BIG POINTS IN C^*-ALGEBRAS AND JB^*-TRIPLES

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1. Introduction

Throughout this paper $\mathbb{K}$ will mean the field of real or complex numbers. Given a normed space X over $\mathbb{K}$, S_X, B_X, and X^* will denote the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X, and $\mathcal{G}_X$ will stand for the group of all surjective linear isometries from X to X. We say that an element u in a normed space X is a **big point** of X if $\overline{co}(\mathcal{G}_X(u)) = B_X$, where $\overline{co}$ means closed convex hull. Note that big points of X lie in S_X (unless $X = 0$). The normed space X is said to be **convex-transitive** if all elements in S_X are big points of X. The space X is said to be **transitive** (respectively, **almost transitive**) if, for every (equivalently, some) element u in S_X, we have $\mathcal{G}_X(u) = S_X$ (respectively, $\overline{\mathcal{G}_X(u)} = S_X$). The notions just defined provide us with a chain of implications

$$\text{transitivity} \Rightarrow \text{almost transitivity} \Rightarrow \text{convex transitivity}\,,$$

none of which is reversible.

The literature dealing with transitivity conditions on normed spaces is linked to the Banach-Mazur "rotation" problem [2] of whether every transitive separable Banach space is linearly isometric to ℓ_2. The reader is referred to the book of S. Rolewicz [49] and the survey papers of F. Cabello [19] and the authors [11] for a comprehensive view of known results and fundamental questions in relation to this matter.

In [10] we showed that the existence in a Banach space X of a big point u such that

$$\eta(X,u) := \limsup_{\|h\|\to 0} \frac{\|u+h\| + \|u-h\| - 2}{\|h\|} < 2$$

implies that X is superreflexive, and that, if in fact the norm of X is Fréchet differentiable at the big point u (equivalently, $\eta(X,u) = 0$), then X is in addition almost transitive. These results suggest that some other "smooth" behaviours of a Banach space at their big points could imply relevant properties of isomorphic or isometric type. In the present paper we choose as

1991 *Mathematics Subject Classification.* Primary 46B03, 46B04, 46L05, 46L70.
Partially supported by Junta de Andalucía grant FQM 0199.

"smooth" behaviour of a Banach space X at a point $u \in S_X$ the one that $\sigma(X,u)$ is "small". Here $\sigma(X,u)$ is defined by the equality

$$\sigma(X,u) := \sup\{\|\psi - \Pi_X(\psi)\| : \psi \in D(X^{**},u)\},$$

where, for any normed space Y, $D(Y,\cdot)$ denotes the duality mapping of Y, and $\Pi_Y : Y^{***} \to Y^*$ stands for the Dixmier projection. The constant $\sigma(X,u)$ is implicitly considered in [34], where it is shown that, if X is a Banach space, and if there is $0 \leq k < 1$ such that $\sigma(X,u) \leq k$ for every $u \in S_X$, then X is Asplund and no proper closed subspace of X^* is norming for X.

In Theorem 2.5 we prove that, if X is a Banach space, and if there is a big point u of X with $\sigma(X,u) < 1$ and such that the norm of X is strongly subdifferentiable at u, then X is Asplund and no proper closed subspace of X^* is norming for X. The strong subdifferentiability of the norm of a Banach space was introduced in [36], becoming the natural succedaneus of the Fréchet differentiability of the norm when smoothness is not required. We note that not much can be expected from the existence in a Banach space X of a big point u such that the norm of X is strongly subdifferentiable at u. Indeed, for every unital C^*-algebra X, the unit u of X is a big point of X (by the Russo-Dye theorem [15, Theorem 30.2]) such that the norm of X is strongly subdifferentiable at u (by [1, Theorem 2.7] and [48, Proposition 3]). We apply Theorem 2.5 and the tools in its proof, together with some results taken from [8] and [10], to obtain in Proposition 2.9 new characterizations of almost transitive superreflexive Banach spaces. Such spaces were first considered by C. Finet [30] (see also [26, Corollary IV.5.7]), who proves that they are uniformly smooth and uniformly convex, and have been revisited recently by F. Cabello [21] and the authors (in references [8] and [10] just quoted). Theorem 2.5 will also become one of the crucial tools in the proof of one of the main results in the paper (namely, Theorem 4.12).

Section 3 is devoted to collect other results on Banach spaces, which will be useful in the proof of the two main theorems of the paper (namely, Theorems 4.1 and 4.12). We include also a consequence of one of such auxiliary results, which will be not applied later, but has its own interest. Thus, among other things, Corollary 3.5 asserts that, if X is a complex Banach space having a big point u with $\sigma(X,u) < \frac{1}{e}$, then the commutator of $\mathcal{G}_X$ (in the algebra $\mathcal{L}(X)$ of all bounded linear operators on X) is a reflexive Banach space. We note that there exist unital C^*-algebras X such that the commutator of $\mathcal{G}_X$ contains ℓ_∞ [12, Example 3.1].

Section 4 contains the main results of the paper. Centering in a first instance in the setting of C^*-algebras, we prove that, if X is either a C^*-algebra or the predual of a von Neumann algebra, and if there is a big point u of X with $\sigma(X,u) < 2$ and such that the norm of X is strongly subdifferentiable at u, then X is finite-dimensional, and the big points of X are precisely the extreme points of B_X (Corollaries 4.6 and 4.13). In the case that X is a C^*-algebra, the assumption that the norm of X is strongly subdifferentiable at u is superabundant, and the remaining requirements on X actually characterize finite-dimensional C^*-algebras among all C^*-algebras (see again Corollary 4.6). The situation for preduals of von Neumann algebras is rather different. Indeed, although the norm of every finite-dimensional Banach space is strongly subdifferentiable at every point of its unit sphere, the dual of a finite-dimensional C^*-algebra X has big points if and only if X is a finite ℓ_∞-sum of copies of $\mathcal{L}(H)$ for some finite-dimensional complex Hilbert space (see again Corollary 4.13). The results on C^*-algebras and preduals of von Neumann algebras just reviewed follow almost straightforwardly from more general ones on JB^*-triples and preduals of JBW^*-triples (Theorems 4.1 and 4.12, respectively). The formulations of such more general results are very similar to those already reviewed in the C^*-algebra setting. Indeed, if X is either a JB^*-triple or the predual of a JBW^*-triple, and if there is a big point u of X with $\sigma(X,u) < 2$ and such that the norm of X is strongly subdifferentiable at u, then the Banach space of X is isomorphic to a Hilbert space, and the big points of X are precisely the extreme points of B_X. As above, in the case that X is a JB^*-triple, the assumption that the norm of X is strongly subdifferentiable at u is superabundant, and the remaining requirements on X characterize the JB^*-triples whose Banach spaces are isomorphic to Hilbert spaces, whereas in the case that X is the predual of a JBW^*-triple, the requirements on X characterize the preduals of those JB^*-triples which are finite ℓ_∞-sums of copies of a simple JB^*-triple of "finite rank". We note that, in the theory of JB^*-triples, the property of finite rank [42] play a roll similar to that of finite-dimensionality in the C^*-algebra setting. Moreover, we point out the well-known fact that a JB^*-triple X is of finite rank if and only if the Banach space of X is isomorphic to a Hilbert space, if and only if all single-generated subtriples of X are finite-dimensional.

In the concluding section (Section 5) we apply Theorems 4.1 and 4.12 just reviewed to obtain new characterizations of complex Hilbert spaces among JB^*-triples and preduals of JBW^*-triples. We prove that a complex Banach space X is a Hilbert space if (and only if) it is either a JB^*-triple or the predual of a JBW^*-triple, and there exists a big point u in X such that $\eta(X,u) < 2$ (Theorem 5.2). For other results in the same line the reader is referred to [51], [9] and [11].

2. A theorem for Banach spaces

Let X be a normed space over $\mathbb{K}$. For u in B_X, we define the set $D(X,u)$ of all **states** of X relative to u by

$$D(X,u) := \{f \in B_{X^*} : f(u) = 1\},$$

which is nonempty if and only if u belongs to S_X. If this is the case, then $D(X,u)$ is a nonempty w^*-closed face of B_{X^*}. The set valued function

$v \to D(X, v)$ on S_X is called the **duality mapping** of X. We denote by Π_X the canonical projection from X^{***} onto X^*, and, for u in S_X, we put

$$\sigma(X, u) := \sup\{\|\psi - \Pi_X(\psi)\| : \psi \in D(X^{**}, u)\}.$$

Given a non negative number k, we denote by $\sigma^k(X)$ the set of those elements v in S_X such that $\sigma(X, v) \leq k$. We recall that a subspace P of X^* is called a **norming subspace** for X if for every x in X we have

$$\|x\| = \sup\{|f(x)| : f \in S_P\},$$

and that X is said to be **nicely smooth** if X^* contains no proper closed norming subspace. Concerning nicely smooth Banach spaces, the reader is referred to [35] and [3].

Most results in this section are inspired by the Giles-Gregory-Sims paper [34]. In particular, the proof of Proposition 2.1 immediately below follows the lines of that of [34, Theorem 3.3]. There it is shown that, if X is a Banach space, and if there is $0 \leq k < 1$ such that $\sigma^k(X) = S_X$, then X is nicely smooth and Asplund.

Proposition 2.1. *Let X be a Banach space over $\mathbb{K}$ such that there is $0 \leq k < 1$ in such a way that the interior of $\sigma^k(X)$ relative to S_X contains big points of X. Then X is nicely smooth.*

Proof. Assume that there exists a proper closed norming subspace of X^* (say P). By Riesz's lemma, there exists h in S_{X^*} satisfying $\|h + P\| > k$. Now, let u and δ be a big point of X and a positive number, respectively, such that x belongs to $\sigma^k(X)$ whenever $x \in S_X$ and $\|x - u\| \leq \delta$. Since u is a big point of X, the set

$$\{T^*(f) : f \in D(X, x)\, , x \in S_X,\ \|x - u\| \leq \delta\ , T \in \mathcal{G}_X\ \}$$

is dense in S_{X^*} [11, Lemma 5.7]. It follows that there exist $T \in \mathcal{G}_X$, $x \in \sigma^k(X)$, and $f \in D(X, x)$ satisfying $\|T^*(f) + P\| > k$. Then, putting $Q := (T^*)^{-1}(P)$, we have $\|f + Q\| > k$, and hence the Hahn-Banach theorem provides us with some α in $S_{X^{**}}$ such that $\alpha(Q) = 0$ and $|\alpha(f)| > k$. Since Q is a norming subspace of X^*, we have $\|y\| \leq \|y + \beta\|$ for every y in X and every β in the polar Q° of Q in X^{**}. This implies that $X \cap Q^\circ = 0$ and that the unique linear extension of f to $X \oplus Q^\circ$ which vanishes on Q° is in fact a Hahn-Banach extension (say g). Now, take a Hahn-Banach extension of g to X^{**} (say ψ). Since α belongs to Q°, the equality $\psi(\alpha) = 0$ holds. On the other hand, since ψ extends f, we have $\Pi_X(\psi) = f$. It follows

$$\|\psi - \Pi_X(\psi)\| \geq |(\psi - \Pi_X(\psi))(\alpha)| = |f(\alpha)| > k.$$

Since ψ belongs to $D(X^{**}, x)$, and x belongs to $\sigma^k(X)$, the inequality $\|\psi - \Pi_X(\psi)\| > k$ just obtained becomes a contradiction. ∎

Let X be a Banach space fulfilling the requirements in Proposition 2.1. As a consequence of that proposition, we are provided with the relevant isomorphic property that $\text{dens}(X^*) = \text{dens}(X)$, where $\text{dens}(.)$ means density character. In what follows we will prove that all requirements on X are inherited by certain closed separable subspaces of X, the abundance of which is enough to deduce that X becomes an Asplund space. The following lemma is a variant of [20, Theorem 1.2].

Lemma 2.2. *Let X be a normed space over $\mathbb{K}$, M be a separable subspace of X, and u a big point of X. Then there exists a closed separable subspace N of X containing $\mathbb{K}u + M$ and such that u becomes a big point of N.*

Proof. Put $Y_1 := \mathbb{K}u + M$, and choose a countable dense subset D_1 of S_{Y_1}. For d in D_1 and m in $\mathbb{N}$, there exists a finite subset $\mathcal{G}_{d,m}$ of $\mathcal{G}_X$ such that the distance from d to the convex hull of $\mathcal{G}_{d,m}(u)$ is less than $\frac{1}{m}$. Denoting by $\mathcal{G}_1$ the subgroup of $\mathcal{G}_X$ generated by $\bigcup_{(d,m)\in D_1\times\mathbb{N}} \mathcal{G}_{d,m}$, $\mathcal{G}_1$ is a countable set and we have $D_1 \subseteq \overline{co}(\mathcal{G}_1(u))$. Now, denote by Y_2 the closed linear hull of $\mathcal{G}_1(Y_1)$, and choose a countable dense subset D_2 of S_{Y_2} containing D_1. Minor changes in the above argument allow us to show the existence of a countable subgroup $\mathcal{G}_2$ of $\mathcal{G}_X$ containing $\mathcal{G}_1$ and satisfying $D_2 \subseteq \overline{co}(\mathcal{G}_2(u))$. Proceeding in such a way we obtain increasing sequences $\{Y_n\}$, $\{D_n\}$, and $\{\mathcal{G}_n\}$ such that, for every n in $\mathbb{N}$, Y_n is a separable subspace of X containing $\mathbb{K}u + M$, D_n is a dense subset of S_{Y_n}, $\mathcal{G}_n$ is a subgroup of $\mathcal{G}_X$ satisfying $\mathcal{G}_n(Y_n) \subseteq Y_{n+1}$, and the inclusion $D_n \subseteq \overline{co}(\mathcal{G}_n(u))$ holds. Now, the desired subspace N is nothing but the closure of $\bigcup_{n\in\mathbb{N}} Y_n$ in X. Indeed, putting $\mathcal{G}_\infty := \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$, $\mathcal{G}_\infty$ becomes a subgroup of $\mathcal{G}_X$, N is $\mathcal{G}_\infty$-invariant (so that $\mathcal{G}_\infty$ can be regarded as a subgroup of $\mathcal{G}_N$), and we have $\overline{co}(\mathcal{G}_\infty(u)) = B_N$. ∎

For background on Asplund spaces the reader is referred to [16] and [26].

Proposition 2.3. *Let X be a Banach space over $\mathbb{K}$ such that there is $0 \leq k < 1$ in such a way that the interior of $\sigma^k(X)$ relative to S_X contains big points of X. Then X is Asplund.*

Proof. Let M be a separable subspace of X. We are going to show that M^* is separable. Let u be a big point of X in the interior of $\sigma^k(X)$ relative to S_X. By Lemma 2.2, there exists a closed separable subspace N of X containing $\mathbb{K}u+M$ and such that u is a big point of N. On the other hand, as remarked in the proof of [34, Theorem 3.3], we have $\sigma^k(X) \cap N \subseteq \sigma^k(N)$. Now u is a big point of N in the interior of $\sigma^k(N)$ relative to S_N, so that Proposition 2.1 applies with N instead of X to obtain that $\mathrm{dens}(N^*) = \mathrm{dens}(N)$. Thus N^* (and hence M^*) is separable. ∎

Let X be a normed space over $\mathbb{K}$, and let u be in S_X. For x in X, the mapping $\alpha \to \|u + \alpha x\|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex, and hence there exists

$$\tau(u, x) := \lim_{\alpha\to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha}.$$

It is well-known that, for x in X, the equality

$$\tau(u, x) = \max\{\Re e(\lambda) : \lambda \in V(X, u, x)\}$$

holds, where $V(X,u,x)$ (called the **numerical range** of x relative to (X,u)) is defined by

$$V(X,u,x) := \{f(x) : f \in D(X,u)\}$$

(see for instance [28, Theorem V.9.5]). We say that the norm of X is **strongly subdifferentiable** at u if

$$\lim_{\alpha \to 0^+} \frac{\|u+\alpha x\|-1}{\alpha} = \tau(u,x) \text{ uniformly for } x \in B_X.$$

The reader is referred to [36], [1], [31], and [13] for a comprehensive view of the usefulness of the strong subdifferentiability of the norm in the theory of Banach spaces. We note that the Fréchet differentiability of the norm of X at u is nothing but the strong subdifferentiability of the norm of X at u together with the smoothness of X at u.

Lemma 2.4. *Let X be Banach space over $\mathbb{K}$, and let u be in S_X such that the norm of X is strongly subdifferentiable at u. Then the mapping $x \to \sigma(X,x)$ from S_X to $\mathbb{R}$ is upper semicontinuous at u.*

Proof. Let ε be a positive number. By [36, Corollary 4.4] and [34, Corollary 2.1], there exists $\delta > 0$ such that the inclusion

$$D(X^{**},\alpha) \subseteq D(X^{**},u) + \frac{\varepsilon}{3} B_{X^{***}}$$

holds whenever α is in $S_{X^{**}}$ with $\|\alpha - u\| < \delta$. Let x be in S_X such that $\|x-u\| < \delta$. Given ψ in $D(X^{**},x)$, there exists φ in $D(X^{**},u)$ satisfying $\|\psi - \varphi\| \leq \frac{\varepsilon}{3}$, and therefore we have

$$\|\psi - \Pi_X(\psi)\| = \|\psi - \varphi - \Pi_X(\psi - \varphi) + \varphi - \Pi_X(\varphi)\| \leq \frac{2\varepsilon}{3} + \sigma(X,u).$$

Since ψ is arbitrary in $D(X^{**},x)$, we obtain $\sigma(X,x) < \varepsilon + \sigma(X,u)$. ∎

According to Lemma 2.4, if X is a Banach space, if u is a norm-one element of X, and if the norm of X is strongly subdifferentiable at u, then u lies in the interior of $\sigma^k(X)$ relative to S_X for every real number $k > \sigma(X,u)$. Therefore Theorem 2.5 immediately below follows from Propositions 2.1 and 2.3.

Theorem 2.5. *Let X be a Banach space over $\mathbb{K}$. Assume that there exists a big point u of X satisfying $\sigma(X,u) < 1$ and such that the norm of X is strongly subdifferentiable at u. Then X is nicely smooth and Asplund.*

Let X be a normed space, and let u be a norm-one element in X. We denote by $\delta(X,u)$ the diameter of $D(X,u)$. Since $\sigma(X,u) \leq \delta(X^{**},u)$, the next corollary follows straightforwardly from Propositions 2.1 and 2.3.

Corollary 2.6. *Let X be a Banach space over $\mathbb{K}$. Assume that there is $0 \leq k < 1$ such that the interior of the set $\{u \in S_X : \delta(X^{**},u) \leq k\}$ relative to S_X contains big points of X. Then X is nicely smooth and Asplund.*

Let X be a normed space, let u be in S_X, and let τ be a vector space topology on X^*. Following [34], we say that **the duality mapping of X is upper semicontinuous** $(n-\tau)$ **at** u if for every τ-neighborhood of zero (say B) in X^* there exists a norm-neighborhood of u (say C) in S_X such that $D(X,x) \subseteq D(X,u)+B$ whenever x belongs to C. Denote by w and n the weak and norm topologies on X^*, respectively. Corollary 4.4 of [36], applied in the proof of Lemma 2.4, asserts that the strong subdifferentiability of the norm of X at u is equivalent to the upper semicontinuity $(n-n)$ of the duality mapping of X at u, and hence implies the upper semicontinuity $(n-w)$ of the duality mapping of X at u. On the other hand, if X is complete, then the upper semicontinuity $(n-w)$ of the duality mapping of X at u is equivalent to the fact that $D(X,u)$ is dense in $D(X^{**},u)$ for the w^*-topology of X^{***} [36, Theorem 3.1], which implies $\delta(X^{**},u)=\delta(X,u)$ and, consequently $\sigma(X,u) \leq \delta(X,u)$.

Now, let X be a Banach space such that there exists a big point u of X in such a way that the norm of X is strongly subdifferentiable at u. It follows from Theorem 2.5 and the above comments that, if $\delta(X,u)<1$, then X is nicely smooth and Asplund. Actually, a better result holds. Indeed, X is superreflexive whenever $\delta(X,u)<2$ (see Corollary 2.8 below). Through Lemma 2.7 which follows, this result is germinally contained in [10].

For any norm-one element u in a normed space X, we define the **roughness of X at** u, $\eta(X,u)$, by

$$\eta(X,u) := \limsup_{\|h\|\to 0} \frac{\|u+h\|+\|u-h\|-2}{\|h\|}.$$

The absence of roughness of X at u (i.e., $\eta(X,u)=0$) is nothing but the Fréchet differentiability of the norm of X at u [26, Lemma I.1.10].

Lemma 2.7. *Let X be a normed space over $\mathbb{K}$, and u a norm-one element of X. Then $\delta(X,u) \leq \eta(X,u)$. If in addition the norm of X is strongly subdifferentiable at u, then we have $\delta(X,u)=\eta(X,u)$.*

Proof. We may assume that $\mathbb{K}=\mathbb{R}$. Let Γ denote the mapping

$$h \to \frac{\|u+h\|+\|u-h\|-2}{\|h\|}$$

from $X \setminus \{0\}$ into $\mathbb{R}$. For x in S_X we have

$$\max\{\lambda : \lambda \in V(X,u,x)\} - \min\{\mu : \mu \in V(X,u,x)\}$$
$$= \lim_{\alpha\to 0^+} \frac{\|u+\alpha x\|+\|u-\alpha x\|-2}{\alpha},$$

and hence

$$\max\{\lambda : \lambda \in V(X,u,x)\} - \min\{\mu : \mu \in V(X,u,x)\}$$

becomes a cluster point for Γ when $\|h\| \to 0$. Since

$$\delta(X,u) = \sup\{\max\{\lambda : \lambda \in V(X,u,x)\} - \min\{\mu : \mu \in V(X,u,x)\} : x \in S_X\},$$

we deduce that $\delta(X,u)$ is also a cluster point for Γ when $\|h\| \to 0$. Therefore we obtain $\delta(X,u) \le \limsup_{\|h\|\to 0} \Gamma(h) = \eta(X,u)$.

Now suppose that the norm of X is strongly subdifferentiable at u. Let $\varepsilon > 0$. We can find $\rho > 0$ such that

$$\frac{\|u+\alpha x\|-1}{\alpha} < \tau(u,x) + \frac{\varepsilon}{2}$$

whenever x is in B_X and $0 < \alpha < \rho$. Therefore, for $0 < \|h\| < \rho$ we have

$$\Gamma(h) = \frac{\|u+h\|+\|u-h\|-2}{\|h\|} < \tau(u, \frac{h}{\|h\|}) + \tau(u, -\frac{h}{\|h\|}) + \varepsilon$$

$$= \max\{\lambda : \lambda \in V(X,u,\frac{h}{\|h\|})\} - \min\{\mu : \mu \in V(X,u,\frac{h}{\|h\|})\} + \varepsilon \le \delta(X,u) + \varepsilon.$$

The above shows $\delta(X,u) \ge \limsup_{\|h\|\to 0} \Gamma(h) = \eta(X,u)$. ∎

It is proved in [10] that, if X is a Banach space, and if there exists a big point u of X such that $\eta(X,u) < 2$, then X is superreflexive. Together with Lemma 2.7 above, this yields the following corollary.

Corollary 2.8. *Let X be a Banach space over $\mathbb{K}$. Assume that there exists a big point u of X satisfying $\delta(X,u) < 2$ and such that the norm of X is strongly subdifferentiable at u. Then X is superreflexive.*

From now on, $\mathcal{J}$ will denote the class of almost transitive superreflexive Banach spaces. This class is reasonably large (see for example [8, Remark 4.3]). A systematic study of the class $\mathcal{J}$ has been first made by C. Finet [30] (see also [26, Corollary IV.5.7]). She proves that every member of $\mathcal{J}$ is uniformly smooth and uniformly convex. Recently, the class $\mathcal{J}$ has been revisited by F. Cabello [21] and the authors (see [8] and [10]). According to [8, Corollary 3.3], members of $\mathcal{J}$ are nothing but convex-transitive Asplund spaces. The results in [30] and [8] just quoted, together with Proposition 2.3, Theorem 2.5, Corollaries 2.6 and 2.8, and the comments after Corollary 2.6 directly yield some new characterizations of members of $\mathcal{J}$, which we list in the following proposition.

Proposition 2.9. *Let X be a Banach space over $\mathbb{K}$. Each of the following conditions, added to the convex transitivity of X, characterizes X as a member of $\mathcal{J}$.*

1. *There exists $0 \le k < 1$ such that $\sigma^k(X)$ has nonempty interior relative to S_X.*
2. *There is a norm-one element u in X satisfying $\sigma(X,u) < 1$ and such that the norm of X is strongly subdifferentiable at u.*
3. *There exists $0 \le k < 1$ such that the set of those elements u in S_X satisfying $\delta(X^{**},u) \le k$ has nonempty interior relative to S_X.*
4. *There exists $0 \le k < 1$ such that the set of those elements u in S_X such that the duality mapping of X is upper semicontinuous $(n-w)$ at u and $\delta(X,u) \le k$ has nonempty interior relative to S_X.*

5. *There is a norm-one element u in X satisfying $\delta(X,u) < 2$ and such that the norm of X is strongly subdifferentiable at u.*

Of course, suitable strengthenings of Conditions 1 to 5, added to the convex transitivity of a Banach space X, also characterize X as a member of $\mathcal{J}$. For instance, a convex transitive Banach space X lies in $\mathcal{J}$ if (and only if) $\sigma^0(X)$ has nonempty interior relative to S_X, if (and only if) $\sigma^0(X) = S_X$ (i.e. $D(X,u) = D(X^{**},u)$ for every norm-one element u of X). In fact, the last characterization follows straightforwardly from [34] and [8]. Note that non-reflexive Banach spaces X satisfying $\sigma^0(X) = S_X$ do exist. Indeed, take X equal to c_0, or the space of all compact operators on a Hilbert space, or (more generally) any nontrivial "M-embedded" Banach space (see Lemma 3.1 below). Another consequence of Proposition 2.9 is that a convex-transitive Banach space X lies in $\mathcal{J}$ if (and only if) there is a norm-one element u in X such that the norm of X is Fréchet differentiable at u. However, as commented in Section 1, a better result holds. Indeed, a Banach space X is a member of $\mathcal{J}$ if (and only if) there exists a big point u of X such that the norm of X is Fréchet differentiable at u [10]. In relation to Condition 4 in Proposition 2.9, it is worth mentioning that a convex-transitive Banach space X lies in $\mathcal{J}$ if (and only if) the duality mapping of X is upper semicontinuous $(n-w)$ at every element of S_X. This follows from [23] and [8].

3. Other auxiliary results

Let X be a Banach space over $\mathbb{K}$. An **L-projection** on X is a linear projection (say π) on X satisfying

$$\|x\| = \|\pi(x)\| + \|x - \pi(x)\|$$

for every $x \in X$. A subspace M of X is said to be an **L-summand** of X if it is the range of an L-projection on X, and an **M-ideal** of X if M° (the polar of M in X^*) is an L-summand of X^*. X is said to be **L-embedded** (respectively, **M-embedded**) whenever X is an L-summand (respectively, an M-ideal) of X^{**}. According to [37, Proposition III.1.2], X is M-embedded if and only if Π_X is an L-projection on X^{***}. Consequently, if X is M-embedded, then X^* is L-embedded.

Lemma 3.1. *Let X be a Banach space over $\mathbb{K}$, and let u be in S_X. If X^* is L-embedded, then either $\sigma(X,u) = 0$ or $\sigma(X,u) = 2$. If actually X is M-embedded, then $\sigma(X,u) = 0$.*

Proof. Assume that X^* is L-embedded and that $\sigma(X,u) \neq 0$. Then X^* is the range of an L-projection π on X^{***}, and there exists $\psi \in D(X^{**},u) \setminus X^*$. Thus we have

$$1 = \Re e[\pi(\psi)(u)] + \Re e[(1-\pi)(\psi)(u)] \leq \|\pi(\psi)\| + \|(1-\pi)(\psi)\| = 1,$$

so $\Re e[(1-\pi)(\psi)(u)] = \|(1-\pi)(\psi)\|$, and so $\phi := \frac{(1-\pi)(\psi)}{\|(1-\pi)(\psi)\|}$ belongs to $D(X^{**},u)$. Moreover, since ϕ belongs to $\ker(\pi)$, and $\Pi_X(\phi)$ belongs to $\pi(X^{***})$, we have

$$\|\phi - \Pi_X(\phi)\| = \|\phi\| + \|\Pi_X(\phi)\| = 2,$$

and hence $\sigma(X,u) = 2$.

Now assume that X is M-embedded. Let ψ be in $D(X^{**},u)$. Then, since Π_X is an L-projection on X^{***}, we have

$$\|(1-\Pi_X)(\psi)\| = 1 - \|\Pi_X(\psi)\| \leq 1 - |\Pi_X(\psi)(u)| = 1 - \psi(u) = 0.$$

Since ψ is arbitrary in $D(X^{**},u)$, we deduce $\sigma(X,u) = 0$. ■

Let X be a normed space over $\mathbb{K}$, and let u be in S_X. For x in X, the **numerical radius** of x relative to (X,u), denoted by $v(X,u,x)$, is defined by the equality

$$v(X,u,x) := \sup\{|\lambda| : \lambda \in V(X,u,x)\}.$$

The **numerical index** of (X,u), denoted by $n(X,u)$, is the number

$$n(X,u) := \max\{r \geq 0 : r\|x\| \leq v(X,u,x) \text{ for all } x \text{ in } X\}.$$

Lemma 3.2. *Let X be a Banach space over $\mathbb{K}$, and let u be in S_X. Then we have $\|1-\Pi_X\| n(X,u) \leq \sigma(X,u)$.*

Proof. Let (x^{**}, x^{***}) be in $X^{**} \times B_{X^{***}}$. Since $n(X^{**},u) = n(X,u)$ [46, Lemma 4.8], we have

$$n(X,u)|x^{***}(x^{**})| \leq v(X^{**},u,x^{**}).$$

Since x^{**} is arbitrary in X^{**}, it follows from [46, Theorem 3.1] that $n(X,u)x^{***}$ belongs to the norm closure of the absolutely convex hull of $D(X^{**},u)$. Therefore, since x^{***} is arbitrary in $B_{X^{***}}$, for every bounded linear operator $F : X^{***} \to X^{***}$ we have

$$n(X,u)\|F\| \leq \sup\{\|F(\psi)\| : \psi \in D(X^{**},u)\}.$$

Now the result follows by taking $F = 1 - \Pi_X$. ■

Corollary 3.3. *Let X be a Banach space over $\mathbb{K}$, and let u be in S_X. If $\sigma(X,u) < 2n(X,u)$, then X does not contain an isomorphic copy of ℓ_1. If actually $\sigma(X,u) < n(X,u)$, then X is reflexive.*

Proof. If $\sigma(X,u) < 2n(X,u)$, then from Lemma 3.2 we deduce that $\|1-\Pi_X\| < 2$, and therefore, by [22, Proposition 2], X does not contain an isomorphic copy of ℓ_1. If $\sigma(X,u) < n(X,u)$, then again from Lemma 3.2 we deduce that $\|1-\Pi_X\| < 1$, so $\Pi_X = 1$, and so X is reflexive. ■

Corollary 3.4. *Let X denote the real or complex space ℓ_1, and let u be any element in the canonical basis of X. Then $\sigma(X,u) = 2$.*

Proof. From the fact that $\mathbb{K}u$ is an L-summand of X we easily deduce that $n(X,u)=1$. Now apply Corollary 3.3 ∎

The next corollary will not be applied in the remaining part of this paper, but has its own interest. It follows the line of [12, Theorem 3.2], where it is shown that, if X is a complex normed space having a big point u with $\delta(X,u)<\frac{\sqrt{3}}{e}$, then $\mathrm{Com}(\mathcal{G}_X)$ is equal to $\mathbb{C}I_X$. Here $\mathrm{Com}(\mathcal{G}_X)$ stands for the set of those bounded linear operators on X which commute with all elements of $\mathcal{G}_X$, and I_X denotes the identity mapping on X. For any normed space X, define the **normed space numerical index**, $N(X)$, of X by $N(X):=n(\mathcal{L}(X),I_X)$. The arguments in the proof of [12, Theorem 3.2] actually show that $\mathrm{Com}(\mathcal{G}_X)=\mathbb{C}I_X$ whenever X is a real (respectively, complex) normed space having a big point u with $\delta(X,u)<2N(X)$ (respectively, $\delta(X,u)<\sqrt{3}N(X)$). The original formulation of [12, Theorem 3.2] quoted above follows from the refinement just pointed out and the Bohnenblust-Karlin theorem [14, Theorem 4.1] that $N(X)\geq\frac{1}{e}$ when X is a complex normed space.

Corollary 3.5. *Let X be a Banach space over $\mathbb{K}$ having a big point u. Then we have:*

(a) *If $\sigma(X,u)<2N(X)$, then $\mathrm{Com}(\mathcal{G}_X)$ does not contain an isomorphic copy of ℓ_1.*
(b) *If $\sigma(X,u)<N(X)$, then $\mathrm{Com}(\mathcal{G}_X)$ is reflexive.*
(c) *If $\mathbb{K}=\mathbb{C}$, and if $\sigma(X,u)<\frac{2}{e}$, then $\mathrm{Com}(\mathcal{G}_X)$ does not contain an isomorphic copy of ℓ_1.*
(d) *If $\mathbb{K}=\mathbb{C}$, and if $\sigma(X,u)<\frac{1}{e}$, then $\mathrm{Com}(\mathcal{G}_X)$ is reflexive.*
(e) *If X is M-embedded, and if $N(X)>0$, then $\mathrm{Com}(\mathcal{G}_X)$ is reflexive.*
(f) *If $\mathbb{K}=\mathbb{C}$, and if X is M-embedded, then $\mathrm{Com}(\mathcal{G}_X)$ is reflexive.*

Proof. We have clearly

$$N(X)=n(\mathcal{L}(X),I_X)\leq n(\mathrm{Com}(\mathcal{G}_X),I_X).$$

On the other hand, for $F\in\mathrm{Com}(\mathcal{G}_X)$, the set $\{x\in X:\|F(x)\|\leq\|F(u)\|\}$ is closed, convex, and $\mathcal{G}_X$-invariant, so that the bigness of u gives $\|F\|=\|F(u)\|$. Therefore the mapping $F\to F(u)$ from $\mathrm{Com}(\mathcal{G}_X)$ to X is a linear isometry sending I_X into u, and hence we have

$$\sigma(\mathrm{Com}(\mathcal{G}_X),I_X)=\sigma(\mathrm{Com}(\mathcal{G}_X)(u),u)\leq\sigma(X,u).$$

Now, (a) and (b) follow from Corollary 3.3, whereas (c) and (d) follow from (a) and (b), respectively, and the Bohnenblust-Karlin theorem. Finally, keeping in mind Lemma 3.1, (e) and (f) follow from (b) and (d), respectively. ∎

The following lemma is known in the case $\tau=n$ (see [31, Theorem 1.2 and Proposition 3.1]).

Lemma 3.6. *Let X be a Banach space over $\mathbb{K}$, let u, v be in S_X such that $D(X, u) = D(X, v)$, and let τ be a vector space topology on X^*. If the duality mapping of X is upper semicontinuous $(n - \tau)$ at u, then the duality mapping of X is upper semicontinuous $(n - \tau)$ at v.*

Proof. Assume that the duality mapping of X is upper semicontinuous $(n - \tau)$ at u but not at v. Then, by [34, Theorem 2.1], there exists a τ-neighborhood N of zero in X^* such that for every $n \in \mathbb{N}$ we can find $f_n \in B_{X^*}$ satisfying

$$\Re e(f_n(v)) > \frac{n}{n+1} \tag{3.1}$$

and

$$f_n \notin D(X, v) + N, \tag{3.2}$$

and there exists $0 < \delta < 1$ such that

$$\{g \in B_{X^*} : \Re e(g(u)) > \delta\} \subseteq D(X, u) + N. \tag{3.3}$$

Take a cluster point f to the sequence $\{f_n\}$ in the w^*-topology of X^*. Then, by (3.1), we have $f \in D(X, v) = D(X, u)$, so $1 = f(u)$ is a cluster point of $\{\Re e(f_n(u))\}$, and so there is $m \in \mathbb{N}$ with $\Re e(f_m(u)) > \delta$. By (3.3), f_m belongs to $D(X, u) + N$. But, applying again that $D(X, u) = D(X, v)$, this contradicts (3.2). ∎

In Corollaries 3.7 and 3.8 which follow we emphasize the cases $\tau = n$ and $\tau = w$ of Lemma 3.6, respectively, adding some peculiar information for such cases. We recall that strong subdifferentiability of the norm of a normed space X at a point $u \in S_X$ is nothing but the upper semicontinuity $(n - n)$ of the duality mapping of X at u [36, Corollary 4.4]

Corollary 3.7. *Let X be a Banach space over $\mathbb{K}$, and let u, v be in S_X such that $D(X, u) = D(X, v)$. If the norm of X is strongly subdifferentiable at u, then the norm of X is strongly subdifferentiable at v. If in addition u is a big point of X, then v is also a big point of X.*

Proof. Assume that the norm of X is strongly subdifferentiable at u and v, and that u is a big point of X. Then, by the equality $D(X, u) = D(X, v)$ and the equivalence $2 \Longleftrightarrow 6$ in [6, Corollary 3.6], v is a big point of X ∎

Corollary 3.8. *Let X be a Banach space over $\mathbb{K}$, and let u, v be in S_X such that $D(X, u) = D(X, v)$. If the duality mapping of X is upper semicontinuous $(n - w)$ at u, then the duality mapping of X is upper semicontinuous $(n - w)$ at v, and we have $\sigma(X, u) = \sigma(X, v)$.*

Proof. We recall that, since X is complete, the upper semicontinuity $(n - w)$ of the duality mapping of X at a point $x \in S_X$ is equivalent to the fact that $D(X, x)$ is dense in $D(X^{**}, x)$ for the w^*-topology of X^{***} [34, Theorem 3.1]. Therefore, if the duality mapping of X is upper semicontinuous $(n - w)$ at

u and v, then, by the assumption $D(X,u)=D(X,v)$, we have $D(X^{**},u)=D(X^{**},v)$, which implies $\sigma(X,u)=\sigma(X,v)$. ∎

Corollary 3.9. *Let X be a Banach space over $\mathbb{K}$, and let u,v be in S_X such that $D(X,u)=D(X,v)$. If $\sigma(X,u)=0$, then also $\sigma(X,v)=0$.*

Proof. Assume that $\sigma(X,u)=0$. Then we have $D(X^{**},u)=D(X,u)$, and hence, by Theorem 3.1 of [34] just applied in the proof of Corollary 3.8, the duality mapping of X is upper semicontinuous at u. Now apply Corollary 3.8. ∎

4. The main results

We recall that a JB^***-triple** is a complex Banach space X with a continuous triple product $\{\cdot,\cdot,\cdot\}:X\times X\times X\to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

1. For all x in X, the mapping $y\to\{x,x,y\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
2. The **main identity**

   $$\{a,b,\{x,y,z\}\}=\{\{a,b,x\},y,z\}-\{x,\{b,a,y\},z\}+\{x,y,\{a,b,z\}\}$$

   holds for all a,b,x,y,z in X.
3. $\| \{x,x,x\} \| = \| x \|^3$ for every x in X.

Concerning Condition 1 above, we also recall that a bounded linear operator T on a complex Banach space X is said to be **hermitian** if $\exp(irT)$ belongs to $\mathcal{G}_X$ for every r in $\mathbb{R}$.

JB^*-triples are of capital importance in the study of bounded symmetric domains in complex Banach spaces. Indeed, open balls in JB^*-triples are bounded symmetric domains, and every symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a suitable JB^*-triple (see [41] and [43]). Examples of JB^*-triples are all C^* algebras under the triple product

$$\{x,y,z\}:=\frac{1}{2}(xy^*z+zy^*x), \tag{4.1}$$

the spaces $\mathcal{L}(H_1,H_2)$ (bounded linear operators) for arbitrary complex Hilbert spaces H_1 and H_2 (with triple product formally defined as in (4.1)), and the so-called **spin factors**. These are constructed from an arbitrary complex Hilbert space $(H,(\cdot|\cdot))$ of hilbertian dimension ≥ 3, by taking a conjugate-linear involutive isometry σ on H, and then by defining the triple product and the norm by

$$\{x,y,z\}:=(x|y)z+(z|y)x-(x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2:=(x|x)+\sqrt{(x|x)^2-|(x|\sigma(x))|^2},$$

respectively, for all x, y, z in H. Other examples can be obtained by noticing that the class of JB^*-triples is closed under $\ell_\infty-$ and c_0-sums.

Our first main result reads as follows.

Theorem 4.1. *Let X be a JB^*-triple. Then the following conditions are equivalent:*

1. *There exists a big point v of X such that $\sigma(X, v) < 2$.*
2. *The Banach space of X is isomorphic to a Hilbert space.*
3. *X is M-embedded and has big points.*

Moreover, if the above conditions are fulfilled, then big points of X, denting points of B_X, and extreme points of B_X coincide.

The proof of Theorem 4.1 involves a big amount of background on JB^*-triples, a part of which is being recalled before formally attacking such a proof. First of all, we recall that linear mappings preserving triple products between JB^*-triples are called **triple homomorphisms**, and that triple isomorphisms (i.e., bijective triple homomorphisms) between JB^*-triples are nothing but surjective linear isometries [43].

Let X be a JB^*-triple. A **subtriple** (respectively, **triple ideal**) of J is a subspace M of J such that $\{MMM\} \subseteq M$ (respectively, $\{MJJ\} + \{JMJ\} \subseteq M$). We say that X is **simple** (respectively, **topologically simple**) if there are no triple ideals (respectively, closed triple ideals) of X others than $\{0\}$ and X. Now let x be in X, and denote by X_x the closed subtriple of X generated by X. It is well-known that there is a unique couple (S_x, ϕ_x), where S_x is a locally compact subset of $]0, \infty[$ such that $S_x \cup \{0\}$ is compact, and ϕ_x is a triple isomorphism from X_x onto the C^*-algebra $C_0(S_x)$ (of all complex-valued continuous functions on S_x vanishing at infinity), such that $\phi_x(x)$ is the inclusion mapping $S_x \hookrightarrow \mathbb{C}$ (see [41, 4.8], [43, 1.15], and [32]). Following [17], we say that the JB^*-triple X is **elementary** if it is of one of the following types: $K(H_1, H_2)$ (compact operators) for complex Hilbert spaces H_1 and H_2, $\{x \in K(H) : x = -\theta x^* \theta\}$ for a complex Hilbert space H and a conjugation θ on H, $\{x \in K(H) : x = \theta x^* \theta\}$ for H and θ as above, a spin factor, the JB^*-triple consisting of all 1×2 matrices over the complex Cayley numbers, or the JB^*-triple consisting of all hermitian 3×3 matrices over the complex Cayley numbers. An element x of X is said to be weakly compact if the conjugate-linear operator $\{x, \cdot, x\}$ is weakly compact, and the JB^*-triple X is called **weakly compact** whenever every element of X is weakly compact.

Weakly compact JB^*-triples are well-understood thanks to the results in [18]. For instance, elementary JB^*-triples are nothing but topologically simple weakly compact JB^*-triples. Actually, keeping in mind [5, Theorem 3.2], the following characterizations of weakly compact JB^*-triples follow from [18].

Lemma 4.2. *Let X be a JB^* triple. Then the following assertions are equivalent:*

1. *X is weakly compact*
2. *S_x is discrete for every $x \in X$.*
3. *X is M-embedded.*
4. *X is the c_0-sum of a suitable family of elementary JB^*-triples.*

Let X be a JB^*-triple. An element u of X is said to be a **tripotent** if $\{u,u,u\} = u$. Given a tripotent u in X, we have $X = X_0(u) \oplus X_1(u) \oplus X_2(u)$, where, for $j \in \{0,1,2\}$, $X_j(u)$ denotes the eigenspace of the operator $\{u,u,\cdot\}$ corresponding to the eigenvalue $\frac{1}{2}j$. Following [42], the tripotent u is said to be **maximal** (respectively, **minimal**) if $X_0(u) = 0$ (respectively, if $u \neq 0$ and $X_2(u) = \mathbb{C}u$). Two tripotents u, v in X are said to be **orthogonal** if $\{u,v,X\} = \{v,u,X\} = 0$. The results of [18], together with [32, Lemma 2.11] and [4, p. 270], lead to the next lemma.

Lemma 4.3. *Let X be a weakly compact JB^*-triple. Then we have:*

1. *There exists a canonical bijection from the set of all extreme points of B_{X^*} to the set of all minimal tripotents of X, which extends to an injective and contractive and conjugate-linear mapping $\pi : X^* \to X$.*
2. *The mapping $(f,g) \to (f|g) := f(\pi(g))$ from $X^* \times X^*$ to $\mathbb{C}$ becomes an inner product on X^*.*
3. *For x in X there are (possibly finite) sequences $\{\lambda_n\}$ of positive numbers and $\{u_n\}$ of pair-wise orthogonal minimal tripotents of X such that $x = \sum_n \lambda_n u_n$ (which implies $\|x\| = \max_n\{\lambda_n\}$). Moreover, x lies in the range of π if and only if $\sum_n \lambda_n < \infty$, and if this is the case, then, taking f in X^* with $\pi(f) = x$, we have $\|f\| = \sum_n \lambda_n$ and $(f|f) = \sum_n \lambda_n^2$.*

Let X be a JB^*-triple. By a **frame** in X we mean a family $\mathcal{E}$ of pair-wise orthogonal minimal tripotents of X such that $\bigcap_{u \in \mathcal{E}} X_0(u) = 0$. We say that X is of **finite rank** if there exists a finite frame in X. As a by-product of Lemma 4.4 immediately below, we realize that JB^*-triples of finite rank are weakly compact. For the proof of Lemma 4.4 the reader is referred to [42, (2.15) and (4.10)] and [18, Proposition 4.5.(iii) and its proof], noticing that the implication $4 \Rightarrow 5$ in the lemma is clear.

Lemma 4.4. *Let X be a JB^* triple. Then the following assertions are equivalent:*

1. *X is of finite rank.*
2. *X has the Radon-Nikodym property*
3. *S_x is finite for all $x \in X$.*
4. *X is a finite ℓ_∞-sum of closed simple triple ideals which are either finite-dimensional, spin factors, or of the form $\mathcal{L}(H_1, H_2)$ for suitable complex Hilbert spaces H_1 and H_2 with $\dim(H_2) < \infty$.*
5. *The Banach space of X is isomorphic to a Hilbert space.*

In relation to Lemma 4.4 just formulated, it is worth mentioning that certain requirements on a JB^*-triple X, much weaker than the Radon-Nikodym property, also imply that the Banach space of X is isomorphic to a Hilbert space [7].

Let X be a JB^*-triple. Since the elements of $\mathcal{G}_X$ are nothing but the triple automorphisms of X, the set of all maximal tripotents of X is $\mathcal{G}_X$-invariant. If X is finite-dimensional, then in fact G_X acts transitively on the set of all maximal tripotents of X [45, Theorem 5.3.(b)]. But, actually the same is true if X is an arbitrary JB^*-triple of finite rank. Indeed, with the help of Lemma 4.4 above, we can reduce to the case that X is either finite-dimensional (where, as we have just seen, the result is known), a spin factor, or of the form $\mathcal{L}(H_1, H_2)$ for suitable complex Hilbert spaces H_1 and H_2 with $\dim(H_2) < \infty$, and, in the two last cases, the result follows by a direct inspection (see [42, Section 3] and [38]). Therefore we have the following.

Lemma 4.5. *Let X be a JB^*-triple of finite rank. Then G_X acts transitively on the set of all maximal tripotents of X.*

JBW^***-triples** are defined as those JB^*-triples having a (complete) predual. The bidual X^{**} of every JB^*-triple X is a JBW^*-triple under a suitable triple product which extends the one of X [27]. Now, let X be a JBW^*-triple. Then the predual of X (denoted by X_*) is unique, and the triple product of X becomes w^*-continuous in each of its variables [5, Theorem 2.1]. On the other hand, for x in S_X, $D(X, x) \cap X_*$ is a (possibly empty) proper closed face of B_{X_*}, and therefore, by [29, Theorem 4.4], there is a unique tripotent u (possibly equal to zero) such that $D(X, x) \cap X_* = D(X, u) \cap X_*$. Such a tripotent u is called **the support of x in X**, and will be denoted by $u(X, x)$.

Proof of Theorem 4.1.

$1 \Rightarrow 2$.- Let v be the big point of X whose existence is assumed in Condition 1. Since $\sigma(X, v) < 2$, and X^* is L-embedded [5, Proposition 3.4], Lemma 3.1 applies, giving that $\sigma(X, v) = 0$. Now we proceed in several steps.

Step (a).- There exists a tripotent u in X with the same properties as v. Indeed, since $\sigma(X, v) = 0$, the duality mapping of X is upper semicontinuous $(n - w)$ at v (see the proof of Corollary 3.9), and, by [13, Theorem 2.7], this last fact is equivalent to the one that $u := u(X^{**}, v)$ lies in X. Since for x in B_X we have $D(X^{**}, x) \cap X^* = D(X, x)$, the definition of $u(X^{**}, v)$ and the above lead to $D(X, v) = D(X, u)$, and hence, from Corollary 3.9 we deduce $\sigma(X, u) = 0$. Since the norm of X is strongly subdifferentiable at u [13, Corollary 2.4], and v is a big point of X, and $D(X, v) = D(X, u)$, Corollary 3.7 applies, giving that u is a big point of X.

Step (b).- X is weakly compact. Let u be the tripotent of X given by Step (a). It is well known that $X_2(u)$, endowed with the product $x \diamond y := \{x, u, y\}$, becomes a norm unital complete normed (possibly non associative) algebra whose unit is precisely u, and hence we have $n(X_2(u), u) > 0$ [46, p. 617].

Since $0 \leq \sigma(X_2(u), u) \leq \sigma(X, u) = 0$, it follows from Corollary 3.3 that $X_2(u)$ is a reflexive Banach space. Therefore, since $\{uXu\}$ is contained in $X_2(u)$, we realize that u is a weakly compact element of X. Now, recalling that elements of $\mathcal{G}_X$ are triple automorphisms, it follows that every element of $\mathcal{G}_X(u)$ is weakly compact. Since the set of all weakly compact elements of X is a closed subspace of X [18, Proposition 4.7], and u is a big point of X, we obtain that all elements of X are weakly compact, i.e., X is a weakly compact JB^*-triple.

Accordingly to Step (b) and Lemma 4.3, the tripotent u of X given by Step (a) satisfies $u = u_1 + ... + u_m$ for some $m \in \mathbb{N}$ and suitable pair-wise orthogonal minimal tripotents $u_1, ..., u_m \in X$.

Step (c).- Every set of pair-wise orthogonal minimal tripotents of X has at most m elements. Assume on the contrary that we have found pair-wise orthogonal minimal tripotents $v_1, ..., v_{m+1}$ in X. Let T be in $\mathcal{G}_X$. Since $T(u_1), ..., T(u_m)$ are pair-wise orthogonal minimal tripotents in X with $\sum_{j=1}^{m} T(u_j) = T(u)$, Step (b) and Lemma 4.3 apply to find $f, g \in X^*$ satisfying $\pi(f) = T(u)$, $\pi(g) = \sum_{i=1}^{m+1} v_i$, $\|f\| = (f|f) = m$, and $\|g\| = (g|g) = m + 1$. Then we have

$$|g(T(u))| = |g(\pi(f))| = |(g|f)| \leq \sqrt{(g|g)}\sqrt{(f|f)} = \sqrt{(m+1)m}.$$

Since T is arbitrary in $\mathcal{G}_X$, and u is a big point of X, the above shows

$$(m+1) = \|g\| = \sup\{|g(T(u))| : T \in \mathcal{G}_X\} \leq \sqrt{(m+1)m},$$

a contradiction.

Step (d).- X satisfies Condition 2 in the theorem. By Steps (b) and (c), and Lemma 4.3, every element of X belongs to the linear hull of a finite set of pair-wise orthogonal tripotents of X. Since the linear hull of any set of pair-wise orthogonal tripotents of X is a subtriple of X, it follows that X_x is finite-dimensional (equivalently, S_x is finite) for all $x \in X$. By Lemma 4.4, X fulfils Condition 2.

$2 \Rightarrow 3$.- The assumption 2 clearly implies that X is weakly compact. On the other hand, by the assumption 2 and the Krein-Milman theorem, B_X is the norm-closed convex hull of its extreme points. Since extreme points of B_X are precisely the maximal tripotents of X [44, Proposition 3.5], and $\mathcal{G}_X$ acts transitively on the set of all maximal tripotents of X (by the assumption 2 and Lemmas 4.4 and 4.5), it follows that each extreme point of B_X is a big point of X.

$3 \Rightarrow 1$.- By Lemma 3.1.

Now, the equivalence of Conditions 1 to 3 in the theorem has been established. Assume that X fulfills Condition 2. We have just shown that extreme points of B_X are big points of X. Take an extreme point u of B_X. Then, since u is a big point of X, and B_X is dentable, Proposition 4.3 of [6] applies, giving that big points of X and denting points of B_X coincide, and that $\mathcal{G}_X(u)$ is dense in the set of all big points of X. Since $\mathcal{G}_X(u)$ consists only of extreme points of B_X, and the set of all extreme points of B_X is closed (since they are the maximal tripotents of X), it follows that big points of X are extreme points of B_X. ∎

Since C^*-algebras are JB^*-triples, and they are finite-dimensional whenever their Banach spaces are reflexive [50], the next corollary follows straightforwardly from Theorem 4.1.

Corollary 4.6. *Let X be a C^*-algebra. Then the following conditions are equivalent:*

1. *There exists a big point v of X such that $\sigma(X, v) < 2$.*
2. *X is finite-dimensional.*
3. *X is M-embedded and has big points.*

Moreover, if the above conditions are fulfilled, then the big points of X are precisely the extreme points of B_X.

It is well-known that M-embedded C^*-algebras are precisely those of the form $(\bigoplus_{\lambda\in\Lambda} \mathcal{K}(H_\lambda))_{c_0}$ for some family $\{H_\lambda\}_{\lambda\in\Lambda}$ of complex Hilbert spaces. It is also known that extreme points of the closed unit ball of a finite-dimensional C^*-algebra X are the unitary elements of X.

Now, we are going to determine those JBW^*-triples X such that there exists a big point f of X_* with $\sigma(X_*, f) < 2$ and such that the norm of X_* is strongly subdifferentiable at f.

Proposition 4.7. *Let X be a weakly compact JB^*-triple. Then the following conditions are equivalent:*

1. *X^* has big points.*
2. *X is the c_0-sum of a suitable family of copies of some elementary JB^*-triple.*
3. *$\mathcal{G}_X$ acts transitively on the set of all minimal tripotents of X.*
4. *$\mathcal{G}_{X^*}$ acts transitively on the set of all extreme points of B_{X^*}.*

Moreover, if the above conditions are fulfilled, then big points of X^*, denting points of B_{X^*}, and extreme points of B_{X^*} coincide.

Proof. $1 \Rightarrow 2$.- By Lemma 4.2, we have $X = (\bigoplus_{\lambda\in\Lambda} X_\lambda)_{c_0}$ for some family $\{X_\lambda\}_{\lambda\in\Lambda}$ of elementary JB^*-triples, so that we have $X^* = (\bigoplus_{\lambda\in\Lambda} X_\lambda^*)_{\ell_1}$ in the natural manner. Let λ and μ be in Λ. Since X_λ and X_μ are M-embedded (by Lemma 4.2), they are Asplund spaces (by [37, Theorem III.3.1]), so X_λ^* and X_μ^* have the Radon-Nikodym property, and so we may choose f_λ and f_μ denting points of $B_{X_\lambda^*}$ and $B_{X_\mu^*}$, respectively, which are also denting points of B_{X^*} (since X_λ^* and X_μ^* are L-summands of X^*). Now the assumption 1 and [6, Proposition 4.3] provide us with some $T \in \mathcal{G}_{X^*}$ satisfying $\|T(f_\lambda) - f_\mu\| < 2$. On the other hand, since X is M-embedded, we have $T = F^*$ for some F in $\mathcal{G}_X$ (by [37, Proposition III.2.2]), and, since $\{X_\lambda\}_{\lambda\in\Lambda}$ is the family of all minimal closed ideals of X, we must have $F(X_\lambda) = X_\rho$ for some $\rho \in \Lambda$. It follows that $T(X_\lambda^*) = X_\rho^*$, which, together with $f_\lambda \in S_{X_\lambda^*}$,

$f_\mu \in S_{X^*_\mu}$, and $\|T(f_\lambda) - f_\mu\| < 2$, implies that $\rho = \mu$. Now X_μ is a copy of X_λ by means of the restriction of F to X_λ.

$2 \Rightarrow 3$.- Since a minimal tripotent in a c_0-sum of JB^*-triples must lie in some of the summands, and, by the assumption 2, all summands are identical in our case, we may actually assume that the JB^*-triple X is elementary. Then Condition 3 follows from [42, (4.6)] if X is of finite rank, and by a direct inspection from the few remaining examples otherwise (compare [42, Section 3]).

$3 \Rightarrow 4$.- Let π be the mapping from X^* to X given by Lemma 4.3. Since π is canonical, we have $\pi \circ (F^*)^{-1} = F \circ \pi$ for every $F \in \mathcal{G}_X$. Now, let f and g be extreme points of B_{X^*}. By the assumption 3, there exists $F \in \mathcal{G}_X$ satisfying $F(\pi(f)) = \pi(g)$. Since π is injective, it follows that $T(f) = g$ with $T := (F^*)^{-1} \in \mathcal{G}_{X^*}$.

$4 \Rightarrow 1$.- We know that X^* has the Radon-Nikodym property, and hence B_{X^*} is the norm-closed convex hull of its denting points. Since denting points of B_{X^*} are extreme points of B_{X^*}, it follows from the assumption 4 that denting points of B_{X^*} and extreme points of B_{X^*} actually coincide, and that each extreme point of B_{X^*} is a big point of B_{X^*}. Finally, since B_{X^*} is dentable, it follows from [6, Proposition 4.3] that big points of B_{X^*} are denting points of B_{X^*}. ■

Lemma 4.8. *Let X be a JBW^*-triple, and let $\{u_\lambda\}_{\lambda\in\Lambda}$ be a family of pairwise orthogonal nonzero tripotents of X. Then, for $\phi \in \ell_\infty(\Lambda, \mathbb{C})$, the family $\{\phi(\lambda)u_\lambda\}_{\lambda\in\Lambda}$ is w^*-sumable in X, and the mapping $\phi \to \sum_{\lambda\in\Lambda} \phi(\lambda)u_\lambda$ becomes a triple isomorphism from $\ell_\infty(\Lambda, \mathbb{C})$ onto the smallest w^*-closed subtriple of X containing $\{u_\lambda : \lambda \in \Lambda\}$.*

Proof. Put $Y := c_0(\Lambda, \mathbb{C})$. Then, for $\psi \in Y$, the family $\{\psi(\lambda)u_\lambda\}_{\lambda\in\Lambda}$ is norm-sumable in X, and the mapping $\Psi : \psi \to \sum_{\lambda\in\Lambda} \psi(\lambda)u_\lambda$ becomes a triple homomorphism from $c_0(\Lambda, \mathbb{C})$ to X. By [47, Lemma 1.5] and the separate w^*-continuity of the triple product of X, Ψ extends uniquely to a w^*-continuous triple homomorphism $\Phi : Y^{**} = \ell_\infty(\Lambda, \mathbb{C}) \to X$. Denote by $\{e_\lambda\}_{\lambda\in\Lambda}$ the canonical generalized basis of Y. For $\phi \in \ell_\infty(\Lambda, \mathbb{C})$, the family $\{\phi(\lambda)e_\lambda\}_{\lambda\in\Lambda}$ is w^*-sumable in Y with sum ϕ, and hence, since $\Phi(e_\lambda) = u_\lambda$ for all $\lambda \in \Lambda$, the family $\{\phi(\lambda)u_\lambda\}_{\lambda\in\Lambda}$ is w^*-sumable in X with sum $\Phi(\phi)$. If $\Phi(\phi) = 0$ for some ϕ in $\ell_\infty(\Lambda, \mathbb{C})$, then for every $\mu \in \Lambda$ we have

$$0 = \{\Phi(\phi), u_\mu, u_\mu\} = \{\sum_{\lambda\in\Lambda} \phi(\lambda)u_\lambda, u_\mu, u_\mu\} = \sum_{\lambda\in\Lambda} \{\phi(\lambda)u_\lambda, u_\mu, u_\mu\} = \phi(\mu)u_\mu\,,$$

and hence $\phi = 0$. Finally note that, by [40, Proposition 1.2] and [47, Lemma 1.3], w^*-continuous triple homomorphisms between JBW^*-triples have w^*-closed range. ■

Proposition 4.9. *Let X be a JB^*-triple. Then the Banach space of X is isomorphic to a Hilbert space if (and only if) X is a JBW^*-triple whose predual does not contain an isomorphic copy of ℓ_1.*

Proof. Assume that X is a JBW^*-triple whose Banach space is not isomorphic to a Hilbert space. By Lemma 4.4, the Banach space of X is not reflexive. By [39, Theorem 3.23], this implies the existence of a sequence $\{u_n\}_{n\in\mathbb{N}}$ of pairwise orthogonal nonzero tripotents of X. By Lemma 4.8, X contains a w^*-closed subspace linearly isometric to ℓ_∞. Since ℓ_1 is the unique predual of ℓ_∞, this is equivalent to the existence of a closed subspace of P of X_* such that $X_*/P = \ell_1$ isometrically. By lifting the canonical basis of ℓ_1 to a bounded subset of X_*, and passing to the closed linear hull, we obtain an isomorphic copy of ℓ_1 in X_*. ■

Corollary 4.10. *Let X be a JBW^*-triple, and let f be in S_{X_*} such that the duality mapping of X_* is upper semicontinuous $(n-w)$ at f. Then either $\sigma(X_*, f) = 0$ or $\sigma(X_*, f) = 2$.*

Proof. By [29, Theorem 4.6], there exists a nonzero tripotent u in X such that $D(X_*, f) = u + B_{X_0(u)}$. Therefore we have $D(X^*, f) \supseteq u + B_{X_0(u)^{\circ\circ}}$. We identify $X_0(u)^{\circ\circ}$ with $X_0(u)^{**}$, and note that $X_0(u)$ is a JBW^*-triple, and that, in the above identification, the restriction of Π_{X_*} to $X_0(u)^{\circ\circ}$ converts into $\Pi_{(X_0(u))_*}$. Let g be in $B_{X_0(u)^{**}}$. Then $u + g$ lies in $D(X^*, f)$, and the equality $\Pi_{X_*}(u + g) = u + \Pi_{(X_0(u))_*}(g)$ holds. Therefore we have

$$\|(1 - \Pi_{(X_0(u))_*})(g)\| = \|(1 - \Pi_{X_*})(u + g)\| \leq \sigma(X_*, f),$$

and, since g is arbitrary in $B_{X_0(u)^{**}}$, we deduce $\|1 - \Pi_{(X_0(u))_*}\| \leq \sigma(X_*, f)$. Assume that $\sigma(X_*, f) < 2$. Then we have $\|1 - \Pi_{(X_0(u))_*}\| < 2$, and hence, by [22, Proposition 2], $(X_0(u))_*$ does not contain an isomorphic copy of ℓ_1. By Proposition 4.9, $X_0(u)$ is a reflexive Banach space. Since $D(X_*, f) = u + B_{X_0(u)}$, and the duality mapping of X_* is upper semicontinuous $(n-w)$ at f, it follows from [34, Theorem 3.1] that $D(X^*, f) = D(X_*, f)$, i.e. $\sigma(X_*, f) = 0$. ■

Given a normed space X and a subset U of S_X, we say that the norm of X is **uniformly strongly subdifferentiable on** U if

$$\lim_{\alpha\to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} = \tau(u, x) \text{ uniformly for } (u, x) \in U \times B_X.$$

Proposition 4.11. *Let X be a JB^*-triple of finite rank. Then the norm of X^* is uniformly strongly subdifferentiable on the set of all extreme points of B_{X^*}.*

Proof. Since X is a finite ℓ_∞-sum of simple JB^*-triples of finite rank (by Lemma 4.4), we may assume that X is simple.

Assume in addition that X is finite-dimensional. Then the norm of X^* is strongly subdifferentiable at every point of S_{X^*} [1, p. 123]. But, since $\mathcal{G}_{X^*}$ acts transitively on the set of all extreme points of B_{X^*} (by Proposition 4.7), we have in fact that the norm of X^* is uniformly strongly subdifferentiable on such a set.

Now assume that $X = \mathcal{L}(H_1, H_2)$ for suitable complex Hilbert spaces H_1 and H_2 with $\dim(H_1) = \infty$ and $\dim(H_2) < \infty$. Let K be the complex Hilbert space of dimension equal to $\dim(H_2) + 1$, and let Y stand for $\mathcal{L}(K, H_2)$. Let $\varepsilon > 0$. By the above paragraph, there exists $\delta > 0$ such that

$$\frac{\|\widehat{f} + \alpha\widehat{g}\| - 1}{\alpha} - \tau(\widehat{f}, \widehat{g}) < \varepsilon$$

whenever $\widehat{f}$ is an extreme point of B_{Y^*}, $\widehat{g}$ belongs to B_{Y^*}, and $0 < \alpha < \delta$. Now, let f be an extreme point of B_{X^*}, and let g be in B_{X^*}. Since X^* equals the projective tensor product $H_1 \otimes_\pi H_2$, and $f = x_1 \otimes x_2$ for some $(x_1, x_2) \in H_1 \times H_2$, there exists a copy of K in H_1 such that $K \otimes H_2$ contains f and g. Moreover, since K is one-complemented in H_1, the natural mapping $K \otimes_\pi H_2 \to H_1 \otimes_\pi H_2$ is an isometry [25, Proposition 3.9]. Since $Y^* = K \otimes_\pi H_2$, it follows

$$\frac{\|f + \alpha g\| - 1}{\alpha} - \tau(f, g) < \varepsilon$$

whenever $0 < \alpha < \delta$.

In view of Lemma 4.4, to conclude the proof it is enough to consider the case that X is an infinite-dimensional spin factor. Then we have $X = \mathbb{C} \otimes_\pi H$, where H is an infinite-dimensional real Hilbert space [47, pp. 438-441]. Let K be the real Hilbert space of dimension 4, and let Y stand for the 4-dimensional spin factor $\mathbb{C} \otimes_\pi K$. Let $\varepsilon > 0$. We know that there exists $\delta > 0$ such that

$$\frac{\|\widehat{f} + \alpha\widehat{g}\| - 1}{\alpha} - \tau(\widehat{f}, \widehat{g}) < \varepsilon$$

whenever $\widehat{f}$ is an extreme point of B_{Y^*}, $\widehat{g}$ belongs to B_{Y^*}, and $0 < \alpha < \delta$. Now, let f be an extreme point of B_{X^*}, and let g be in B_{X^*}. Since X^* equals the injective tensor product $\mathbb{C} \otimes_\epsilon H$, there exists a copy of K in H such that $\mathbb{C} \otimes K$ contains f and g. Since the natural mapping $\mathbb{C} \otimes_\epsilon K \to \mathbb{C} \otimes_\epsilon H$ is an isometry [25, Proposition 4.3], and $Y^* = \mathbb{C} \otimes_\epsilon K$, it follows

$$\frac{\|f + \alpha g\| - 1}{\alpha} - \tau(f, g) < \varepsilon$$

whenever $0 < \alpha < \delta$. ■

We recall that a JBW^*-triple X is said to be **purely atomic** if X_* is the closed linear hull of the set of all extreme points of B_{X_*}.

Theorem 4.12. *Let X be a JBW^*-triple. Then the following conditions are equivalent:*

1. *There exists a big point f of X_* with $\sigma(X_*, f) < 2$ and such that the norm of X_* is strongly subdifferentiable at f.*
2. *The Banach space of X is isomorphic to a Hilbert space, and X_* has big points.*
3. *X is purely atomic and there exists a big point f of X_* satisfying $\sigma(X_*, f) < 2$.*

4. *X is a finite ℓ_∞-sum of copies of a simple JB^*-triple of finite rank.*

Moreover, if the above conditions are fulfilled, then big points of X_*, denting points of B_{X_*}, and extreme points of B_{X_*} coincide.

Proof. $1 \Rightarrow 2$.- Recalling that the strong subdifferentiability of the norm at a point implies the upper semicontinuity $(n-w)$ of the duality mapping at that point, the proof of the present implication reduces to putting together Corollary 4.10, Theorem 2.5, and Lemma 4.4.

$2 \Rightarrow 3$.- This implication is clear.

$3 \Rightarrow 4$.- Assume that Condition 3 is fulfilled. Since X is atomic, we can apply [33, Theorem E and Proposition 2] and Lemma 4.2 to realize that $X = Y^{**}$ for some weakly compact JB^*-triple Y. Let f be the big point of $X_* = Y^*$ whose existence is assumed in 3. Since $\sigma(Y^*, f) < 2$, it follows from Corollary 3.4 that Y^* cannot contain an isometric copy of ℓ_1 in such a way that f becomes an element of the canonical basis. Since the big points of Y^* are the extreme points of B_{Y^*}, and $\mathcal{G}_{Y^*}$ acts transitively on the set of all extreme points of B_{Y^*} (by Proposition 4.7), the fact just shown for f remains true when f is replaced with any extreme point of B_{Y^*}. By Lemma 4.3, this implies that every set of pair-wise orthogonal minimal tripotents of Y is finite. Then, applying again Lemma 4.3, we deduce that S_y is finite for every $y \in Y$. Now Condition 4 follows from Lemma 4.4 and Proposition 4.7.

$4 \Rightarrow 1$.- By Lemma 4.4 and Propositions 4.7 and 4.11.

When Conditions 1 to 4 are fulfilled, the coincidence of big points of X_*, denting points of B_{X_*}, and extreme points of B_{X_*} follows from Proposition 4.7. ∎

Corollary 4.13. *Let X be a von Neumann algebra. Then the following conditions are equivalent:*

1. *There exists a big point f of X_* with $\sigma(X_*, f) < 2$ and such that the norm of X_* is strongly subdifferentiable at f.*
2. *X is finite-dimensional, and X_* has big points.*
3. *X is purely atomic and there exists a big point f of X_* satisfying $\sigma(X_*, f) < 2$.*
4. *X is a finite ℓ_∞-sum of copies of $\mathcal{L}(H)$ for some finite-dimensional complex Hilbert space H.*

Moreover, if the above conditions are fulfilled, then big points of X_* and extreme points of B_{X_*} coincide.

5. Characterizing Hilbert spaces

Every complex Hilbert space H becomes a simple JB^*-triple of finite rank (indeed, we have $H = \mathcal{L}(H, \mathbb{C})$), whose triple product is given by

$$\{x, y, z\} = \frac{1}{2}((x|y)z + (z|y)x)$$

for all $x, y, z \in H$. More precisely, it follows from [42] that all frames in a JB^*-triple X of finite rank have the same cardinal (called the **rank** of X), and that complex Hilbert spaces are precisely the JB^*-triples of rank one.

In [9] we obtained several characterizations of complex Hilbert spaces, among either the JB^*-triples or the preduals of JBW^*-triples, in terms of transitivity conditions. A detailed review of such characterizations and other new characterizations in the same line can be seen in [11, Section 5]. On the other hand, since almost transitive JBW^*-triples are Hilbert spaces [9, Corollary 2.6], all characterizations obtained in [8] and [10] of members X of the class $\mathcal{J}$ (of all almost transitive superreflexive Banach spaces), as well as the new ones shown in Proposition 2.9 of the present paper, become in fact characterizations of complex Hilbert spaces when we require that X is in addition a JB^*-triple. We are going to conclude this paper showing deeper characterizations of complex Hilbert spaces, among either the JB^*-triples or the preduals of JBW^*-triples, again in terms of transitivity conditions.

JB^***-algebras** are defined as those complete normed Jordan complex algebras X endowed with a conjugate-linear algebra-involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in X, where, for x in X, the operator $U_x : X \to X$ is defined by $U_x(y) = 2x.(x.y) - x^2.y$. It is well-known that, if X is a JB^*-triple, and if u is a tripotent of X, then $X_2(u)$ becomes a JB^*-algebra with unit u for suitable product an involution.

Lemma 5.1. *Let X be a nonzero JB^*-algebra with a unit* $\mathbf{1}$ *such that $X \neq \mathbb{C}\mathbf{1}$. Then we have $\delta(X, \mathbf{1}) = 2$.*

Proof. Take $x = x^* \in X \setminus \mathbb{R}\mathbf{1}$. Then, since Jordan algebras are power-associative, the closed subalgebra of X generated by $\{x, \mathbf{1}\}$ (say Y) is associative and $*$-invariant, and hence it is a commutative C^*-algebra different from $\mathbb{C}\mathbf{1}$. Write $Y = C(\Omega)$ for some Hausdorff compact topological space Ω, choose $a, b \in \Omega$ such that $x(a) = \min\{x(t) : t \in \Omega\}$ and $x(b) = \max\{x(t) : t \in \Omega\}$, and put

$$y := \frac{2x - x(b) - x(a)}{x(b) - x(a)} \in S_Y.$$

Since $y(a) = 1$ and $y(b) = -1$, the unit point measures at a and b on Ω are elements of $D(Y, \mathbf{1})$ whose distance is equal to 2. Since $\delta(X, \mathbf{1}) \geq \delta(Y, \mathbf{1})$, we have $\delta(X, \mathbf{1}) = 2$. ∎

Theorem 5.2. *For a complex Banach space X, the following assertions are equivalent:*

1. *X is a JB^*-triple and there exists a big point u of X with $\eta(X,u)<2$.*
2. *X is a JB^*-triple and there exists a big point u of X with $\delta(X,u)<2$ and such that the duality mapping of X is upper semicontinuous $(n-w)$ at u.*
3. *X is the predual of a JBW^*-triple and there exists a big point f of X with $\eta(X,f)<2$.*
4. *X is the predual of a JBW^*-triple and there exists a big point f of X with $\delta(X,f)<2$ and such that the norm of X is strongly subdifferentiable at f.*
5. *X is a Hilbert space.*

Proof. $1\Rightarrow 2$.- By [10, Lemma 1] and [24, Theorem VII.4.4], the assumption 1 implies that X is a superreflexive Banach space. As a consequence, X is M-embedded. Since, by the assumption 1, X has big points, Theorem 4.1 applies giving that such big points are extreme points of B_X, and hence maximal tripotents. Since the norm of every JB^*-triple is strongly sudifferentiable at any nonzero tripotent [13, Corollary 2.4], and, for the big point u of X whose existence is assumed in 1, we have $\delta(X,u)<2$ (by Lemma 2.7), Assertion 2 follows.

$2\Rightarrow 5$.- Let u be the big point of X whose existence is assumed in 2. By the comments after Corollary 2.6, the assumption 2 implies that $\sigma(X,u)<2$. Then, by Theorem 4.1, u is a maximal tripotent of X. Now, since $\delta(X,u)<2$, $X_2(u)$ is a JB^*-algebra with unit u such that $\delta(X_2(u),u)<2$. By Lemma 5.1, we have $X_2(u)=\mathbb{C}u$, i.e., u is a minimal tripotent of X. It follows that $\{u\}$ is a frame in X, and therefore X is a JB^*-triple of rank one.

$5\Rightarrow 1$.- This implication is clear.

$3\Rightarrow 4$.- By [10, Lemma 1] and [24, Theorem VII.4.4], the assumption 3 implies that X is a superreflexive Banach space. Then, by Lemma 4.4, the Banach space of the JBW^*-triple X^* is isomorphic to a Hilbert space. Since, by the assumption 3, X has big points, Theorem 4.12 applies giving that such big points are extreme points of B_X. Since the norm of X is strongly sudifferentiable at every extreme point of B_X (by Proposition 4.11), and, for the big point f of X whose existence is assumed in 3, we have $\delta(X,f)<2$ (by Lemma 2.7), Assertion 4 follows.

$4\Rightarrow 5$.- Let f be the big point of X whose existence is assumed in 4. By the comments after Corollary 2.6, the assumption 4 implies that $\sigma(X,f)<2$. Then, by Theorem 4.12, the JBW^*-triple X^* is of finite rank, and f is an extreme point of B_X. Since $\mathcal{G}_X$ acts transitively on the set of all extreme points of B_X (by Proposition 4.7), and $\delta(X,f)<2$, it follows that $\delta(X,g)<2$ for every extreme point g of B_X. Now, assume that 5 does not hold. Then X^* is of finite rank >1, and hence there are two orthogonal minimal tripotents in X^*. With the help of Lemma 4.3, we can find extreme points g_1,g_2 of B_X such that the linear hull of $\{g_1,g_2\}$ (say Y) becomes an isometric copy of ℓ_1^2 in such a way that $\{g_1,g_2\}$ converts into the canonical basis. Therefore we have

$$2 = \delta(Y, g_1) \leq \delta(X, g_1) < 2,$$

a contradiction.

$5 \Rightarrow 3$.- This implication is clear. ■

Corollary 5.3. *For a complex Banach space X, the following assertions are equivalent:*

1. *X is a C^*-algebra and there exists a big point u of X with $\eta(X, u) < 2$.*
2. *X is a C^*-algebra and there exists a big point u of X with $\delta(X, u) < 2$ and such that the duality mapping of X is upper semicontinuous $(n{-}w)$ at u.*
3. *X is the predual of a von Neumann algebra and there exists a big point f of X with $\eta(X, f) < 2$.*
4. *X is the predual of a von Neumann algebra and there exists a big point f of X with $\delta(X, f) < 2$ and such that the norm of X is strongly subdifferentiable at f.*
5. *$X = \mathbb{C}$.*

Acknowledgements.- The authors are very grateful to W. Kaup for crucial remarks concerning the main results of this paper. They also want to thank A. Peralta for interesting comments.

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