is reflexive, and a bijective bounded linear operator from X/M to X.*
3. *There exists a closed subspace M of X such that X/M is reflexive, and a surjective bounded linear operator from X/M to X.*

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