BANACH SPACE CHARACTERIZATIONS OF UNITARIES

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Abstract. We invoke the early paper of H. F. Bohnenblust and S. Karling [3] to provide a very short proof of the recent theorem of C. Akemann and N. Weaver [1] characterizing unitary elements of a unital C^*-algebra A as those norm-one elements u of A such that the dual space A^* is the linear hull of the set of states S_u of u. Moreover, we generalize such a theorem to the setting of JB^*-triples.

1. Discussing the Akemann-Weaver theorem

In [1], C. Akemann and N. Weaver point out how the celebrated paper of R. V. Kadison [10] implicitly contains a Banach space characterization of unitary elements in unital C^*-algebras. Then they ask for explicit characterizations of such a kind, and provide those given by the following theorem (see Theorem 2 of [1] and its proof).

Theorem 1.1. *Let A be a unital C^*-algebra, and let u be a norm-one element of A. Then the following conditions are equivalent:*

(1) *u is unitary.*
(2) *The dual space A^* is the linear hull of the set of states S_u of u.*
(3) *u is a vertex of the closed unit ball of A.*

We recall that, given a norm-one element u of a Banach space X, the states of u (relative to X) are defined as those norm-one elements f of the dual space X^* satisfying $f(u) = 1$. We also recall that vertices of the closed unit ball of a Banach space X are defined as those norm-one elements u of X such that the set of states S_u of u separates the points of X.

The proof provided in [1] of the implication (1) $\Rightarrow$ (2) is really easy. Indeed, Condition (2) is known to be fulfilled in the case that u equals the unit $\mathbf{1}$ of A, and hence it also remains fulfilled for every unitary u because unitary elements of A lie in the orbit of $\mathbf{1}$ under the group of all surjective linear isometries on A. By the way, an alternative proof of (1) $\Rightarrow$ (2) can be given by noticing that, if u is a unitary element of A, then A becomes a C^*-algebra with unit u under the product $x\Box y := xu^*y$ and the involution $x \to ux^*u$. On the other hand, the implication (2) $\Rightarrow$ (3) is clear. The main aim of this section is to point out that, as a matter of fact, the implication (3) $\Rightarrow$ (1) is proved in Example 4.1 of the early paper of H. F. Bohnenblust and S. Karling [3] (see also [16, Theorem 9.5.16.(c)]). By

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the way, Bohnenblust and Karlin also point out in [3] how the equivalence (3) $\Longleftrightarrow$ (1) drastically simplifies Kadison's original arguments in [10].

We conclude this section by listing, in Remark 1.2 immediately below, some other known results related to Akemann-Weaver Theorem 1.1. To be short, norm-one elements u of a Banach space X such that X^* is the linear hull of S_u will be called geometrically unitary elements of X.

Remark 1.2. (a).- Let X be a Banach space, let u be a norm-one element of X, and define the numerical index, $n(X,u)$, of X at u by

$$n(X,u) := \inf_{\|x\|=1} \sup_{f\in S_u} |f(x)|$$

(equivalently, $n(X,u)$ is the maximum nonnegative number k satisfying $k \sup_{f\in S_u} |f(x)| \leqslant \|x\|$ for every $x \in X$). Then u is a geometrically unitary element of X if and only if $n(X,u) > 0$ [15, Theorem 3.2].

(b).- The above result becomes an abstract version of the Moore-Sinclair theorem [5, Theorem 31.1] that, if A is a complex Banach algebra with a norm-one unit $\mathbf{1}$, then $\mathbf{1}$ is a geometrically unitary element of A. Indeed, it is enough to apply the Bohnenblust-Karlin theorem that $n(A,\mathbf{1}) \geqslant \frac{1}{e}$ [3] (see also [4, Theorem 4.1]). Part (a) of the present remark also implies that the requirement of associativity of A in the Moore-Sinclair theorem can be altogether removed because, as pointed out in [15, p. 617], the Bohnenblust-Karlin theorem remains true in the non-associative context.

(c).- Let A be a complex Banach algebra with a norm-one unit $\mathbf{1}$ (associativity of A is now required), and let u be an algebraically unitary element of A (i.e., an invertible element satisfying $\|u\| = \|u^{-1}\|$). Since the operator of left multiplication by u on A is a surjective linear isometry taking $\mathbf{1}$ to u, it follows from the Moore-Sinclair theorem that u is a geometrically unitary element of A.

(d).- Parts (a) and (c) of the present remark have been recently rediscovered (see [2, Theorem 3.1] and [2, Corollary 3.5], respectively).

(e).- Let X be a Banach space, and let u be a norm-one element of X. Then we have $n(X^{**},u) = n(X,u)$ [15, Lemma 4.8]. Consequently, by Part (a) of the present remark, u is a geometrically unitary element of X^{**} if and only if it is a geometrically unitary element of X [2, Corollary 3.4].

(f).- Let A be a unital C^*-algebra. It follows from [7, Theorem 1] that, for a norm-one element u in A, each of Conditions (1) to (3) in Theorem 1.1 is equivalent to

(4) $n(A,u)$ is equal to 1 or $\frac{1}{2}$.

Moreover, the existence of a norm-one element u of A with $n(A,u) = 1$ is equivalent to the commutativity of A.

(g).- Vertices u of the closed unit ball of a Banach space X need not be geometrically unitary. Indeed, there exists a real Banach space E such that, taking X equal to the space of all bounded linear operators on E, and u equal to the identity mapping on E, u becomes a vertex of the closed ball of

X but we have $n(X, u) = 0$ [14, Example 3.b]. A less natural example can be found in [2, Example 3.7].

2. Generalizing the Akemann-Weaver theorem

JB^*-triples are defined as those complex Banach spaces X endowed with a continuous triple product $\{\cdot\cdot\cdot\} : X \times X \times X \longrightarrow X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all x in X, the mapping $y \to \{xxy\}$ from X to X is a hermitian operator on X (in the sense of [4, Definition 5.1]) and has nonnegative spectrum.
(2) The equality
$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$
holds for all a, b, x, y, z in X.
(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Every C^*-algebra becomes a JB^*-triple under the triple product
$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$
More generally, C^*-algebras are JB^*-algebras (under the product $x \circ y := \frac{1}{2}(xy + yx)$), and JB^*-algebras become JB^*-triples (under a triple product naturally derived from their binary products and involutions) [6, 18]. We recall that JB^*-algebras are defined as those complete normed Jordan complex algebras A endowed with a conjugate-linear algebra-involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in A, where, for x in A, the operator $U_x : A \to A$ is defined by $U_x(y) = 2x \circ (x \circ y) - x^2 \circ y$.

The main interest of JB^*-triples relies on the fact that, up to biholomorphic equivalence, there are no bounded symmetric domains in complex Banach spaces others than the open unit balls of JB^*-triples [12]. Unitary elements of a JB^*-triple X are defined as those elements u of X satisfying $\{u\{uxu\}u\} = x$ for every $x \in X$. It is easily seen that, if a C^*-algebra A has a unitary element in the JB^*-sense, then A has a unit, and unitary elements in the JB^*-sense coincide with unitary elements in the usual C^*-meaning. Now, the main result in this section reads as follows.

Theorem 2.1. *For a norm-one element u in a JB^*-triple X, the following conditions are equivalent:*

(1) *u is unitary.*
(2) *$n(X, u)$ is equal to 1 or $\frac{1}{2}$.*
(3) *u is geometrically unitary.*
(4) *u is a vertex of the closed unit ball of A.*

The proof of Theorem 2.1 goes as follows. If Condition (1) is fulfilled, then X, endowed with the product $x \circ y := \{xuy\}$ and the involution $x^* := \{uxu\}$, becomes a unital JB^*-algebra whose unit is precisely u (see [6]), and hence (2) holds by [17, Theorem 26] (see also [9, Theorem 4]). On the other hand, the implication (2) $\Rightarrow$ (3) follows from Remark 1.2.(a), and the one (3) $\Rightarrow$ (4) is clear. Finally, the implication (4) $\Rightarrow$ (1) follows from [6, Proposition 4.3] and [11, Lemma 3.1]. An alternative proof of (4) $\Rightarrow$ (1) can be given by keeping in mind that vertices of the closed unit ball of a Banach space are extreme points, that extreme points of the closed unit ball of a JB^*-triple are well-understood [13, Proposition 3.5], and then by selecting (with the help of [8, Proposition 1.(a)]) those extreme points which are in fact vertices.

In relation to the above proof, it is worth mentioning that, contrarily to what happens in the case of C^*-algebras, the group of all surjective linear isometries on a JB^*-triple X need not act transitively on the set of all unitary elements of X [6, Example 5.7]. Let us also notice that, by the references applied above, the existence in a JB^*-triple X of a norm-one element u with $n(X, u) = 1$ is equivalent to the fact that X is triple-isomorphic to a unital commutative C^*-algebra.

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