CHARACTERIZATIONS OF ALMOST TRANSITIVE SUPERREFLEXIVE BANACH SPACES

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1.- Introduction

Throughout this paper X will denote a Banach space over the field $\mathbb{K}$ of real or complex numbers, $S = S(X)$ and $B = B(X)$ will be the unit sphere and the closed unit ball of X, respectively, and $\mathcal{G} = \mathcal{G}(X)$ will stand for the group of all surjective linear isometries on X. The Banach space X is said to be *almost transitive* whenever, for every (equivalently, some) element u in S, $\mathcal{G}(u)$ is dense in S. We denote by $\mathcal{J}$ the class of almost transitive superreflexive Banach spaces. This class has been first considered by C. Finet [7] (see also [6; Corollary IV.5.7]) and, very recently, has been revisited by F. Cabello [4] and the authors [2].

According to [7], every member of $\mathcal{J}$ is uniformly smooth and uniformly convex. By his part, F. Cabello shows that, for an almost transitive Banach space, superreflexivity is equivalent to reflexivity (and even to either enjoy the Radon-Nikodym property or be Asplund). He also proves, that, for a superreflexive Banach space, the notion of almost transitivity is equivalent to that (in general weaker) of convex transitivity. We recall that the Banach space X is said to be *convex transitive* if, for every u in S, we have $\overline{co}\mathcal{G}(u) = B$, where $\overline{co}$ means closed convex hull. In [2], we show that members of $\mathcal{J}$ can be characterized as those convex transitive Banach spaces which either have the Radon-Nikodym property or are Asplund.

Actually, the result just reviewed follows from a more general fact involving the concept of a rough space. For u in S, we put

$$\eta(X,u) \ := \ \limsup_{\|h\|\to 0} \frac{\|u+h\| + \|u-h\| - 2}{\|h\|} \ .$$

Given $\epsilon > 0$, the Banach space X is said to be *ϵ-rough* if, for every u in S, we have $\eta(X,u) \geq \epsilon$. We say that X is *rough* whenever it is ϵ-rough for some $\epsilon > 0$, and *extremely rough* whenever it is 2-rough. Since, for u in S, the Fréchet differentiability of the norm of X at u can be characterized by the equality $\eta(X,u) = 0$ [6; Lemma I.1.3], it follows that the roughness of X can be seen as a uniform non Fréchet-differentiability of the norm, and hence becomes the extremely opposite situation to that of the uniform smoothness. We proved in [2] that a Banach space X is a member of $\mathcal{J}$ if (and only if) it is convex transitive and either X or X^* is non rough.

As main result, we show in the present paper that the Banach space X is a member of $\mathcal{J}$ if (and only if) it is convex transitive and either X or X^* is not extremely rough. Through a technical lemma, namely Lemma 1, the main tool in the proof is a theorem, essentially due to R. C. James, establishing that uniformly non-square Banach spaces are superreflexive [5; Theorem VII.4.4]. We also find another remarkable characterization of members of $\mathcal{J}$ involving the notion of a big point. Let us say that an element u of X is a *big point* of X if u belongs to S and $\overline{co}\mathcal{G}(u) = B$ (so that X is convex transitive precisely when all elements in S are big points of X). We prove that X lies in $\mathcal{J}$ if (and only if) there exists a big point u in X such that the norm of X is Fréchet differentiable at u.

2.- The results

The Banach space X is said to be *uniformly non-square* if there exists $0 < a < 1$ such that $\|x-y\| < 2a$ whenever x, y are in B with $\|x+y\| \geq 2a$.

LEMMA 1.- *Assume that there exists a big point u in X such that $\eta(X,u) < 2$. Then X is uniformly non-square.*

Proof.- Let us fix ϵ satisfying $\eta(X,u) < \epsilon < 2$. Then there is $0 < \delta < 1$ such that

$$\frac{\|u+h\| + \|u-h\| - 2}{\|h\|} \ \leq \ \epsilon$$

whenever h is in $X\backslash\{0\}$ and $\|h\| \leq \delta$. Now

$$\{v \in X : \frac{\|v+h\| + \|v-h\| - 2}{\|h\|} \ \leq \ \epsilon \text{ whenever } h \text{ is in } X\backslash\{0\} \text{ with } \|h\| \leq \delta\}$$

is a closed, convex, and $\mathcal{G}$-invariant subset of X containing u. It follows from the bigness of u that

$\frac{\|v+h\|+\|v-h\|-2}{\|h\|} \ \leq \ \epsilon$ whenever v is in B and h is in $X\backslash\{0\}$ with $\|h\| \leq \delta$. (*)

Take σ with $\epsilon < \sigma < 2$, and a with

$$\frac{1}{2}\max\{2-(\sigma-\epsilon)\delta, 2-(2-\sigma)\delta\} \ < \ a \ < \ 1 \ .$$

Let x, y be in B such that $\|x+y\| \geq 2a$. Then we have

$$\|x + y\| \ \geq \ 2 - (\sigma - \epsilon)\delta \ ,$$

and hence

$$\|x + \delta y\| \ \geq \ 2 - (\sigma - \epsilon)\delta - (1 - \delta) \ .$$

Since, on the other hand, the equality

$$\|x - \delta y\| \ \geq \ \|x - y\| - (1 - \delta)$$

holds, we obtain

$$\|x + \delta y\| \ + \ \|x - \delta y\| \ \geq \ \|x - y\| \ + \ (2 - \sigma + \epsilon)\delta \ .$$

It follows from (*) that

$$\|x - y\| \ + \ (2 - \sigma + \epsilon)\delta \ \leq \ 2 + \epsilon\delta \ ,$$

and therefore

$$\|x - y\| \ \leq \ 2 - (2 - \sigma)\delta \ < \ 2a \ . \blacksquare$$

We say that an element f of X^* is a w^*-big point of X if f belongs to $S(X^*)$ and the convex hull of $\mathcal{G}(X^*)(f)$ is w^*-dense in $B(X^*)$. By keeping in mind that the norm of X^* is lower w^*-semicontinuous, the proof of Lemma 2 below is similar to that of Lemma 1.

LEMMA 2.- *Assume that there exists a w^*-big point f in X^* such that $\eta(X^*, f) < 2$. Then X^* is uniformly non-square.*

Let u be in S. For x in X, the number $\lim_{\alpha \to 0^+} \frac{\|u+\alpha x\|-1}{\alpha}$ (which always exists because the mapping $\alpha \to \|u + \alpha x\|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex) is usually denoted by $\tau(u, x)$. We say that the norm of X is strongly subdifferentiable at u if

$$\lim_{\alpha \to 0^+} \frac{\|u+\alpha x\|-1}{\alpha} \ = \ \tau(u, x) \text{ uniformly for } x \text{ in } B.$$

The reader is referred to [1] and [8] for a comprehensive view of the usefulness of the strong subdifferentiability of the norm in the theory of Banach spaces.

For u in S, we put

$$D(X, u) := \{g \in X^* \ : \ \|g\| = g(u) = 1\} \ .$$

LEMMA 3.- *Let u be a big point of X such that the norm of X is strongly subdifferentiable at u. Then the set*

$$\{T^*(f)\ :\ f \in D(X,u),\ T \in \mathcal{G}\}$$

is norm-dense in $S(X^*)$.

Proof.- Let ϵ be a positive number. Since the norm of X is strongly subdifferentiable at u, we can apply [8; Theorem 1.2.$(iv) \Rightarrow (i)$] to find $0 < \delta$ such that $d(g, D(X,u)) < \epsilon$ whenever g belongs to $B(X^*)$ and $|g(u)-1| < \delta$. Now, let h be in $S(X^*)$. Since u is a big point of X, there exits T in $\mathcal{G}$ satisfying $|h(T(u)) - 1| < \delta$. Now $T^*(h)$ lies in $B(X^*)$ and satisfies $|T^*(h)(u)-1| < \delta$, and hence there is f in $D(X,u)$ such that $\|T^*(h)-f\| < \epsilon$. For such an f, we have $\|h - T^{*-1}(f)\| < \epsilon$. ■

The dual X^* of the Banach space X is said to be convex w^*-transitive if every element of $S(X^*)$ is a w^*-big point of X^*. An easy and well-known consequence of the Hahn-Banach theorem is that convex transitivity of X implies convex w^*-transitivity of X^*. Recall that the symbol $\mathcal{J}$ stands for the class of almost transitive superreflexive Banach spaces.

THEOREM 1.- *The following assertions are equivalent:*

1. *X is a member of $\mathcal{J}$.*
2. *There exists a big point u in X such that the norm of X is Fréchet differentiable at u.*
3. *There exists a w^*-big point f in X^* such that the norm of X^* is Fréchet differentiable at f.*
4. *X is convex transitive and the norm of X is not extremely rough.*
5. *X^* is convex w^*-transitive and the norm of X^* is not extremely rough.*

Proof.- Certainly the implications $1 \Rightarrow 4$ and $1 \Rightarrow 5$ are true.

$2 \Rightarrow 1$.- Since the norm of X is Fréchet differentiable at u, we have $\eta(X,u) = 0 < 2$, so that, since u is a big point of X, we can apply Lemma 1 and the already quoted James' theorem [5; Theorem VII.4.4] to obtain that X is superreflexive. On the other hand, the Fréchet differentiability of the norm of X at u implies that the norm of X is strongly subdifferentiable at u and that $D(X,u)$ reduces to a singleton, so that, by Lemma 3, X^* is almost transitive. Now, surely, there exits in the unit sphere of the reflexive Banach space X^* some point g such that the norm of X^* is Fréchet differentiable at g, and such a point is a big point of X^* (because X^* is almost transitive). Repeating the argument with (X^*, g) instead of (X, u), we obtain that X is almost transitive.

$3 \Rightarrow 1$.- With X^* instead of X, and Lemma 2 instead of Lemma 1, we can argue as in the proof of $2 \Rightarrow 1$ above to obtain that X^* (and hence also X) is superreflexive, and that X is almost transitive.

$4 \Rightarrow 2$.- Since the norm of X is not extremely rough, there exists v in S such that $\eta(X, v) < 2$. Since X is convex transitive, such an v is a big point of X. By Lemma 1, X is reflexive, so that there is some u in S such that the norm of X is Fréchet differentiable at u. Applying again that X is convex transitive, we obtain that u is a big point of X.

$5 \Rightarrow 3$.- With X^* instead of X and Lemma 2 instead of Lemma 1, the proof is similar to that of $4 \Rightarrow 2$ above. ∎

It follows from Theorem 1 (or even from its forerunner [2; Theorem 3.2]) that anyone of the following two assertions is sufficient (and of course necessary) to convert the Banach space X into a member of $\mathcal{J}$:

i) X is convex transitive and either has the Radon-Nikodym property or is Asplund.

ii) X^* is convex w^*-transitive and either X has the Radon-Nikodym property or X is Asplund.

Now, recall that a subset R of a topological space T is said to be nowhere dense in T if the interior of the closure of R in T is empty. Actually, Theorem 3.2 in [2] contains enough information to derive other characterizations of members X of $\mathcal{J}$, like the following:

iii) There exists a non nowhere dense subset of S consisting of big points of X, and X has the Radon-Nikodym property.

iv) There exists a non nowhere dense subset of $S(X^*)$ consisting of w^*-big points of X^*, and X is Asplund.

Now, we can complete the situation by proving the next corollary.

COROLLARY 1.- *The following assertions are equivalent:*

1. *X lies in $\mathcal{J}$.*
2. *There exists a non nowhere dense subset of S consisting of big points of X, and X is Asplund.*
3. *There exists a non nowhere dense subset of $S(X^*)$ consisting of w^*-big points of X^*, and X has the Radon-Nikodym property.*

Proof.- By the Hahn-Banach theorem, an element u in S is a big point of X if and only if, for every g in $S(X^*)$, we have

$$\sup\{|g(T(u))| \ : \ T \in \mathcal{G}\} = 1 \ .$$

Analogously, an element f in $S(X^*)$ is a w^*-big point of X^* if and only if, for every x in S, we have

$$\sup\{|F(f)(x)| \ : \ F \in \mathcal{G}(X^*)\} = 1 \ .$$

Therefore the set of all big points of X is closed in S, and the set of all w^*-big points of X^* is norm-closed in $S(X^*)$. Assume that Assertion 2 holds. Then, by the first requirement, there is a non-empty open subset of S consisting of big points of X. By the second requirement, there must exist a point u in such an open set such that the norm of X is Fréchet differentiable at u. By the implication $2 \Rightarrow 1$ in Theorem 1, X is a member of $\mathcal{J}$. Now, assume that Assertion 3 holds. Then there is a non-empty open subset (say A) of $S(X^*)$ consisting of w^*-big points of X^*. Since X has the Radon-Nikodym property, we can apply [3; Theorem 5.7.4] to find some f in A such that the norm of X^* is Fréchet differentiable at f. Then X lies in $\mathcal{J}$ by $3 \Rightarrow 1$ in Theorem 1. ∎

Given $1 \leq p \leq \infty$, a subspace M of the Banach space X is said to be an *L^p-summand* of X if there is a linear projection π from X onto M such that, for every x in X, we have

$$\|x\|^p = \|\pi(x)\|^p + \|x - \pi(x)\|^p \quad (1 \leq p < \infty) \ ,$$
$$\|x\| = \max\{\|\pi(x)\|, \|x - \pi(x)\|\} \quad (p = \infty) \ .$$

If M is an L^p-summand of X, then the projection π above is uniquely determined by M, and is called the *L^p-projection* from X onto M.

COROLLARY 2.- *Assume that there exists a big point u in X such that $\mathbb{K}u$ is an L^p-summand of X for some $1 < p \leq \infty$. Then X is a Hilbert space. If in addition $p \neq 2$, then X is one-dimensional.*

Proof.- First of all, note that a Hilbert space of dimension ≥ 2 cannot have one-dimensional L^p-summands for $p \neq 2$, so that it is enough to show that X is a Hilbert space. Since $1 < p \leq \infty$, and $\mathbb{K}u$ is an L^p-summand of X, the norm of X is Fréchet differentiable at u. It follows from the bigness of u and the implication $2 \Rightarrow 1$ in Theorem 1 that X is almost transitive. Assume that $\mathbb{K} = \mathbb{C}$. Then, since L^p-projections on complex Banach spaces are hermitian operators, the result follows from [10; Theorem 6.4]. Now assume that $\mathbb{K} = \mathbb{R}$. Then, denoting by π the L^p-projection from X onto $\mathbb{K}u$, $\mathbf{1}-2\pi$ becomes an isometric reflexion on X. It follows from [11; Theorem 2.a)] that X is a Hilbert space. ∎

Corollary 2 above does not remain true for $p = 1$. Indeed, for X equal to either ℓ_1 or ℓ_1^n ($n \in \mathbb{N}$), every element u in the natural basis of X is a big point of X such that $\mathbb{K}u$ is an L^1-summand of X. In any case, if X is convex transitive and has a one-dimensional L^1-summand, then X is one-dimensional [2; Corollary 3.5].

We conclude this paper with two remarks related to the matter we have developed.

REMARK 1.- Concerning Lemma 1, it is worth mentioning that, if the Banach space X is uniformly non-square, then we have $\eta(X, u) < 2$ for every u in S. To verify this assertion, assume that there exists some u in S satisfying $\eta(X, u) = 2$. By the proof of [6; Proposition I.1.11], for every n in $\mathbb{N}$, there are f_n, g_n in $B(X^*)$ satisfying $Re(f_n(u)) > 1 - \frac{1}{n}$, $Re(g_n(u)) > 1 - \frac{1}{n}$, and $\|f_n - g_n\| \geq 2 - \frac{1}{n}$. Now, assume additionally that X is uniformly non-square. Then so is X^* [5; p. 173], and hence there is $0 < a < 1$ such that $\|f + g\| < 2a$ whenever f, g are in $B(X^*)$ with $\|f - g\| \geq 2a$. Taking n big enough to have $\|f_n - g_n\| \geq 2a$, $Re(f_n(u)) > a$, and $Re(g_n(u)) > a$, it follows

$$2a \ < \ Re(f_n(u) + g_n(u)) \ \leq \ \|f_n + g_n\| \ < \ 2a \ ,$$

a contradiction. ∎

REMARK 2.- We say that the Banach space X has the *Mazur's intersection property* whenever every bounded closed convex subset of X can be represented as an intersection of closed balls in X. Analogously, we say that X^* has the *Mazur's w^*-intersection property* whenever every bounded w^*-closed convex subset of X^* can be expressed as an intersection of closed balls in X^*. We proved in [2; Theorem 3.4] that X lies in $\mathcal{J}$ if and only if there exists a big point in X, and the set of all denting point of B is dense in S. Applying [9; Theorem 3.1], we have:

i) X is a member of $\mathcal{J}$ if and only if X^* has the Mazur's w^*-intersection property and there is a big point in X.

We also proved in [2; Theorem 3.4] that X lies in $\mathcal{J}$ if and only if there exists a w^*-big point in X^*, and the set of all w^*-denting points of $B(X^*)$ is norm-dense in $S(X^*)$. With [9; Theorem 2.1] in the mind, this result reads as follows:

ii) X is a member of $\mathcal{J}$ if and only if X has the Mazur's intersection property and there is a w^*-big point in X^*.

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