CLOSED DERIVATIONS OF BANACH ALGEBRAS

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0. Introduction.

The theory of closed *-derivations of C^*-algebras has been recently fully developed, with special attention to the study of the properties of the domain of such a derivation (see [4]). For general Banach algebras a similar theory is missing, although we know some precedents (see [6]). In this paper we give the first grounds for a development of a theory of closed derivations of arbitrary Banach algebras. Precisely, we prove that the domain of a densely defined closed derivation of a unital Banach algebra is a full subalgebra of the given Banach algebra and, more generally, if a is an element in the domain of such a derivation, then also $f(a)$ (in the sense of the holomorphic functional calculus) lies in the domain, where f is any complex-valued holomorphic mapping on the spectrum of a.

1. Algebraic preliminaries.

Let A be an associative (complex) algebra, and X be a two-sided A-module. By an X-valued derivation of A we mean a linear mapping D from a subalgebra $dom(D)$ of A into X satisfying

$$D(ab)=aD(b)+D(a)b$$

for all a,b in $dom(D)$. In the case $X=A$, we say simply that D is a derivation of A. We will assume that A has a unit (denoted by I) and that the A-module X is unital, that is: $Ix=xI=x$ for all x in X. When I belongs to the domain of an X-valued derivation D of A, it is clear that $D(I)=0$. On the other hand, if I is not in $dom(D)$, then it is straightforward to see that the mapping $D_1:zI+a\longrightarrow D(a)$ is an X-valued

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derivation of A whose domain (the subalgebra $\mathbb{C}I\oplus dom(D)$) contains the unit of A. So we have:

1.1. *Every X-valued derivation of A can be extended to an X-valued derivation of A whose domain contains the unit of A. This extension vanishes at the unit.*

For the study of a partially defined mapping, the graph of such a mapping becomes an interesting tool. In our situation we need previously to endow with a structure of associative algebra the product vector space $A\mathbf{x}X$. Define a product on $A\mathbf{x}X$ by

$$(a,x)(b,y):=(ab,ay+xb).$$

With this product $A\mathbf{x}X$ becomes an associative algebra called the split null X-extension of A.

The following assertions are of direct verification:

1.2. *The natural imbedding of A into the split null X-extension of A is a unit-preserving algebra homomorphism.*

1.3. *The range of X under the natural imbedding of X into the split null X-extension of A is a two-sided ideal whose square is zero.*

1.4. *An element (a,x) is invertible in the split null X-extension of A if and only if a is invertible in A. Moreover in this case we have*

$$(a,x)^{-1}=(a^{-1},-a^{-1}xa^{-1}).$$

The following properties, which are also inmediate, show that the structure of the split null X-extension of A is the appropiate one for the study of X-valued derivations of A.

1.5. *A mapping from a subset of A into X is an X-valued derivation of A if and only if its graph is a subalgebra of the split null X-extension of A.*

1.6. *If D is an X-valued derivation of A, then the mapping $a\longrightarrow(a,D(a))$ is an algebra isomorphism from $dom(D)$ onto the graph of D.*

We recall that a subalgebra B of A is called a full subalgebra of A if $I \in B$ and B contains the inverses of its elements that are invertible in A. By using 1.4 and the equality $D(a^{-1})=-a^{-1}D(a)a^{-1}$, which is true for any X-valued derivation D of A and any invertible element a with $a,a^{-1} \in dom(D)$, we obtain:

1.7. *Let D be an X-valued derivation of A. Then $dom(D)$ is a full subalgebra of A if and only if the graph of D is a full subalgebra of the split null X-extension of A.*

2. Some analytic tools.

Now let A be a unital Banach algebra, and X be a unital Banach two-sided A-module (that is: X is a complex Banach space and a unital two-sided A-module satisfying $\|ax\| \le \|a\|\|x\|$ and $\|xa\| \le \|x\|\|a\|$ for all a in A and x in X). Then the split null X-extension of A can and will be regarded as a new unital Banach algebra under the norm

$$\|(a,x)\|:=\|a\|+\|x\|.$$

We recall that the spectrum of an element b in a unital algebra B is defined as the set

$$sp(b,B):=\{z \in \mathbb{C} : b-zI \text{ is not invertible in } B\}.$$

From 1.2 and 1.4, we obtain:

2.1. *For all (a,x) in the split null X-extension of A we have*

$$sp((a,x),A \times X) = sp(a,A).$$

Recall that, if a is an element in the unital Banach algebra A, and W is an open subset of $\mathbb{C}$ containing the spectrum of a, then there exists a closed neighbourhood of $sp(a,A)$ contained in W and which has rectifiable boundary (say γ). Thus, if f is any complex valued holomorphic mapping on W, the element $f(a)$ of A is unambiguously defined by the equality

$$f(a):=\frac{1}{2\pi i}\int_{\gamma} f(z)(zI-a)^{-1}dz$$

(see [3; Section 7]). Clearly $f(a)$ lies in the closed subspace of A generated by $\{(zI-a)^{-1} : z \in \mathbb{C} \setminus sp(a,A)\}$, so, as a consequence, we have:

2.2. *$f(a)$ belongs to any closed full subalgebra of A which contains a.*

If x is any element in the Banach A-module X, then by 2.1

$$sp((a,x),A\times X) = sp(a,A) \subset W.$$

Therefore, we can build the elements $f(a)$ and $f((a,x))$ of the Banach algebras A and $A\times X$, respectively. We will prove:

2.3. $$f((a,x))=\left(f(a) \ , \ \frac{1}{2\pi i}\int_{\gamma} f(z)(zI-a)^{-1}x(zI-a)^{-1}dz \right).$$

In fact, we have:

$$f((a,x))=\frac{1}{2\pi i}\int_{\gamma} f(z)((z(I,0)-(a,x))^{-1}dz=\frac{1}{2\pi i}\int_{\gamma} f(z)(zI-a,-x)^{-1}dz.$$

By applying 1.4, we obtain:

$$f((a,x))=\frac{1}{2\pi i}\int_{\gamma} f(z)((zI-a)^{-1},(zI-a)^{-1}x(zI-a)^{-1})dz=$$
$$\frac{1}{2\pi i}\int_{\gamma} (f(z)(zI-a)^{-1},f(z)(zI-a)^{-1}x(zI-a)^{-1})dz.$$

Now the proof of 2.3 is concluded by using that the natural projections from $A\times X$ onto A and X, respectively, are continuous linear mappings, and the commutation of the integral with such mappings.

2.4. Remark. If we assume $ax=xa$, then $(zI-a)^{-1}x=x(zI-a)^{-1}$ for all z in $\mathbb{C}\backslash sp(a,A)$. Hence we can use [3; Theorem 7.11] to obtain:

$$\frac{1}{2\pi i}\int_{\gamma} f(z)(zI-a)^{-1}x(zI-a)^{-1}dz=\frac{1}{2\pi i}\left(\int_{\gamma} f(z)(zI-a)^{-2}dz\right)x=f'(a)x.$$

Thus we have the following simpler form of the equality 2.3:

$$f((a,x))=(f(a),f'(a)x) \ , \quad if \quad ax=xa.$$

2.5. Remark. Another case in which 2.3 can be formally improved arises when the A-module X agrees with the given Banach algebra A. Then

$$\frac{1}{2\pi i}\int_{\gamma} f(z)(zI-a)^{-1}x(zI-a)^{-1}dz$$

is nothing but the valuation at x of the differential at a of the mapping $\mathbf{f}:c\longrightarrow f(c)$ from $\{c\in A : sp(c,A)\subset W\}$ into A [10; Theorem 10.38]. Therefore 2.3 can be written in this case as follows:

$$f((a,b))=(f(a),\mathbf{f}'(a)(b)) \quad ((a,b)\in A\times A).$$

The following result will be used in our discussion of closed derivations of Banach algebras, but it has its own interest. If B is a unital normed algebra and b is an element in B, we will write

$$\mathbf{r}_B(b):=sup\{|z| \ : \ z\in sp(b,B)\},$$

the "algebraic" spectral radius of b relative to B.

2.6. Proposition. *Let A be a unital Banach algebra, and let B be a dense subalgebra of A containing the unit of A. Then B is a full subalgebra of A if and only if $r_A(b)=r_B(b)$ for all b in B.*

Proof. The "only if" part is clear. For the "if" part, let a be in B such that a^{-1} exists in A. Since B is dense in A, there is b in B such that $\|b-a^{-1}\|<\|a\|^{-1}$, so $\|1-ab\|<1$, and so by the completeness of A $r_A(1-ab)<1$. By the assumption r_A and r_B agree on B. Hence $r_B(1-ab)<1$, so 1 does not lie in $sp(1-ab,B)$, ab is invertible in B, and so a has a right inverse in B. Analogously a has a left inverse in B. Therefore a^{-1} belongs to B, and B is a full subalgebra of A. ■

We conclude this section with the following lemma, the proof of which is easy by induction.

2.7. Lemma. *Let A be a unital Banach algebra, X a unital Banach two-sided A-module, D an X-valued derivation of A, and n be a natural number. Then*

$$\|D(a^n)\|\le n\|a\|^{n-1}\|D(a)\|,$$

for all a in $dom(D)$.

3. The main result.

Through this section A will denote a unital Banach algebra, X a unital Banach two-sided bimodule, and D a densely defined closed X-valued derivation of A.

3.1. Lemma. *The unit of A lies in $dom(D)$.*

Proof. By the density of $dom(A)$ in A there is a in $dom(D)$ such that $\|1-a\|<1$. For a natural number n, let us write $a_n:=1-(1-a)^n$. Clearly a_n lies in $dom(D)$ and $\{a_n\}\longrightarrow 1$. Let D_1 denote the extesion of D assured by 1.1. Then $D(a_n)=-D_1((1-a)^n)$ so, by Lemma 2.7,

$$\|D(a_n)\|\le n\|1-a\|^{n-1}\|D(a)\|\longrightarrow 0.$$

Now we have $a_n\in dom(D)$, $\{a_n\}\longrightarrow 1$, and $\{D(a_n)\}\longrightarrow 0$. Therefore 1 lies in $dom(D)$ because D is closed.

3.2. Lemma. *$dom(D)$ is a full subalgebra of A.*

Proof. By the above lemma, the density of $dom(D)$ in A, and Proposition 2.6, it is enough to show that $r_A(a)=r_{dom(D)}(a)$ for all a in $dom(D)$. But $r_A(a)=r_{A\mathbf{x}X}((a,D(a))$ by 2.1, and if G denotes the graph of D we have $r_{A\mathbf{x}X}((a,D(a))=r_G((a,D(a))$ because G is a closed subalgebra of the Banach algebra $A\mathbf{x}X$. Also $r_G^{\bullet}((a,D(a))=r_{dom(D)}(a)$ by 1.6. Therefore $r_A(a)=r_{dom(D)}(a)$, as required.

3.3 Theorem. *Let a be in dom(D), and f be a complex valued holomorphic mapping on an open neighbourhood W of sp(a,A). Then f(a) lies in dom(D) and*

$$D(f(a))=\frac{1}{2\pi i}\int_{\gamma} f(z)(zI-a)^{-1}D(a)(zI-a)^{-1}dz,$$

where γ is the rectificable boundary of a suitable closed neighbourhood of sp(a,A) contained in W.

Poof. By the above lemma and 1.7, the graph of D (say G as above) is a (closed) full subalgebra of $A\mathbf{x}X$. Since $(a,D(a))$ belongs to G and $sp((a,D(a)),A\mathbf{x}X)=sp(a,A)$ (see 2.1), by 2.2 we have that $f((a,D(a)))$ lies in G. Now the proof is concluded by applying 2.3.

3.4. Remark. If in the theorem we assume additionally $aD(a)=D(a)a$, then $D(f(a))=f'(a)D(a)$ (see Remark 2.4). If in the theorem we assume $X=A$, then $D(f(a))=\mathbf{f}'(a)(D(a))$ (see Remark 2.5).

Since the algebra $dom(D)$ is isomorphic to the graph of D (see 1.6), which is a closed subalgebra of the Banach algebra $A\mathbf{x}X$, actually $dom(D)$ is a Banach algebra under the norm $\|a\|_1:=\|a\|+\|D(a)\|$. In this way, by Lemma 3.2, the pair of Banach algebras $(A,dom(D))$, with norms $\|.\|$ and $\|.\|_1$, respectively, is a Wiener pair (see definition in [7; 11.7]). But it is known that, if (R,R_1) is a Wiener pair of commutative Banach algebras with R_1 dense in R, then the mapping $M\longrightarrow M\cap R_1$ is a homeomorphism from the space of maximal ideals of R onto the space of maximal ideals of R_1 [7; 11.7.V]. Therefore we obtain:

3.5. Corollary [6]. *Assume that A is commutative. Then the mapping $M\longrightarrow M\cap dom(D)$ is a homeomorphism from the space of maximal ideals of A onto the space of maximal ideals of dom(D).*

The following corollary shows in particular that, when A splits in two direct summands, then all densely defined closed derivations of A arise in a natural way from the knowledge of densely defined closed derivations of the given direct summands.

3.6. Corollary. *Every central idempotent in A lies in $dom(D)$.*

Proof. Let e be a central idempotent in A, and let us consider the open subset W of $\mathbb{C}$ given by $W:=B(0,\frac{1}{2})\cup B(1,\frac{1}{2})$ (where $B(z,r)$ denotes the open ball in $\mathbb{C}$ with center z and radius r), and the complex valued holomorphic mapping f on W defined by $f(z)=0$ if $z\in B(0,\frac{1}{2})$ and $f(z)=1$ if $z\in B(1,\frac{1}{2})$. Then the set $A_W:=\{a\in A : sp(a,A)\subset W\}$ is open in A, e lies in A_W, and the mapping $a\longrightarrow f(a)$ from A_W into A is continuous. Therefore, by the density of $dom(D)$ in A, there exists a in $dom(D)\cap A_W$ such that $\|f(e)-f(a)\|<1$. But, clearly, $f(e)=e$ and $f(a)$ is an idempotent. Since e is a central idempotent, we have $e-f(a)=(e-f(a))^3$, hence $\|e-f(a)\|\leq\|e-f(a)\|^3$. Therefore $e=f(a)$ because $\|e-f(a)\|<1$, and e lies in $dom(D)$ by Theorem 3.3.

3.7. Concluding remark. This paper was written in spanish around 1980, and was submitted for publication to "Revista Matemática Hispano-Americana". With date June 27, 1984, the corrected galley proofs were sent to the Editors of this Journal, and, since then, no news I have received about it, and the paper remained peaceful in my file. The fact is that the Journal vanished from the mathematical firmament, without any notice to the authors of papers submitted in the concluding step of its existence. Now, with the occasion of the homage offered by the "Departamento de Análisis Matemático de la Universidad de Granada" and the "Academia de Ciencias de Granada" to the memory of Professor Pablo Bobillo, I have thought to rewrite the paper in english, and to contribute to the homage with this new version. In this way, I want to remember wrong times for the mathematics in Spain, today fortunately overcome thanks to several people, among which Pablo Bobillo must be taken into account.

Though I have lightly revised the old manuscript, I have wanted to preserve its original spirit. Therefore some comments, about how the passage of time has affected its content, seem appropiate. With some standard results in [12], Proposition 2.6 becomes an affirmative