C^*- and JB^*-algebras generated by a non self-adjoint idempotent

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Abstract. Let A be a C^*-algebra generated by a non self-adjoint idempotent e, and put $K := sp(\sqrt{e^*e}) \setminus \{0\}$. It is known that K is a compact subset of $[1, \infty[$ whose maximum element is greater than 1, and that, in general, no more can be said about K. We prove that, if 1 does not belong to K, then A is $*$-isomorphic to the C^*-algebra $C(K, M_2(\mathbb{C}))$ of all continuous functions from K to the C^*-algebra $M_2(\mathbb{C})$ (of all 2×2 complex matrices), and that, if 1 belongs to K, then A is $*$-isomorphic to a distinguished proper C^*-subalgebra of $C(K, M_2(\mathbb{C}))$. By replacing C^*-algebra with JB^*-algebra, $sp(\sqrt{e^*e}) \setminus \{0\}$ with the triple spectrum $\sigma(e)$ of e, and $M_2(\mathbb{C})$ with the three-dimensional spin factor C_3, similar results are obtained.

1. Introduction

Let A be a C^*-algebra generated by a non self-adjoint idempotent e, and put $K := sp(\sqrt{e^*e}) \setminus \{0\}$, where $sp(\cdot)$ means spectrum. We proved in [**1**] that K is a compact subset of $[1, \infty[$ whose maximum element is greater than 1, and that, in general, no more can be said about K. Moreover we got an "almost description" of A (collected in Proposition 2.4 of the present paper) in terms of a Banach $*$-algebra $\mathcal{A}(K)$, which consists of all 2×2 matrices over $C(K)$ with an unusual but natural multiplication. In the present paper we obtain a complete description of A. We prove that, if 1 does not belong to K, then A is $*$-isomorphic to the C^*-algebra $C(K, M_2(\mathbb{C}))$ of all continuous functions from K to the C^*-algebra $M_2(\mathbb{C})$ of all 2×2 complex matrices (Theorem 2.8). To study the case that 1 belongs to K, we need to introduce a distinguished proper C^*-subalgebra of $C(K, M_2(\mathbb{C}))$, namely the one (denoted by $C_p(K, M_2(\mathbb{C}))$) consisting of all elements $\alpha \in C(K, M_2(\mathbb{C}))$ such that $\alpha(1)$ belongs to $\mathbb{C}p$, for a given self-adjoint idempotent $p \in M_2(\mathbb{C})$ different from 0 and 1. It is easy to see that $C_p(K, M_2(\mathbb{C}))$ does not depend

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structurally on p. We prove that, if 1 belongs to K, then A is $*$-isomorphic to $C_p(K, M_2(\mathbb{C}))$, for p as above (Theorem 3.3).

Among the consequences of the results reviewed in the preceding paragraph, we emphasize the one asserting that a C^*-algebra contains a non central self-adjoint idempotent if and only if it contains a copy of either $M_2(\mathbb{C})$ or $C_p([1,2], M_2(\mathbb{C}))$ for any self-adjoint idempotent $p \in M_2(\mathbb{C})$ different from 0 and 1 (Corollary 4.3 and Remark 4.4). It is also worth mentioning the fact that, if a C^*-algebra A contains a non central idempotent e, then there exists a continuous mapping $r \to e_r$ from $[1, \infty[$ to the set of idempotents of A satisfying $e_{\|e\|} = e$ and $\|e_r\| = r$ for every $r \in [1, \infty[$ (Proposition 4.5).

The concluding sections of the paper (Sections 5 and 6) are devoted to prove the appropriate variants, for JB^*-algebras, of the results previously obtained for C^*-algebras. We show that, by replacing C^*-algebra with JB^*-algebra, $sp(\sqrt{e^*e}) \setminus \{0\}$ with the triple spectrum $\sigma(e)$ of e (for a given idempotent e), and $M_2(\mathbb{C})$ with the three-dimensional (complex) spin factor C_3, all results reviewed above remain true. As a consequence, a JB-algebra contains a non central idempotent if and only if it contains a copy of either $\mathcal{S}_3$ or $C_p([1,2], \mathcal{S}_3)$ for any idempotent $p \in \mathcal{S}_3$ different from 0 and 1, where $\mathcal{S}_3$ stands for the three-dimensional real spin factor (Corollary 6.8).

Turning out to the world of C^*-algebras, let us review the fact, proved in Corollary 4.7, that a C^*-algebra contains a non central self-adjoint idempotent if and only if it contains a non normal partial isometry. In the case of JB^*-algebras, we have been able to prove the "only if" part of the appropriate variant of the fact just reviewed (see Corollary 6.6), but have been unable to prove or disprove the "if" part. We note that, if such an "if" part were proved, then, in particular, we would be provided with an affirmative answer to the unsolved question whether every JB^*-algebra containing a nonzero tripotent must contain a nonzero self-adjoint idempotent.

2. The case of C^*-algebras: the first theorem

Let A be an associative complex algebra. In the case that A has not a unit, we denote by A_1 the algebra obtained by adjoining a unit to A. Otherwise, we put $A_1 := A$. As usual, for $a \in A$, we define the spectrum of a (relative to A) as the subset $sp(A, a)$ of $\mathbb{C}$ given by

$$sp(A, a) := \{\lambda \in \mathbb{C} : a - \lambda \ \text{ is not invertible in } A_1\},$$

and we recall that, if A is in fact a Banach algebra, then $sp(A, a)$ is a nonempty compact subset of $\mathbb{C}$.

From now on, $M_2(\mathbb{C})$ will denote the C^*-algebra of all 2×2 matrices with entries in $\mathbb{C}$.

Lemma 2.1. *Let e be an idempotent in $M_2(\mathbb{C})$ different from 0 and 1, and put $e_{11} := \|e\|^{-2}e^*e$, $e_{12} := \|e\|^{-1}e^*$, $e_{21} := \|e\|^{-1}e$, and $e_{22} := \|e\|^{-2}ee^*$. Then, for $i, j, k, l \in \{1, 2\}$, we have $e_{ij}^* = e_{ji}$, $e_{ij}e_{kl} = e_{il}$ if $j = k$, and $e_{ij}e_{kl} = \|e\|^{-1}e_{il}$ if $j \neq k$.*

Proof. The equality $e_{ij}^* = e_{ji}$ is clear. On the other hand, we have $sp(M_2(\mathbb{C}), e^*e) = \{0, \|e\|^2\}$, and hence $(e^*e - \|e\|^2)e^*e = 0$, which reads as $e_{11}^2 = e_{11}$. Analogously, $e_{22}^2 = e_{22}$. Now we have

$$\begin{aligned}(ee^*e - \|e\|^2 e)^*(ee^*e - \|e\|^2 e) &= (e^*ee^* - \|e\|^2 e^*)(ee^*e - \|e\|^2 e) \\ &= (e^*e)^3 - 2\|e\|^2(e^*e)^2 + \|e\|^4 e^*e = 0,\end{aligned}$$

and hence $ee^*e - \|e\|^2 e = 0$, which reads as both $e_{21}e_{11} = e_{21}$ and $e_{22}e_{21} = e_{21}$. By taking adjoints, we deduce $e_{11}e_{12} = e_{12}$ and $e_{12}e_{22} = e_{12}$. The remaining assertions in the lemma are either obvious or easily deducible from the above computations. ∎

The mapping $\eta : [1, \infty[\to M_2(\mathbb{C})$, which is introduced in Lemma 2.2 immediately below, will play a crucial role through the paper.

Lemma 2.2. *Let t be in $[1, \infty[$, and let $\eta(t)$ denote the element of $M_2(\mathbb{C})$ defined by*

$$\eta(t) := \frac{1}{2}\begin{pmatrix} 1 & t + \sqrt{t^2 - 1} \\ t - \sqrt{t^2 - 1} & 1 \end{pmatrix}.$$

Then $\eta(t)$ is an idempotent satisfying $\|\eta(t)\| = t$. As a consequence, putting $\eta_{11}(t) := t^{-2}\eta(t)^*\eta(t)$, $\eta_{12}(t) := t^{-1}\eta(t)^*$, $\eta_{21}(t) := t^{-1}\eta(t)$, and $\eta_{22}(t) := t^{-2}\eta(t)\eta(t)^*$, we have $\eta_{ij}(t)^* = \eta_{ji}(t)$, $\eta_{ij}(t)\eta_{kl}(t) = \eta_{il}(t)$ if $j = k$, and $\eta_{ij}(t)\eta_{kl}(t) = t^{-1}\eta_{il}(t)$ if $j \neq k$.

Proof. That $\eta(t)$ is an idempotent in $M_2(\mathbb{C})$ is straightforward. Moreover, computing its norm accordingly to the formula in the introduction of [**4**], we have $\|\eta(t)\| = t$. The consequence follows from Lemma 2.1. ∎

Let K be a subset of $[1, \infty[$. We denote by η_K the restriction to K of the continuous mapping $t \to \eta(t)$ from $[1, \infty[$ to $M_2(\mathbb{C})$, given by Lemma 2.2. Moreover, for $i, j \in \{1, 2\}$, we denote by η_{ij}^K the restriction to K of the continuous mapping $t \to \eta_{ij}(t)$ from $[1, \infty[$ to $M_2(\mathbb{C})$, given by that lemma.

Now, let K be a compact subset of $[1, \infty[$. Let u stand for the element of $C(K)$ defined by $u(t) := t$ for every $t \in K$. We denote by $\mathcal{A}(K)$ the complex Banach $*$-algebra whose vector space is that of all 2×2 matrices with entries in $C(K)$, whose (bilinear) product is determined by the equalities $(f[ij])(g[kl]) := (fg)[il]$ if $j = k$ and $(f[ij])(g[kl]) := (u^{-1}fg)[il]$ if $j \neq k$, whose norm is given by $\|(f_{ij})\| := \|f_{11}\| + \|f_{12}\| + \|f_{21}\| + \|f_{22}\|$, and whose (conjugate-linear) involution $*$ is determined by $(f[ij])^* := \overline{f}[ji]$. Here, as usual, for $f \in C(K)$ and $i, j \in \{1, 2\}$, $f[ij]$ means the matrix having f in the (i, j)-position and 0's elsewhere. It is useful to see $\mathcal{A}(K)$ as a $C(K)$-module in the natural manner, namely by defining the product of a function $f \in C(K)$ and a matrix $(f_{ij}) \in \mathcal{A}(K)$ by $f(f_{ij}) := (ff_{ij})$. In this regarding, we straightforwardly realize that $\mathcal{A}(K)$ becomes in fact an algebra over $C(K)$, i.e., the operators of left and right multiplication by arbitrary elements of $\mathcal{A}(K)$ are $C(K)$-module homomorphisms. Moreover, the symbol $f[ij]$ can now be read as the product of the function $f \in C(K)$ and the matrix $[ij] \in \mathcal{A}(K)$, where, for $i, j \in \{1, 2\}$, $[ij]$ stands for the matrix having the constant function equal to one in the (i, j)-position and 0's elsewhere.

For K as a above, we denote by $C(K, M_2(\mathbb{C}))$ the C^*-algebra of all continuous functions from K to $M_2(\mathbb{C})$. We will see $C(K, M_2(\mathbb{C}))$ as a $C(K)$-module in the natural manner. From now on, u will always stand for the element of $C(K)$ defined by $u(t) := t$ for every $t \in K$

Proposition 2.3. *Let K be a compact subset of $[1, \infty[$ whose maximum element is greater than 1. Then η_K is a non self-adjoint idempotent in $C(K, M_2(\mathbb{C}))$ satisfying $sp(C(K, M_2(\mathbb{C})), \sqrt{\eta_K^* \eta_K}) \setminus \{0\}) = K$, and the mapping $\mathcal{F}$ from $\mathcal{A}(K)$ to $C(K, M_2(\mathbb{C}))$, defined by*

$$\mathcal{F}((f_{ij})) := \sum_{i,j \in \{1,2\}} f_{ij} \eta_{ij}^K,$$

becomes a continuous $*$-homomorphism satisfying $\mathcal{F}(u[21]) = \eta_K$.

Proof. By the first part of Lemma 2.2, for $t \in K$, $\eta(t)$ is an idempotent in $M_2(\mathbb{C})$ satisfying $\|\eta(t)\| = t$, which implies

$$sp(M_2(\mathbb{C}), \sqrt{\eta(t)^* \eta(t)}) \setminus \{0\} = \{t\}.$$

It follows that η_K is a non self-adjoint idempotent of $C(K, M_2(\mathbb{C}))$ satisfying $sp(C(K, M_2(\mathbb{C})), \sqrt{\eta_K^* \eta_K}) \setminus \{0\} = K$. On the other hand, the mapping

$$\mathcal{F} : \mathcal{A}(K) \to C(K, M_2(\mathbb{C}))$$

is a $*$-homomorphism if (and only if), for every $t \in K$, the composition of $\mathcal{F}$ with the valuation at t is a $*$-homomorphism from $\mathcal{A}(K)$ to $M_2(\mathbb{C})$. But this last fact follows from the definition of the operations on $\mathcal{A}(K)$, and the second part of Lemma 2.2. Finally, both the continuity of $\mathcal{F}$ (it is in fact contractive) and that $\mathcal{F}(u[21]) = \eta_K$ become obvious facts. ∎

Now, we invoke one of the main results in [**1**], namely the following.

Proposition 2.4. *Let A be a C^*-algebra, and let e be a non self-adjoint idempotent in A. Then $K := sp(A, \sqrt{e^* e}) \setminus \{0\}$ is a compact subset of $[1, \infty[$ whose maximum element (namely $\|e\|$) is grater than 1, and there exists a unique continuous $*$-homomorphism $F : \mathcal{A}(K) \to A$ such that $F(u[21]) = e$. Moreover we have:*

1. *The closure in A of the range of F coincides with the C^*-subalgebra of A generated by e.*
2. *F is injective if and only if either 1 does not belong to K or 1 is an accumulation point of K.*
3. *If 1 is an isolated point of K, then $\ker(F)$ consists precisely of those matrices $(f_{ij}) \in \mathcal{A}(K)$ which vanish at every $t \in K \setminus \{1\}$ and satisfy*

$$f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1) = 0.$$

As an immediate consequence of Propositions 2.3 and 2.4, we obtain the following.

COROLLARY 2.5. *Let K be a compact subset of $[1,\infty[$ whose maximum element is greater than 1, and let $\mathcal{F} : \mathcal{A}(K) \to C(K, M_2(\mathbb{C}))$ be the $*$-homomorphism given by Proposition 2.3. Then we have:*

(1) *The closure in $C(K, M_2(\mathbb{C}))$ of the range of $\mathcal{F}$ coincides with the C^*-subalgebra of $C(K, M_2(\mathbb{C}))$ generated by η_K.*
(2) *$\mathcal{F}$ is injective if and only if either 1 does not belong to K or 1 is an accumulation point of K.*
(3) *If 1 is an isolated point of K, then $\ker(\mathcal{F})$ consists precisely of those matrices $(f_{ij}) \in \mathcal{A}(K)$ which vanish at every $t \in K \setminus \{1\}$ and satisfy*

$$f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1) = 0.$$

LEMMA 2.6. *Let X be a complex normed space, let Ω be a Hausdorff compact topological space, and let f be a function from Ω to $\mathbb{C}$ such that there are continuous mappings $\alpha, \beta : \Omega \to X$ satisfying $\beta(t) \neq 0$ and $\alpha(t) = f(t)\beta(t)$ for every $t \in \Omega$. Then f is continuous.*

PROOF. Put $M := \max\{\|\alpha(t)\| : t \in \Omega\}$ and $m := \min\{\|\beta(t)\| : t \in \Omega\}$. Then we have $m > 0$, and hence $|f(t)| \leq m^{-1}M$ for every $t \in \Omega$, so that f is bounded. Let t be in Ω, and let $\{t_\lambda\}$ be a net in Ω converging to t. Take a cluster point z of the net $\{f(t_\lambda)\}$ in $\mathbb{C}$. Then $(z, \alpha(t))$ is a cluster point of the net $\{(f(t_\lambda), \alpha(t_\lambda))\}$ in $\mathbb{C} \times X$, and therefore we have $\alpha(t) = z\beta(t)$, which implies (since $\beta(t) \neq 0$) $z = f(t)$. In this way we have shown that $f(t)$ is the unique cluster point of $\{f(t_\lambda)\}$ in $\mathbb{C}$. Since $\{f(t_\lambda)\}$ is bounded, we deduce that $\{f(t_\lambda)\}$ converges to $f(t)$. ■

LEMMA 2.7. *Let K be a compact subset of $]1,\infty[$. Then the $*$-homomorphism $\mathcal{F} : \mathcal{A}(K) \to C(K, M_2(\mathbb{C}))$, given by Proposition 2.3, is surjective. As a consequence, $C(K, M_2(\mathbb{C}))$ is generated by η_K as a C^*-algebra.*

PROOF. Let us fix $t \in K$. By Lemma 2.2, the linear hull of

$$\{\eta_{ij}(t) : i,j \in \{1,2\}\}$$

is a $*$-invariant subalgebra of $M_2(\mathbb{C})$. Moreover, since $t \in]1,\infty[$, such a subalgebra is not commutative (indeed, $\eta_{12}(t)$ does not commute with $\eta_{21}(t)$). If follows that such a subalgebra is the whole algebra $M_2(\mathbb{C})$, and, consequently, that $\{\eta_{ij}(t) : i,j \in \{1,2\}\}$ becomes a basis of $M_2(\mathbb{C})$.

Let α be in $C(K, M_2(\mathbb{C}))$. It follows from the above that, for each $t \in K$, there are complex numbers $f_{11}(t), f_{12}(t), f_{21}(t), f_{22}(t)$ uniquely determined by the condition

$$\alpha(t) = f_{11}(t)\eta_{11}(t) + f_{12}(t)\eta_{12}(t) + f_{21}(t)\eta_{21}(t) + f_{22}(t)\eta_{22}(t). \tag{2.1}$$

Moreover, applying again Lemma 2.2, for every $t \in K$ we have:

$$\eta_{11}(t)\alpha(t)\eta_{11}(t) = (f_{11}(t) + t^{-1}f_{12}(t) + t^{-1}f_{21}(t) + t^{-2}f_{22}(t))\eta_{11}(t),$$
$$\eta_{12}(t)\alpha(t)\eta_{12}(t) = (t^{-1}f_{11}(t) + t^{-2}f_{12}(t) + f_{21}(t) + t^{-1}f_{22}(t))\eta_{12}(t),$$
$$\eta_{21}(t)\alpha(t)\eta_{21}(t) = (t^{-1}f_{11}(t) + f_{12}(t) + t^{-2}f_{21}(t) + t^{-1}f_{22}(t))\eta_{21}(t),$$
$$\eta_{22}(t)\alpha(t)\eta_{22}(t) = (t^{-2}f_{11}(t) + t^{-1}f_{12}(t) + t^{-1}f_{21}(t) + f_{22}(t))\eta_{22}(t).$$

Since, for $i,j \in \{1,2\}$, $\eta_{ij}^K \alpha \eta_{ij}^K$ and η_{ij}^K are continuous functions on K, and $\eta_{ij}(t) \neq 0$ for every $t \in K$, it follows from Lemma 2.6 that the mappings

$$t \to f_{11}(t) + t^{-1}f_{12}(t) + t^{-1}f_{21}(t) + t^{-2}f_{22}(t),$$
$$t \to t^{-1}f_{11}(t) + t^{-2}f_{12}(t) + f_{21}(t) + t^{-1}f_{22}(t),$$
$$t \to t^{-1}f_{11}(t) + f_{12}(t) + t^{-2}f_{21}(t) + t^{-1}f_{22}(t),$$
$$t \to t^{-2}f_{11}(t) + t^{-1}f_{12}(t) + t^{-1}f_{21}(t) + f_{22}(t)$$

from K to $\mathbb{C}$ are continuous. Since, for $t \in K$ we have

$$\begin{vmatrix} 1 & t^{-1} & t^{-1} & t^{-2} \\ t^{-1} & t^{-2} & 1 & t^{-1} \\ t^{-1} & 1 & t^{-2} & t^{-1} \\ t^{-2} & t^{-1} & t^{-1} & 1 \end{vmatrix} = t^{-8} \begin{vmatrix} t^2 & t & t & 1 \\ t & 1 & t^2 & t \\ t & t^2 & 1 & t \\ 1 & t & t & t^2 \end{vmatrix} = -t^{-8}(t^2-1)^4 \neq 0,$$

we deduce that, for all $i,j \in \{1,2\}$, the function $f_{ij} : t \to f_{ij}(t)$ from K to $\mathbb{C}$ is continuous. Therefore, we can consider the element (f_{ij}) of $\mathcal{A}(K)$, which, in view of (2.1), satisfies $\mathcal{F}((f_{ij})) = \alpha$. Since α is arbitrary in $C(K, M_2(\mathbb{C}))$, the surjectivity of $\mathcal{F}$ is proved. Now, it follows from Assertion (1) in Corollary 2.5 that $C(K, M_2(\mathbb{C}))$ is generated by η_K as a C^*-algebra. ∎

Now we are ready to prove the main result in this section.

Theorem 2.8. *Let A be a C^*-algebra, and let e be a non self-adjoint idempotent in A. Put $K := sp(A, \sqrt{e^*e}) \setminus \{0\}$ (which, in view of Proposition 2.4, is a compact subset of $[1, \infty[$ whose maximum element is greater than 1), and assume that 1 does not belong to K. Then the C^*-subalgebra of A generated by e is $*$-isomorphic to $C(K, M_2(\mathbb{C}))$. More precisely, we have*

(1) *There exists a unique $*$-homomorphism $\Phi : C(K, M_2(\mathbb{C})) \to A$ such that $\Phi(\eta_K) = e$.*
(2) *Such a $*$-homomorphism is isometric, and its range coincides with the C^*-subalgebra of A generated by e.*

Proof. Let $\mathcal{F} : \mathcal{A}(K) \to C(K, M_2(\mathbb{C}))$ and $F : \mathcal{A}(K) \to A$ be the $*$-homomorphisms given by Propositions 2.3 and 2.4, respectively. By Assertion (2) in Corollary 2.5 (respectively, Proposition 2.3) $\mathcal{F}$ (respectively, F) is injective. On the other hand, by the first conclusion in Lemma 2.7, $\mathcal{F}$ is surjective. It follows that $\Phi := F \circ \mathcal{F}^{-1}$ is an injective $*$-homomorphism from $C(K, M_2(\mathbb{C}))$ to A satisfying $\Phi(\eta_K) = e$. As any injective $*$-homomorphism between C^*-algebras, Φ is isometric, and hence has closed range. Now, that Φ is the unique $*$-homomorphism from $C(K, M_2(\mathbb{C}))$ to A satisfying

$\Phi(\eta_K) = e$, as well as that the range of Φ coincides with the C^*-subalgebra of A generated by e, follows from the fact (given also by Lemma 2.7) that $C(K, M_2(\mathbb{C}))$ is generated by η_K as a C^*-algebra. ■

3. The case of C^*-algebras: the second theorem

Let A be an associative complex algebra. The quasi-product $a \circ b$ of two elements a, b of A is defined by $a \circ b := ab - a - b$. An element $a \in A$ is said to be quasi-invertible in A if there exists $b \in A$ satisfying $a \circ b = b \circ a = 0$. It is well-known and easy to see that the element $a \in A$ is quasi-invertible in A if and only if $1 - a$ is invertible in A_1, if and only if there exists a unique element $b \in A$ satisfying $a \circ b = 0$.

Lemma 3.1. *Let K be a compact subset of $[1, \infty[$ whose maximum element is greater than 1, and let $\mathcal{F} : \mathcal{A}(K) \to C(K, M_2(\mathbb{C}))$ be the $*$-homomorphism given by Proposition 2.3. Then an element $x \in \mathcal{A}(K)$ is quasi-invertible in $\mathcal{A}(K)$ if and only if $\mathcal{F}(x)$ is quasi-invertible in $C(K, M_2(\mathbb{C}))$.*

Proof. Let $x = (f_{ij})$ be in $\mathcal{A}(K)$. We claim that x is quasi-invertible in $\mathcal{A}(K)$ if and only if $\lambda_x(t) \neq 0$ for every $t \in K$, where $\lambda_x(t) :=$

$$\frac{t^2-1}{t^2}(f_{11}(t)f_{22}(t) - f_{12}(t)f_{21}(t)) - \frac{1}{t}(f_{12}(t) + f_{21}(t)) - f_{11}(t) - f_{22}(t) + 1.$$

Assume that x is quasi-invertible in $\mathcal{A}(K)$. Let us fix $t \in K$, and identify complex-valued continuous functions on $\{t\}$ with complex numbers. Then, since the restriction mapping $\mathcal{A}(K) \to \mathcal{A}(\{t\})$ is a homomorphism, $(f_{ij}(t))$ is a quasi-invertible element of $\mathcal{A}(\{t\})$, and hence there are complex numbers $g_{11}(t), g_{12}(t), g_{21}(t), g_{22}(t)$ uniquely determined by the condition $(f_{ij}(t)) \circ (g_{ij}(t)) = 0$. This means that the linear system in the indeterminates $x_{11}, x_{12}, x_{21}, x_{22} \in \mathbb{C}$

$$\left\{\begin{array}{l} (f_{11}(t) + t^{-1}f_{12}(t) - 1)x_{11} + (f_{12}(t) + t^{-1}f_{11}(t))x_{21} = f_{11}(t) \\ (f_{11}(t) + t^{-1}f_{12}(t) - 1)x_{12} + (f_{12}(t) + t^{-1}f_{11}(t))x_{22} = f_{12}(t) \\ (f_{21}(t) + t^{-1}f_{22}(t))x_{11} + (f_{22}(t) + t^{-1}f_{21}(t) - 1)x_{21} = f_{21}(t) \\ (f_{21}(t) + t^{-1}f_{22}(t))x_{12} + (f_{22}(t) + t^{-1}f_{21}(t) - 1)x_{22} = f_{22}(t) \end{array}\right. \tag{3.1}$$

has a unique solution (namely $x_{ij} = g_{ij}(t)$), and hence that the principal determinant of the system (by the way, equal to $\lambda_x(t)^2$) is nonzero. Conversely, assume that $\lambda_x(t) \neq 0$ for every $t \in K$. Then, for each $t \in K$, the system (3.1) has a unique solution $x_{ij} = g_{ij}(t)$, and, since the function $t \to \lambda_x(t)$ from K to $\mathbb{C}$ is continuous, the functions $g_{ij} : t \to g_{ij}(t)$ from K to $\mathbb{C}$ are continuous. Then we easily realize that $y := (g_{ij}) \in \mathcal{A}(K)$ is the unique element of $\mathcal{A}(K)$ satisfying $x \circ y = 0$, which implies that x is quasi-invertible in $\mathcal{A}(K)$. Now, the claim is proved.

On the other hand, $\mathcal{F}(x)$ is quasi-invertible in $C(K, M_2(\mathbb{C}))$ if and only if $1 - \mathcal{F}(x)$ is invertible in $C(K, M_2(\mathbb{C}))$, if and only if $1 - \mathcal{F}(x)(t)$ is invertible in $M_2(\mathbb{C})$ for every $t \in K$, if and only if $\det(1 - \mathcal{F}(x)(t)) \neq 0$ for every $t \in K$, where $\det(\cdot)$ means determinant. But, for $t \in K$, a straightforward but tedious computation shows that $\det(1-\mathcal{F}(x)(t)) = \lambda_x(t)$. Therefore, $\mathcal{F}(x)$ is quasi-invertible in $C(K, M_2(\mathbb{C}))$ if and only if $\lambda_x(t) \neq 0$. By invoking the claim proved in the preceding paragraph, the result follows. ∎

Let K be a compact subset of $[1, \infty[$ with $1 \in K$, and let p be a self-adjoint idempotent in $M_2(\mathbb{C})$, different from 0 and 1. Then $\mathbb{C}p$ is a self-adjoint subalgebra of $M_2(\mathbb{C})$, and hence

$$C_p(K, M_2(\mathbb{C})) := \{\alpha \in C(K, M_2(\mathbb{C})) : \alpha(1) \in \mathbb{C}p\}$$

is a proper C^*-subalgebra of $C(K, M_2(\mathbb{C}))$. We note that, in the construction of the C^*-algebra $C_p(K, M_2(\mathbb{C}))$, the choice of the idempotent p is structurally irrelevant. Indeed, if, for $i \in \{1,2\}$, p_i is a self-adjoint idempotent in $M_2(\mathbb{C})$, different from 0 and 1, then there exists a norm-one element χ_i in the Hilbert space $\mathbb{C}^2$ such that p_i is the operator $\chi \to (\chi|\chi_i)\chi_i$ on $\mathbb{C}^2$, and hence, since there exists a unitary element $v \in M_2(\mathbb{C})$ with $v\chi_1 = \chi_2$ (by transitivity of Hilbert spaces), the mapping $\alpha \to v\alpha v^*$ becomes a $*$-automorphism of $C(K, M_2(\mathbb{C}))$ sending $C_{p_1}(K, M_2(\mathbb{C}))$ onto $C_{p_2}(K, M_2(\mathbb{C}))$. We also note that, if we take $p = \eta(1)$, then $C_p(K, M_2(\mathbb{C}))$ contains η_K.

Lemma 3.2. *Let K be a compact subset of $[1, \infty[$ with $1 \in K$, and whose maximum element is greater than 1, and let $\mathcal{F} : \mathcal{A}(K) \to C(K, M_2(\mathbb{C}))$ be the $*$-homomorphism given by Proposition 2.3. Then the closure in $C(K, M_2(\mathbb{C}))$ of the range of $\mathcal{F}$ coincides with $C_{\eta(1)}(K, M_2(\mathbb{C}))$. As a consequence, $C_{\eta(1)}(K, M_2(\mathbb{C}))$ is generated by η_K as a C^*-algebra.*

Proof. For $x = (f_{ij})$ in $\mathcal{A}(K)$, we have

$$\mathcal{F}(x)(1) = (f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1))\eta(1) \in \mathbb{C}\eta(1),$$

and therefore $\mathcal{F}(x)$ lies in $C_{\eta(1)}(K, M_2(\mathbb{C}))$. This shows that the range of $\mathcal{F}$ (say B) is contained in $C_{\eta(1)}(K, M_2(\mathbb{C}))$.

To continue our argument, it is useful to identify $C(K, M_2(\mathbb{C}))$ with $C(K) \otimes M_2(\mathbb{C})$ in the natural manner. Then we have:

$$2 \otimes \eta(1) = 1 \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = (1+u)^{-1}u(\eta_{11}^K + \eta_{12}^K + \eta_{21}^K + \eta_{22}^K) \in B, \tag{3.2}$$

$$\sqrt{u^2-1} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = u(\eta_{21}^K - \eta_{12}^K) \in B, \tag{3.3}$$

$$\sqrt{u^2-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = u(\eta_{22}^K - \eta_{11}^K) \in B, \tag{3.4}$$

$$(u^2-1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = u^2(\eta_{22}^K + \eta_{11}^K) - u(\eta_{12}^K + \eta_{21}^K) \in B. \tag{3.5}$$

Now, keep in mind that B is a $C(K)$-submodule of $C(K, M_2(\mathbb{C}))$, and denote by $C_1(K)$ the closed ideal of $C(K)$ consisting of those complex-valued continuous functions on K vanishing at 1. It follows from (3.2) that

$$C(K) \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \subseteq B,$$

and, by invoking the Stone-Weierstrass theorem, it follows from (3.3), (3.4), and (3.5) that

$$C_1(K)\otimes\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \subseteq \overline{B},\ C_1(K)\otimes\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \subseteq \overline{B},\ C_1(K)\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \subseteq \overline{B}.$$

Since

$$\left\{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

is a basis of $M_2(\mathbb{C})$, we deduce that $C_1(K) \otimes M_2(\mathbb{C}) \subseteq \overline{B}$. Since

$$C_{\eta(1)}(K, M_2(\mathbb{C})) = [\mathbb{C} \otimes \eta(1)] \oplus [C_1(K) \otimes M_2(\mathbb{C})],$$

and $\mathbb{C} \otimes \eta(1) \subseteq B$ (by (3.2)), we obtain $C_{\eta(1)}(K, M_2(\mathbb{C})) \subseteq \overline{B}$. By invoking the first paragraph in the present proof, we have $C_{\eta(1)}(K, M_2(\mathbb{C})) = \overline{B}$.

Now, it follows from Assertion (1) in Corollary 2.5 that $C_{\eta(1)}(K, M_2(\mathbb{C}))$ is generated by η_K as a C^*-algebra. ■

Now we are ready to prove the main result in this section.

Theorem 3.3. *Let A be a C^*-algebra, and let e be a non self-adjoint idempotent in A. Put $K := sp(A, \sqrt{e^*e}) \setminus \{0\}$ (which, in view of Proposition 2.4, is a compact subset of $[1, \infty[$ whose maximum element is greater than 1), and assume that 1 belongs to K. Then the C^*-subalgebra of A generated by e is $*$-isomorphic to $C_p(K, M_2(\mathbb{C}))$ for any self-adjoint idempotent $p \in M_2(\mathbb{C})$ different from 0 and 1. More precisely, we have:*

1. *There exists a unique $*$-homomorphism $\Phi : C_{\eta(1)}(K, M_2(\mathbb{C})) \to A$ such that $\Phi(\eta_K) = e$.*
2. *Such a $*$-homomorphism is isometric, and its range coincides with the C^*-subalgebra of A generated by e.*

Proof. For every element c in a complex Banach algebra C, put

$$r(C, c) := \max\{|\lambda| : \lambda \in sp(C, c)\},$$

and note that, since

$$\{0\} \cup sp(C, c) = \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}c \notin q - inv(C)\}$$

(where $q - inv(C)$ stands por the set of all quasi-invertible elements of C), we have

$$r(C, c) = \max[\{0\} \cup \{|\lambda| : \lambda \in \mathbb{C} \setminus \{0\}, \lambda^{-1}c \notin q - inv(C)\}]. \tag{3.6}$$

Now, let $\mathcal{F} : \mathcal{A}(K) \to C(K, M_2(\mathbb{C}))$ and $F : \mathcal{A}(K) \to A$ be the $*$-homomorphisms given by Propositions 2.3 and 2.4, respectively. Then, for $x \in \mathcal{A}(K)$ we have

$$\|F(x)\|^2 = r(A, F(x^*x)) \leq r(\mathcal{A}(K), x^*x),$$

and, by keeping in mind Lemma 3.1 and (3.6), we have also

$$r(\mathcal{A}(K), x^*x) = r(C(K, M_2(\mathbb{C})), \mathcal{F}(x^*x)) = \|\mathcal{F}(x)\|^2,$$

so that the inequality $\|F(x)\| \leq \|\mathcal{F}(x)\|$ holds. Therefore $\mathcal{F}(x) \to F(x)$ $(x \in \mathcal{A}(K))$ becomes a (well-defined) continuous $*$-homomorphism from the range of $\mathcal{F}$ to A. Then, by the first conclusion in Lemma 3.2, such a $*$-homomorphism extends by continuity to a $*$-homomorphism

$$\Phi : C_{\eta(1)}(K, M_2(\mathbb{C})) \to A$$

satisfying $\Phi \circ \mathcal{F} = F$, and hence $\Phi(\eta_K) = e$. Now, that Φ is the unique $*$-homomorphism from $C_{\eta(1)}(K, M_2(\mathbb{C}))$ to A satisfying $\Phi(\eta_K) = e$, as well as that the range of Φ coincides with the C^*-subalgebra of A generated by e, follows from the fact (given also by Lemma 3.2) that $C_{\eta(1)}(K, M_2(\mathbb{C}))$ is generated by η_K as a C^*-algebra.

To conclude the proof, it is enough to show that Φ is injective. Let α be in $\ker(\Phi)$. Then, by Lemma 3.2, there exists a sequence $x_n = (f_{ij}^n)$ in $\mathcal{A}(K)$ such that $\mathcal{F}(x_n) \to \alpha$. For $n \in \mathbb{N}$ and $i, j \in \{1, 2\}$, define $g_{ij}^n \in C(K)$ by

$$\begin{aligned}
g_{11}^n &:= f_{11}^n + u^{-1} f_{12}^n + u^{-1} f_{21}^n + u^{-2} f_{22}^n,\\
g_{12}^n &:= u^{-1} f_{11}^n + u^{-2} f_{12}^n + f_{21}^n + u^{-1} f_{22}^n,\\
g_{21}^n &:= u^{-1} f_{11}^n + f_{12}^n + u^{-2} f_{21}^n + u^{-1} f_{22}^n,\\
g_{22}^n &:= u^{-2} f_{11}^n + u^{-1} f_{12}^n + u^{-1} f_{21}^n + f_{22}^n.
\end{aligned}$$

Then we have $[ij]x_n[ij] = g_{ij}^n[ij]$. Now, since the restriction of F to $C(K)[ij]$ is an isometry (by the proof of Theorem 2.6 of [**1**]), we deduce

$$\begin{aligned}
\|g_{ij}^n\| = \|g_{ij}^n[ij]\| &= \|F(g_{ij}^n[ij])\| = \|F([ij]x_n[ij])\| = \|F([ij])F(x_n)F([ij])\|\\
&= \|F([ij])\Phi(\mathcal{F}(x_n))F([ij])\| \to \|F([ij])\Phi(\alpha)F([ij])\| = 0.
\end{aligned}$$

As a consequence, $g_{ij}^n(t) \to 0$ for every $t \in K$. Since for $t \in K \setminus \{1\}$, we have

$$\begin{vmatrix} 1 & t^{-1} & t^{-1} & t^{-2} \\ t^{-1} & t^{-2} & 1 & t^{-1} \\ t^{-1} & 1 & t^{-2} & t^{-1} \\ t^{-2} & t^{-1} & t^{-1} & 1 \end{vmatrix} = -t^{-8}(t^2 - 1)^4 \neq 0,$$

it follows from the definition of g_{ij}^n that $f_{ij}^n(t) \to 0$ for every $t \in K \setminus \{1\}$. Now, since for $t \in K \setminus \{1\}$ we have $\mathcal{F}(x_n)(t) \to \alpha(t)$ and

$$\mathcal{F}(x_n)(t) = \sum_{i,j \in \{1,2\}} f_{ij}^n(t)\eta_{ij}(t) \to 0,$$

for such a t we obtain $\alpha(t) = 0$. Therefore, if 1 is an accumulation point of K, then $\alpha = 0$, as desired. Assume that 1 is an isolated point of K. Then the function $\chi : K \to \mathbb{C}$, defined by $\chi(1) := 1$ and $\chi(t) := 0$ for $t \in K \setminus \{1\}$, is continuous, and, since there exists $\lambda \in \mathbb{C}$ such that $\alpha(1) = \lambda\eta(1)$, for such a λ we have $\alpha = \lambda\chi\eta_{21}^K = \mathcal{F}(\lambda\chi[21])$. Therefore

$$0 = \Phi(\alpha) = \Phi(\mathcal{F}(\lambda\chi[21])) = F(\lambda\chi[21])),$$

which, in view of Assertion (3) in Proposition 2.4, implies $\lambda = 0$, and hence $\alpha = 0$. ∎

4. The case of C^*-algebras: some consequences

In this section, we combine Theorems 2.8 and 3.3 to derive some attractive consequences. We begin with an easy corollary to Theorem 3.3.

Corollary 4.1. *Let A be a C^*-algebra generated by a non self-adjoint idempotent e, and put $K := sp(A, \sqrt{e^*e}) \setminus \{0\}$. If 1 is an isolated point of the compact set K, then A is $*$-isomorphic to the C^*-algebra*

$$\mathbb{C} \times C(K \setminus \{1\}, M_2(\mathbb{C})).$$

Proof. If 1 belongs to K, then, for α in $C_{\eta(1)}(K, M_2(\mathbb{C}))$, there exists a unique complex number $\lambda(\alpha)$ such that $\alpha(1) = \lambda(\alpha)\eta(1)$, and the mapping

$$\alpha \to (\lambda(\alpha), \alpha_{|K\setminus\{1\}})$$

becomes an injective $*$-homomorphism from $C_{\eta(1)}(K, M_2(\mathbb{C}))$ to

$$\mathbb{C} \times C^b(K \setminus \{1\}, M_2(\mathbb{C})),$$

where $C^b(K \setminus \{1\}, M_2(\mathbb{C}))$ stands for the C^*-algebra of all bounded continuous function from $K \setminus \{1\}$ to $M_2(\mathbb{C})$. Moreover, if 1 is in fact an isolated point of K, then we have that

$$C^b(K \setminus \{1\}, M_2(\mathbb{C})) = C(K \setminus \{1\}, M_2(\mathbb{C})),$$

and that the above $*$-homomorphism is surjective. Finally, apply Theorem 3.3. ∎

Corollary 4.2. *Let A be a C^*-algebra generated by a non self-adjoint idempotent e, and put $K := sp(A, \sqrt{e^*e}) \setminus \{0\}$. Then A has a unit if and only if either 1 does not belong to K or 1 is an isolated point of K.*

Proof. In view of Theorems 2.8 and 3.3, and Corollary 4.1, it is enough to show that, if 1 is an accumulation point of K, then $C_{\eta(1)}(K, M_2(\mathbb{C}))$ has not a unit. Assume that 1 belongs to K. We claim that, given $t_0 \in K \setminus \{0\}$, the valuation at t_0 (as a mapping from $C_{\eta(1)}(K, M_2(\mathbb{C}))$ to $M_2(\mathbb{C})$) is surjective. Indeed, if $a = (\lambda_{ij})$ is an arbitrary element of $M_2(\mathbb{C})$, then, for $i, j \in \{1, 2\}$, there exists $f_{ij} \in C(K)$ such that $f_{ij}(1) = 0$ and $f_{ij}(t_0) = \lambda_{ij}$, and hence the element α of $C(K, M_2(\mathbb{C}))$, defined by $\alpha(t) := (f_{ij}(t))$ for every $t \in K$, lies in $C_{\eta(1)}(K, M_2(\mathbb{C}))$ and satisfies $\alpha(t_0) = a$. Assume in addition that $C_{\eta(1)}(K, M_2(\mathbb{C}))$ has a unit $\mathbf{1}$. Then, by the claim just proved, for every $t \in K \setminus \{1\}$, $\mathbf{1}(t)$ must be equal to the unit of $M_2(\mathbb{C})$. Now, if 1 is in fact an accumulation point of K, then $\mathbf{1}(1)$ is the unit of $M_2(\mathbb{C})$, which is not possible because $\mathbf{1}(1)$ is a complex multiple of $\eta(1)$. ∎

Corollary 4.3. *Let A be a C^*-algebra. Then A has a non self-adjoint idempotent (if and) only if it contains (as a C^*-subalgebra) a copy of either $M_2(\mathbb{C})$ or $C_p([1,2], M_2(\mathbb{C}))$ for any self-adjoint idempotent $p \in M_2(\mathbb{C})$ different from 0 and 1.*

Proof. Assume that A has a non self-adjoint idempotent e, and put $K := sp(A, \sqrt{e^*e}) \setminus \{0\}$. We may suppose that A is generated by e. If 1 does not belong to K, then, by Theorem 2.8, A contains a copy of $M_2(\mathbb{C})$. Assume that 1 belongs to K, and that K is disconnected. Take a clopen proper subset U of K with $1 \in U$. Then, arguing as in the proof of Corollary 4.1, we realize that A is $*$-isomorphic to $C_p(U, M_2(\mathbb{C})) \times C(K \setminus U, M_2(\mathbb{C}))$, for some self-adjoint idempotent $p \in M_2(\mathbb{C})$ different from 0 and 1, and hence it contains a copy of $M_2(\mathbb{C})$. Finally, assume that 1 belongs to K, and that K is connected. Then we have $K = [1, \|e\|]$, and therefore, by Theorem 3.3, A is isomorphic to $C_p([1, \|e\|], M_2(\mathbb{C}))$, for some p as above. But, taking a homeomorphism ϕ from $[1, \|e\|]$ onto $[1,2]$ with $\phi(1) = 1$, ϕ induces a $*$-isomorphism from $C([1, \|e\|], M_2(\mathbb{C}))$ onto $C([1,2], M_2(\mathbb{C}))$ sending $C_p([1, \|e\|], M_2(\mathbb{C}))$ onto $C_p([1,2], M_2(\mathbb{C}))$. ∎

We Remark that $C_p([1,2], M_2(\mathbb{C}))$ does not contain any copy of $M_2(\mathbb{C})$. To realize this, we argue by contradiction, and hence we assume that $C_p([1,2], M_2(\mathbb{C}))$ contains a copy (say B) of $M_2(\mathbb{C})$. For $\alpha \in C_p([1,2], M_2(\mathbb{C}))$, let $\lambda(\alpha)$ stand for the unique complex number satisfying $\alpha(1) = \lambda(\alpha)p$. Then, since $\lambda : C_p([1,2], M_2(\mathbb{C})) \to \mathbb{C}$ is a homomorphism, by the simplicity of B we have $\lambda(B) = 0$. Therefore B is contained in the ideal (say M) of $C([1,2], M_2(\mathbb{C}))$ consisting of those continuous functions from $[1,2]$ to $M_2(\mathbb{C})$ vanishing at 1. Now, since (clearly) M has no nonzero idempotent, and the unit of B is a nonzero idempotent of M, the contradiction is clear.

Remark 4.4. In relation to Corollary 4.3 above, it is worth mentioning that a C^*-algebra contains a non self-adjoint idempotent if and only if it contains a non central self-adjoint idempotent [**1**]. By the way, the "only if" part of the result in [**1**] just quoted follows easily from Corollary 4.3, whereas the "if part" is a consequence of Proposition 4.5 immediately below.

In relation to Proposition 4.5 immediately below, we note that non self-adjoint idempotents in a C^*-algebra are non central.

Proposition 4.5. *Let A be a C^*-algebra containing a non central idempotent e. Then there exists a continuous mapping $r \to e_r$ from $[1, \infty[$ to the set of idempotents of A satisfying $e_{\|e\|} = e$ and $\|e_r\| = r$ for every $r \in [1, \infty[$.*

Proof. First assume that e is not self-adjoint. Then, by Theorems 2.8 and 3.3, we may assume that A is of the form $C(K, M_2(\mathbb{C}))$ or $C_{\eta(1)}(K, M_2(\mathbb{C}))$, where, in the first case, K is a compact subset of $]1, \infty[$ and, in the second case, K is a compact subset of $[1, \infty[$ whose maximum element is greater than 1 and such that $1 \in K$. In any case, put $\rho := \max K > 1$. Let r be in

$[1, \infty[$, and let e_r denote the element of $C(K, M_2(\mathbb{C}))$ defined by

$$e_r(t) := \eta(1 + \frac{(r-1)(t-1)}{\rho - 1})$$

for every $t \in K$. Noticing that, in the case that 1 belongs to K, e_r lies in $C_{\eta(1)}(K, M_2(\mathbb{C}))$, it turns out that, in any case e_r is an element of A. Moreover, keeping in mind Lemma 2.2, we easily realize that e_r is an idempotent, and that $\|e_r\| = r$. On the other hand, since $\|\eta_K\| = \rho$, we have $e_{\|\eta_K\|} = \eta_K$. Now it only remains to show that the mapping $r \to e_r$ is continuous. Fix $r \in [1, \infty[$ and $\varepsilon > 0$, and take $\delta > 0$ such that $\|\eta(s) - \eta(r)\| < \varepsilon$ whenever s is in $[1, \infty[$ with $|s - r| < \delta$. Then, for $s \in [1, \infty[$ with $|s - r| < \delta$, we have for every $t \in K$,

$$\left| \left[1 + \frac{(s-1)(t-1)}{\rho - 1} \right] - \left[1 + \frac{(r-1)(t-1)}{\rho - 1} \right] \right| = \frac{|s-r|(t-1)}{\rho - 1} \leq |s - r| < \delta,$$

so $\|e_s(t) - e_r(t)\| < \varepsilon$ for every $t \in K$, and so $\|e_s - e_r\| \leq \varepsilon$.

Now assume that e is self-adjoint. Since e is non central, we may choose a self-adjoint element $a \in A$ with $ea - ae \neq 0$. Then the mapping $D : A \to A$ defined by $D(b) := ba - ab$ for every $b \in A$ becomes a continuous derivation satisfying $D(e) \neq 0$ and $D(b^*) = -D(b)^*$ for every $b \in A$. Therefore, for $s \in \mathbb{R}$, $\exp(sD)$ is a continuous automorphism of A satisfying

$$[\exp(sD)(b)]^* = \exp(-sD)(b^*)$$

for every $b \in A$, and consequently

$$g(s) := \exp(sD)(e)$$

is a nonzero idempotent in A, and we have

$$g(s)^* = g(-s). \tag{4.1}$$

Now, consider the continuous mapping $f : \mathbb{R} \to [1, \infty[$ defined by

$$f(s) := \|g(s)\|.$$

By (4.1), we have

$$f(-s) = f(s) \tag{4.2}$$

for every $s \in \mathbb{R}$. Let r, s be in $\mathbb{R}$. Then, keeping in mind (4.1), (4.2), and that $\exp(\frac{s-r}{2}D)$ is an automorphism of A, we have

$$\begin{aligned} f(\frac{r+s}{2})^2 &= \|g(\frac{r+s}{2})\|^2 = \|g(\frac{r+s}{2})^* g(\frac{r+s}{2})\| = r(A, g(\frac{r+s}{2})^* g(\frac{r+s}{2})) \\ &= r(A, g(-\frac{r+s}{2}) g(\frac{r+s}{2})) = r[A, \exp(\frac{s-r}{2}D)(g(-\frac{r+s}{2}) g(\frac{r+s}{2}))] \\ &= r[A, [\exp(\frac{s-r}{2}D)(g(-\frac{r+s}{2}))][\exp(\frac{s-r}{2}D) g((\frac{r+s}{2}))]] \\ &= r(A, g(-r)g(s)) \leq \|g(-r)\| \|g(s)\| = f(-r)f(s) = f(r)f(s), \end{aligned}$$

and therefore

$$f(\frac{r+s}{2}) \leq \sqrt{f(r)f(s)} \leq \frac{f(r) + f(s)}{2}.$$

In this way we have shown that f is convex. Assume that $f(r) = 1$ for some $r \in]0, \infty[$. Then, by (4.2) and the convexity of f, we have $f(s) = 1$ for every $s \in [-r, r]$. Therefore, for $s \in [-r, r]$, the idempotent $g(s)$ has norm equal to 1, so it is self-adjoint, and so, by (4.1) the equality $g(s) = g(-s)$ holds. Since g is differentiable at 0 with $g'(0) = D(e)$, the above implies $D(e) = 0$, which is a contradiction. Thus, $f(r) > 1$ for every $r \in]0, \infty[$. Now, let $0 < r < s$. Noticing that $f(0) = 1$ and that then, by the convexity of f, the mapping $t \to \frac{f(t)-1}{t}$ is incresing, we have

$$0 < f(r) - 1 < \frac{s}{r}(f(r) - 1) \leq f(s) - 1.$$

In this way, we have shown that $f_{|[0,\infty[}$ is strictly increasing and non bounded. As a consequence, the range of $f_{|[0,\infty[}$ is $[1, \infty[$, and the inverse mapping $h : [1, \infty[\to [0, \infty[$ is continuous. Now, for $r \in [1, \infty[$, let e_r be the idempotent of A defined by $e_r := g(h(r))$. Then, clearly, the mapping $r \to e_r$ is continuous, and we have $e_1 = e$. Moreover, by the definition of g and h, we have also that $\|e_r\| = f(h(r)) = r$ for every $r \in [1, \infty[$. ∎

We recall that partial isometries in a C^*-algebra A are defined as those elements $a \in A$ satisfying $aa^*a = a$.

LEMMA 4.6. *Let A be a C^*-algebra, and let a be a partial isometry in A such that both a^*a and aa^* lie in the centre of A. Then a is normal.*

PROOF. For $x, y \in A$, put $[x, y] := xy - yx$. Since a^*a and aa^* lie in the centre of A, we have $[a^*a, a] = 0$ and $[aa^*, a] = 0$, which reads as $a^*a^2 = a$ and $a^2a^* = a$, respectively. The two las equalities, together with the one $aa^*a = a$, and those obtained by taking adjoints, imply $[[a, a^*], a] = 0$. By Proposition 18.13 of [**2**], we have $r(A, [a, a^*]) = 0$, and hence, since $[a, a^*]$ is self-adjoint, we actually have $[a, a^*] = 0$. ∎

Let A denote the C^*-algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space H, let $b : H \to H$ be any non surjective linear isometry, and put $a := b$ (respectively $a := b^*$). Then a is a non normal partial isometry in A such that a^*a (respectively, aa^*) lies in the centre of A.

COROLLARY 4.7. *Let A be a C^*-algebra. Then the following assertions are equivalent:*

(1) *A contains a non central self-adjoint idempotent.*
(2) *There exists a non normal partial isometry $a \in A$ such that a belongs to a^2Aa^2.*
(3) *A contains a non normal partial isometry.*

PROOF. (1) $\Rightarrow$ (2).- By the assumption (1), Remark 4.4, and Theorems 2.8 and 3.3, we may assume that A is of the form $C(K, M_2(\mathbb{C}))$ or $C_{\eta(1)}(K, M_2(\mathbb{C}))$, where, in the first case, K is a compact subset of $]1, \infty[$

and, in the second case, K is a compact subset of $[1,\infty[$ whose maximum element is greater than 1 and such that $1 \in K$. In any case, by Lemma 2.2, η_{21}^K is a non normal partial isometry in A, and we have $\eta_{21}^K = (\eta_{21}^K)^2(u^2\eta_{12}^K)(\eta_{21}^K)^2$.

(2) $\Rightarrow$ (3).- This is clear.

(3) $\Rightarrow$ (1).- Let a be the partial isometry whose existence is assumed in (3). Then, keeping in mind that both a^*a and aa^* are self-adjoint idempotents, it follows from Lemma 4.6 that A contains a non central self-adjoint idempotent. ■

Put $A := M_2(\mathbb{C})$ and $a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then a is a non normal partial isometry in A, which does not belong to a^2Aa^2.

5. The case of JB^*-algebras: the main results

Over fields of characteristic different from two, Jordan algebras are defined as those (possibly non associative) commutative algebras satisfying the identity $(x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2)$. For a and b in a Jordan algebra, we put $U_a(b) := 2a \cdot (a \cdot b) - a^2 \cdot b$. Let A be an associative algebra. Then A becomes a Jordan algebra under the Jordan product defined by

$$a \cdot b := \frac{1}{2}(ab + ba).$$

Moreover, for all $a, b \in A$ we have

$$U_a(b) := 2a \cdot (a \cdot b) - a^2 \cdot b = aba. \tag{5.1}$$

Jordan subalgebras of A are, by definition, those subspaces J of A satisfying $J \cdot J \subseteq J$.

Let K be a compact subset of $[1,\infty[$. Then the linear mapping $\Theta : \mathcal{A}(K) \to \mathcal{A}(K)$, determined by

$$\Theta(f[ij]) := f[ij] \text{ if } i \neq j,\ \Theta(f[11]) := f[22],\ \Theta(f[22]) := f[11]$$

for every $f \in C(K)$, becomes an isometric involutive $*$-antiautomorphism of $\mathcal{A}(K)$. Therefore, the set of fixed elements for Θ is a closed $*$-invariant Jordan subalgebra of $\mathcal{A}(K)$, and hence a Banach-Jordan $*$-algebra. Such a Banach-Jordan $*$-algebra will be denoted by $\mathcal{J}(K)$. Note that elements of $\mathcal{J}(K)$ are precisely those matrices $(f_{ij}) \in \mathcal{A}(K)$ satisfying $f_{11} = f_{22}$, or equivalently, those elements of $\mathcal{A}(K)$ of the form $f([11]+[22])+g[12]+h[21]$ with $f, g, h \in C(K)$.

We take from [**1**] the following.

Lemma 5.1. *Let K be a compact subset of $[1,\infty[$. Then $\mathcal{J}(K)$ is generated by $u[21]$ as a Jordan-Banach $*$-algebra.*

JB^*-algebras are defined as those Banach-Jordan $*$-algebras J satisfying $\|U_a(a^*)\| = \|a\|^3$ for every $a \in J$. By keeping in mind (5.1), it is easy to realize that C^*-algebras are JB^*-algebras under their Jordan products.

The mapping

$$\theta : \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \to \begin{pmatrix} \lambda_{22} & \lambda_{12} \\ \lambda_{21} & \lambda_{11} \end{pmatrix}$$

is an involutive $*$-antiautomorphism of $M_2(\mathbb{C})$. Therefore, the set of fixed elements for θ is a $*$-invariant Jordan subalgebra of the C^*-algebra $M_2(\mathbb{C})$, and hence a JB^*-algebra. Such a JB^*-algebra is called the three-dimensional spin factor, and is denoted by $\mathcal{C}_3$.

Let K be a compact subset of $[1,\infty[$. We denote by $C(K,\mathcal{C}_3)$ the JB^*-algebra of all continuous functions from K to $\mathcal{C}_3$. We will identify $C(K,\mathcal{C}_3)$ with the JB^*-subalgebra of $C(K,M_2(\mathbb{C}))$ consisting of those continuous functions from K to $M_2(\mathbb{C})$ whose range is contained in $\mathcal{C}_3$.

Lemma 5.2. *Let K be a compact subset of $[1,\infty[$ whose maximum element is greater than 1, let $\mathcal{F}:\mathcal{A}(K)\to C(K,M_2(\mathbb{C}))$ be the $*$-homomorphism given by Proposition 2.3, and let $\mathcal{G}$ denote the restriction to $\mathcal{J}(K)$ of $\mathcal{F}$. Then $\mathcal{G}$ is a $*$-homomorphism from $\mathcal{J}(K)$ to the JB^*-algebra underlying $C(K,M_2(\mathbb{C}))$, and the closure in $C(K,M_2(\mathbb{C}))$ of the range of $\mathcal{G}$ coincides with the JB^*-subalgebra of $C(K,\mathcal{C}_3)$ generated by η_K.*

Proof. Noticing that $\mathcal{G}(u[21])=\eta_K$, and keeping in mind Lemma 5.1, it is enough to show that the range of $\mathcal{G}$ is contained in $C(K,\mathcal{C}_3)$. But this follows from the fact that η_K actually belongs to $C(K,\mathcal{C}_3)$, and a new application of Lemma 5.1. ■

Lemma 5.3. *Let K be a compact subset of $]1,\infty[$. Then $C(K,\mathcal{C}_3)$ is generated by η_K as a JB^*-algebra.*

Proof. Identifying $C(K,M_2(\mathbb{C}))$ with $C(K)\otimes M_2(\mathbb{C})$ in the natural manner, the operator $\widehat{\theta}:=1\otimes\theta$ becomes an involutive $*$-antiautomorphism of $C(K,M_2(\mathbb{C}))$, whose set of fixed points is precisely $C(K,\mathcal{C}_3)$. Moreover, since $\mathcal{A}(K)$ is generated by $u[21]$ as a Banach $*$-algebra (by Lemma 2.5 of [**1**]), and $\mathcal{F}(\Theta(u[21]))=\widehat{\theta}(\mathcal{F}(u[21]))$, we have $\mathcal{F}\circ\Theta=\widehat{\theta}\circ\mathcal{F}$. On the other hand, by Lemma 2.7, $\mathcal{F}:\mathcal{A}(K)\to C(K,M_2(\mathbb{C}))$ is surjective. Since $\mathcal{J}(K)$ is the set of fixed points for Θ, and $C(K,\mathcal{C}_3)$ is the set of fixed points for $\widehat{\theta}$, and $\mathcal{G}$ is the restriction to $\mathcal{J}(K)$ of $\mathcal{F}$, it follows that $\mathcal{G}$ (as a mapping from $\mathcal{J}(K)\to C(K,\mathcal{C}_3)$) is surjective. Now, apply Lemma 5.2. ■

We recall that a JB^*-triple is a complex Banach space X endowed with a continuous triple product $\{\cdot,\cdot,\cdot\}:X\times X\times X\to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

1. For all x in X, the mapping $y\to\{x,x,y\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
2. The main identity

$$\{a,b,\{x,y,z\}\}=\{\{a,b,x\},y,z\}-\{x,\{b,a,y,\},z\}+\{x,y,\{a,b,z\}\}$$

holds for all a, b, x, y, z in X.

(3) $\|\{x, x, x\}\| = \|x\|^3$ for every x in X.

Concerning Condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be hermitian if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$. Examples of JB^*-triples are all C^*-algebras under the triple product $\{\cdot,\cdot,\cdot\}$ determined by $\{a, b, a\} := ab^*a$.

Let X be a JB^*-triple, and let x be in X. It is well-known that there is a unique couple (K, ϕ), where K is a compact subset of $[0, \infty[$ with $0 \in K$, and ϕ is an isometric triple homomorphism from $C_0(K)$ to X, such that the range of ϕ coincides with the JB^*-subtriple of X generated by x, and $\phi(v) = x$, where v stands for the mapping $t \to t$ from K to $\mathbb{C}$ (see [**8**, 4.8], [**9**, 1.15], and [**5**]). The locally compact subset $K \setminus \{0\}$ of $]0, \infty[$ is called the triple spectrum of x, and will be denoted by $\sigma(x)$. We note that $\sigma(x)$ does not change when we replace X with any JB^*-subtriple of X containing x.

We take from [**1**] the following.

Lemma 5.4. *Let A be a C^*-algebra, and let a be in A such that $0 \in sp(a^*a)$. Then we have $\sigma(a) = sp(A, \sqrt{a^*a}) \setminus \{0\}$.*

As in the particular case of C^*-algebras, already commented, JB^*-algebras are JB^*-triples under the triple product $\{\cdot,\cdot,\cdot\}$ determined by $\{a, b, a\} := U_a(b^*)$ (see [**3**] and [**13**]). For later reference, we remark that, if a JB^*-algebra J has a unit $\mathbf{1}$, then for $a, b \in J$ we have

$$a \cdot b = \{a, \mathbf{1}, b\} \text{ and } a^* = \{\mathbf{1}, a, \mathbf{1}\}. \tag{5.2}$$

Theorem 5.5. *Let J be a JB^*-algebra, and let e be a non self-adjoint idempotent in J. Put $K := \sigma(e)$, and assume that 1 does not belong to K. Then K is a compact subset of $]1, \infty[$, and the JB^*-subalgebra of J generated by e is $*$-isomorphic to $C(K, \mathcal{C}_3)$. More precisely, we have:*

(1) *There exists a unique $*$-homomorphism $\Psi : C(K, \mathcal{C}_3) \to J$ such that $\Psi(\eta_K) = e$.*

(2) *Such a $*$-homomorphism is isometric, and its range coincides with the JB^*-subalgebra of J generated by e.*

Proof. Let J_e denote the JB^*-subalgebra of J generated by e. By [**13**] and [**12**], there exists a C^*-algebra A containing J_e as a JB^*-subalgebra. Therefore, by Lemma 5.4 and Proposition 2.4, $K := \sigma(e)$ is a compact subset of $]1, \infty[$. By Theorem 2.8, there exists an isometric $*$-homomorphism $\Phi : C(K, M_2(\mathbb{C})) \to A$ such that $\Phi(\eta_K) = e$. Let Ψ stands for the restriction of Φ to $C(K, \mathcal{C}_3)$. Then, clearly, Ψ is an isometric $*$-homomorphism from $C(K, \mathcal{C}_3)$ to the JB^*-algebra underlying A, which satisfies $\Psi(\eta_K) = e$. Noticing that the JB^*-subalgebras of A and J generated by e coincide, it follows from Lemma 5.3 that the range of Ψ is J_e. This last fact allows us to see Ψ as a $*$-homomorphisms from $C(K, \mathcal{C}_3)$ to J. That Ψ is the unique (automatically continuous [**12**]) $*$-homomorphism from $C(K, \mathcal{C}_3)$ to J with $\Psi(\eta_K) = e$ follows from a new application of Lemma 5.3. ∎

Let K be a compact subset of $[1,\infty[$ with $1 \in K$, and let p be a self-adjoint idempotent in $\mathcal{C}_3$, different from 0 and 1. Then

$$C_p(K,\mathcal{C}_3) := \{\alpha \in C(K,\mathcal{C}_3) : \alpha(1) \in \mathbb{C}p\}$$

is a proper JB^*-subalgebra of $C(K,\mathcal{C}_3)$. As in the case of the C^*-algebra $C_p(K,M_2(\mathbb{C}))$, the JB^*-algebra $C_p(K,\mathcal{C}_3)$ does not depend structurally on p. Indeed, if, for $i \in \{1,2\}$, p_i is a self-adjoint idempotent in $M_2(\mathbb{C})$, different from 0 and 1, then $\{p_i, 1-p_i\}$ is a "frame of tripotents" in the simple JB^*-triple underlying $\mathcal{C}_3$, and therefore, by Theorem 5.9 of [**10**], there exists a triple automorphism ϕ of $\mathcal{C}_3$ satisfying $\phi(p_1) = p_2$ and $\phi(1-p_1) = 1-p_2$. This implies that $\phi(1) = 1$, and then, by (5.2), that ϕ is actually an algebra $*$-automorphism. Such a $*$-automorphism of $\mathcal{C}_3$ induces a $*$-automorphism of $C(K,\mathcal{C}_3)$ sending $C_{p_1}(K,\mathcal{C}_3)$ onto $C_{p_2}(K,\mathcal{C}_3)$

Lemma 5.6. *Let K be a compact subset of $[1,\infty[$ with $1 \in K$, and whose maximum element is greater than 1. Then $C_{\eta(1)}(K,\mathcal{C}_3)$ is generated by η_K as a JB^*-algebra.*

Proof. Argue as in the proof of Lemma 5.3, invoking Lemma 3.2 instead of Lemma 2.7. ∎

By invoking Theorem 3.3 and Lemma 5.6 instead of Theorem 2.8 and Lemma 5.3, respectively, the proof of the following theorem is similar to that of Theorem 5.5, and hence is omitted.

Theorem 5.7. *Let J be a JB^*-algebra, and let e be a non self-adjoint idempotent in J. Put $K := \sigma(e)$, and assume that 1 belongs to K. Then K is a compact subset of $[1,\infty[$ whose maximum element is greater than 1, and the JB^*-subalgebra of J generated by e is $*$-isomorphic to $C_p(K,\mathcal{C}_3)$ for any self-adjoint idempotent $p \in \mathcal{C}_3$ different from 0 and 1. More precisely, we have:*

1. *There exists a unique $*$-homomorphism $\Psi : C_{\eta(1)}(K,\mathcal{C}_3) \to J$ such that $\Psi(\eta_K) = e$.*
2. *Such a $*$-homomorphism is isometric, and its range coincides with the JB^*-subalgebra of J generated by e.*

6. The case of JB^*-algebras: some consequences

In this section, we deal with the main corollaries to Theorems 5.5 and 5.7.

Corollary 6.1. *Let J be a JB^*-algebra generated by a non self-adjoint idempotent e, and put $K := \sigma(e)$. If 1 is an isolated point of the compact set K, then J is $*$-isomorphic to the JB^*-algebra*

$$\mathbb{C} \times C(K \setminus \{1\}, \mathcal{C}_3).$$

Proof. Argue as in the proof of Corollary 4.1, invoking Theorem 5.7 instead of Theorem 3.3. ∎

Corollary 6.2. *Let J be a JB^*-algebra generated by a non self-adjoint idempotent e, and put $K := \sigma(e)$. Then J has a unit if and only if either* 1 *does not belong to K or* 1 *is an isolated point of K.*

Proof. Argue as in the proof of Corollary 4.2, invoking Theorems 5.5 and 5.7, and Corollary 6.1 instead of Theorems 2.8 and 3.3, and Corollary 4.1, respectively. ■

Corollary 6.3. *Let J be a JB^*-algebra. Then J has a non self-adjoint idempotent (if and) only if it contains (as a JB^*-subalgebra) a copy of either $\mathcal{C}_3$ or $C_p([1,2],\mathcal{C}_3)$ for any self-adjoint idempotent $p \in \mathcal{C}_3$ different from* 0 *and* 1.

Proof. Argue as in the proof of Corollary 4.3, invoking Theorems 5.5 and 5.7, and Corollary 6.1 instead of Theorems 2.8 and 3.3, and Corollary 4.1, respectively. ■

Arguing as in the comment following Corollary 4.3, one can realize that the JB^*-algebra $C_p([1,2],\mathcal{C}_3)$ does not contain any copy of $\mathcal{C}_3$.

Let J be a Jordan algebra. For $a, b, c \in J$, we put

$$[a, b, c] := (a \cdot b) \cdot c - a \cdot (b \cdot c).$$

The centre of J is defined as the set of those elements $a \in J$ such that $[a, J, J] = 0$. It is well-known and easy to see that central elements a of J satisfy $[J, J, a] = [J, a, J] = 0$.

Remark 6.4. In relation to Corollary 6.3 above, it is worth mentioning that a JB^*-algebra contains a non self-adjoint idempotent if and only if it contains a non central self-adjoint idempotent [**1**]. Actually, the “only if” part of the result in [**1**] just quoted follows easily from Corollary 6.3, whereas the “if part” is a consequence of Proposition 6.5 immediately below.

Proposition 6.5. *Let J be a JB^*-algebra containing a non central idempotent e. Then there exists a continuous mapping $r \to e_r$ from $[1, \infty[$ to the set of idempotents of J satisfying $e_{\|e\|} = e$ and $\|e_r\| = r$ for every $r \in [1, \infty[$.*

Proof. First assume that e is not self-adjoint. Then, invoking Theorems 5.5 and 5.7 instead of Theorems 2.8 and 3.3, respectively, and keeping in mind that, for every $t \in [1, \infty[$, $\eta(t)$ lies in $\mathcal{C}_3$, the first part of the proof of Proposition 4.5 works verbatim.

Now assume that e is self-adjoint. Since e is non central, we may apply Lemma 2.5.5 of [**6**] to find $c \in J$ such that $U_e(c) \neq e \cdot c$ or, equivalently, $[e, e, c] \neq 0$. Moreover, clearly, such an element c can be chosen self-adjoint. There is no loss of generality in assuming that J is generated by $\{e, c\}$ as a JB^*-algebra. Then, by [**12**], there exists a C^*-algebra A containing J as a JB^*-subalgebra. Put $a := i(ec - ce) \in A$, and consider the mapping $D : A \to A$ defined by $D(b) := ba - ab$ for every $b \in A$. Then D becomes a continuous derivation of A satisfying $D(b^*) = -D(b)^*$ for every $b \in A$ (since a is self-adjoint). Moreover, for every $b \in J$ we have

$$D(b) = 4i[e, b, c] \in J, \tag{6.1}$$

and consequently $D(e) \neq 0$. By the second part of the proof of Proposition 4.5, there exists a continuous function $h : [1, \infty[\to \mathbb{R}$ such that the continuous mapping $e \to e_r := \exp(h(r)D)(e)$, from $[1, \infty[$ to the set of idempotents of A, satisfies $e_1 = e$ and $\|e_r\| = r$ for every $r \in [1, \infty[$. Therefore, the proof is concluded by realizing that, for every $r \in [1, \infty[$, e_r lies in J. But this follows from the fact that, by (6.1), J is invariant under D. ∎

An element a in a JB^*-algebra J is said to be normal if the equality $[a, a, a^*] = 0$ is satisfied. In the case that the JB^*-algebra J is a JB^*-subalgebra of a given C^*-algebra A, the equality $[a, a, a^*] = 0$ in J reads in A as $[[a, a^*], a] = 0$, and hence, by arguing as in the conclusion of the proof of Lemma 4.4, it is equivalent to the usual normality in A, namely $[a, a^*] = 0$.

An element x in a JB^*-triple is said to be a tripotent if the equality $\{x, x, x\} = x$ holds. Thus, the tripotents in a C^*-algebra are precisely the partial isometries, and, more generally, the tripotents in a JB^*-algebra are precisely those elements a satisfying $U_a(a^*) = a$.

Corollary 6.6. *Let J be a JB^*-algebra. Then the following assertions are equivalent:*

1. *J contains a non central self-adjoint idempotent.*
2. *There exists a non normal tripotent $a \in J$ such that a belongs to $U_{a^2}(J)$.*

Proof. (1) $\Rightarrow$ (2).- By the assumption (1), Remark 6.4, and Theorems 5.5 and 5.7, we may assume that J is of the form $C(K, \mathcal{C}_3)$ or $C_{\eta(1)}(K, \mathcal{C}_3)$, where, in the first case, K is a compact subset of $]1, \infty[$ and, in the second case, K is a compact subset of $[1, \infty[$ whose maximum element is greater than 1 and such that $1 \in K$. In any case, by Lemma 2.2, η_{21}^K is a non normal partial isometry in J, and we have $\eta_{21}^K = U_{(\eta_{21}^K)^2}(u^2\eta_{12}^K)$, with $u^2\eta_{12}^K \in J$.

(2) $\Rightarrow$ (1).- Assume that Assertion (2) holds. We may suppose that J is generated by a as a JB^*-algebra. Since a belongs to $U_{a^2}(J)$, Lemma 1 of [**11**] applies, giving the existence of an idempotent $e \in J$ such that

$$U_a(J) = U_e(J).$$

Note that, by [**7**, pages 118-119], $U_e(J)$ is a subalgebra of J, and that e is a unit for such a subalgebra. Assume that e is self-adjoint. Then $U_e(J)$ is a JB^*-subalgebra of J, and hence, since

$$a = U_a(a^*) \in U_a(J) = U_e(J),$$

and J is generated by a as a JB^*-algebra, we deduce that $U_a(J) = J$ and that e is a unit for J. It follows from [**7**, Theorem 13 in page 52] that there exists a unique element $b \in J$ (called the "inverse" of a) such that $a = U_a(b)$, and that such a b satisfies $[a, x, b] = 0$ for every $x \in J$. Therefore we have that $b = a^*$, and then that $[a, a, a^*] = 0$, contrarily to the assumption that a is not normal. In this way we have shown that the idempotent e is not self-adjoint, and the proof is concluded by applying Remark 6.4. ∎

Comparing Corollary 6.6 with Corollary 4.7, one is tempted to conjecture that the equivalent assertions (1) and (2) in Corollary 6.6 are also equivalent to the following:

(3) J contains a non normal tripotent.

As a matter of fact, we have been unable to prove or disprove the conjecture just formulated. Actually, an eventual verification of such a conjecture would provide in particular an affirmative answer to the following unsolved question.

Problem 6.7. Let J be a JB^*-algebra containing a nonzero tripotent. Does J contain a nonzero self-adjoint idempotent?

We conclude the paper with an application to the theory of JB-algebras. JB-algebras are defined as those Banach-Jordan real algebras J satisfying $\|a\|^2 \leq \|a^2 + b^2\|$ for all $a, b \in J$. The basic reference for JB-algebras is [**6**]. By Proposition 3.8.2 of [**6**], the self-adjoint part of every JB^*-algebra becomes a JB-algebra. In particular, the self-adjoint part of the three-dimensional (complex) spin factor $\mathcal{C}_3$ is a JB-algebra, which is called the three-dimensional real spin factor, and is denoted by $\mathcal{S}_3$. We denote by $C([1, 2], \mathcal{S}_3)$ the JB-algebra of all continuous functions from $[1, 2]$ to $\mathcal{S}_3$. Moreover, given an idempotent $p \in \mathcal{S}_3$ different from 0 and 1, we denote by $C_p([1, 2], \mathcal{S}_3)$ the JB-subalgebra of $C([1, 2], \mathcal{S}_3)$ consisting of all elements $\alpha \in C([1, 2], \mathcal{S}_3)$ such that $\alpha(1)$ belongs to $\mathbb{R}p$.

Now, we have the following.

Corollary 6.8. *Let J be a JB-algebra. Then J has a non central idempotent (if and) only if it contains (as a JB-subalgebra) a copy of either $\mathcal{S}_3$ or $C_p([1, 2], \mathcal{S}_3)$ for any idempotent $p \in \mathcal{S}_3$ different from 0 and 1.*

Proof. By [**12**] and [**13**], there exists a JB^*-algebra whose self-adjoint part is equal to J. Now apply Remark 6.4 and Corollary 6.3. ∎

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