Banach algebras with large groups of unitary elements

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1. Introduction

By a normed algebra we mean a real or complex (possibly nonassociative) algebra A endowed with a norm $\|\cdot\|$ satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. A complete normed associative algebra will be called a Banach algebra. A normed algebra is called norm-unital if it has a unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$. Unitary elements of a norm-unital normed associative algebra A are defined as those invertible elements u of A satisfying $\|u\| = \|u^{-1}\| = 1$. By a unitary normed associative algebra we mean a norm-unital associative normed algebra A such that the convex hull of the set of its unitary elements is norm-dense in the closed unit ball of A. In the sequel we will denote by U_A the set of unitary elements of A. Relevant examples of unitary Banach algebras are all unital C^*-algebras and the discrete group algebras $\ell_1(G)$ for every group G.

The study of unitary Banach algebras is quite recent (see [**2, 8, 14, 15, 20, 51**]). They were first considered by E. R. Cowie in her P.D. thesis [**14**]. However, with the exception of some facts concerning discrete group algebras [**15**], her results were not published elsewhere. Fifteen years later, unitary Banach algebras were reconsidered by M. L. Hansen and R. V. Kadison [**20**], who were unaware of Cowie's work. Both [**14**] and [**20**] were mainly concerned with the achievement of characterizations of unital C^*-algebras among unitary Banach algebras. Recently, G. V. Wood [**51**] recovers some of Cowie's unpublished results, surveys the Hansen-Kadison paper, and proves some new results about discrete group algebras. In [**8**], complex Banach spaces whose algebras of operators are unitary are studied, and it is proved that, under certain additional conditions, they turn out to be Hilbert spaces. In [**2**], unitary Banach algebras are considered by themselves, showing that

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all unitary Banach algebras are quotients of discrete group algebras, proving different characterizations of them in terms of numerical ranges, studying dentability of their closed unit balls, and characterizing unital C^*-algebras among them by means of holomorphic conditions.

In the present paper, we continue the line of [**2**], thus devoting the most part of it to the development of a general theory of unitary Banach algebras. We also retake the study, begun in [**8**], of those complex Banach spaces whose algebras of operators are unitary, and, by the first time, we extend such a study to the case of real spaces. Moreover, at the end of the paper, we leave the associative scope in order to deal by the first time with nonassociative unitary normed algebras.

The content of the paper is organized as follows. In Section 2 we revisit the concepts of maximality and unique maximality (which are closely related to that of unitarity), introduced in [**20**] and [**14**] (see also [**15**]), respectively. By the sake of usefulness, we introduce the notions of strong maximality and strong unique maximality, and clarify how all these notions are related among them, as well as with that of unitarity. To this end, we also introduce the concept of minimality of the equivalent norm (a weakening of the classical notion of minimality of the norm [**9**]), and prove in Corollary 2.2 that a norm-unital associative normed algebra is uniquely maximal (respectively, strongly uniquely maximal) if and only if it is unitary and has minimality of the equivalent norm (respectively, minimality of the norm). Consequently, from the known facts that unital C^*-algebras are unitary (by the Russo-Dye theorem) and have minimality of the norm, we deduce that they are strongly uniquely maximal (Corollary 2.3). Moreover, applying a result of Cowie (collected in our Corollary 3.11), we realize that, in the commutative case, unital C^*-algebras are nothing other than strongly uniquely maximal complex Banach algebras (Remark 2.6.(b)). We remark how some known results, proved in the literature under the requirement of unique maximality, actually remain true under the weaker one of minimality of the equivalent norm (see Theorem 2.5).

For a real (respectively, complex) norm unital Banach algebra A, consider Property $(\mathcal{S})$ which follows:

$(\mathcal{S})$ There exists a linear (respectively, conjugate-linear) algebra involution on A mapping each unitary element to its inverse.

It is known that Property $(\mathcal{S})$ is fulfilled in the case that A is a unital C^*-algebra, a discrete group algebra, or a finite-dimensional unitary Banach algebra. However, in general, unitary Banach algebras need not satisfy Property $(\mathcal{S})$, even if they are complex and commutative [**2**]. The main result in Section 3 (Theorem 3.2) asserts that unitary semisimple commutative complex Banach algebras satisfy Property $(\mathcal{S})$, and that, endowed with the involution given by such a property, they become hermitian $*$-algebras. From this theorem we deduce that, in the commutative case, unital C^*-algebras are nothing other than strongly maximal unitary semisimple complex Banach algebras (Corollary 3.7). We emphasize also Theorem 3.10, which shows that a uniquely maximal complex Banach algebra "close enough to be commutative" is isometrically isomorphic to a commutative C^*-algebra.

Section 4 is mainly devoted to study Property $(\mathcal{S})$ in the noncommutative case. To this end, we introduce "good" groups as those groups G such that every primitive ideal of the complex Banach $*$-algebra $\ell_1(G)$ is $*$-invariant, and prove that, if A is a real or complex unitary semisimple Banach algebra such that the group U_A is good, then A satisfies Property $(\mathcal{S})$ (Theorem 4.2 and Corollary 4.7). It seems to be an open problem whether or not every group is good. Anyway, we show that this problem has an affirmative answer if (and only if) every primitive unitary complex Banach algebra satisfies Property $(\mathcal{S})$, if (and only if) every primitive unitary real Banach algebra satisfies Property $(\mathcal{S})$ (Proposition 4.8). The section ends with several characterizations, involving strong maximality, of unital C^*-algebras (Proposition 4.9). It follows from such characterizations that, if G is a non trivial group, then the unitary semisimple complex Banach algebra $\ell_1(G)$ is not strongly maximal (much less strongly uniquely maximal) (Remark 4.10). We note that there are choices of G such that $\ell_1(G)$ is uniquely maximal [**15**], as well as other choices such that G is commutative and $\ell_1(G)$ is maximal but not uniquely maximal [**51**].

In Section 5 we translate to the real case some of the results obtained for complex algebras in Sections 2 and 3. Thus, we prove that every uniquely maximal norm-unital commutative real Banach algebra is isometrically isomorphic to a real C^*-algebra (Proposition 5.1). In [**2**] it is shown that every finite-dimensional real C^*-algebra is unitary. However, in general, unital real C^*-algebras need not be unitary. Anyway, since real C^*-algebras have minimality of the norm (Corollary 5.3), it follows that, for real C^*-algebras, the concepts of unitarity and strong unique maximality are equivalent. As a consequence, every finite-dimensional real C^*-algebra is strongly uniquely maximal. We conclude this section by proving that every maximal semisimple finite-dimensional real Banach algebra is isometrically isomorphic to a real C^*-algebra (Theorem 5.8). This generalizes to the real case the corresponding result for complex algebras, first proved in [**14**] (see also [**51**]).

If X is a complex Hilbert space, then the algebra $\mathcal{L}(X)$ (of all bounded linear operators on X) is a (complex) C^*-algebra, and hence it is unitary. It seems to be an open problem whether or not all complex Banach spaces X such that $\mathcal{L}(X)$ is unitary are in fact Hilbert spaces. Some partial affirmative answers to this problem have been given in [**8**]. We devote Section 6 to provide the reader with some new partial affirmative answers to this problem, to formulate the actual variant of the problem for real spaces, and to give partial affirmative answers to such a variant. We prove that a complex Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is unitary and satisfies Property $(\mathcal{S})$ (Theorem 6.4). Therefore, according to Proposition 4.8 already commented, if every group is good, then all complex Banach spaces

X such that $\mathcal{L}(X)$ is unitary are in fact Hilbert spaces. Given a real or complex Banach space X, and a vector space topology τ on $\mathcal{L}(X)$ stronger than the weak-operator topology (in short, w_{op}), let us say that $\mathcal{L}(X)$ is τ-unitary if the τ-closed convex hull of $U_{\mathcal{L}(X)}$ is equal to the closed unit ball of $\mathcal{L}(X)$. It seems to be an unsolved problem whether or not $\mathcal{L}(X)$ is unitary whenever X is an infinite-dimensional real Hilbert space. Anyway, we prove that the problem just raised answers affirmatively if, in its formulation, unitarity is replaced with w_{op}-unitarity (Corollary 6.6). On the other hand, if X is a complex Banach space such that $\mathcal{L}(X)$ is w_{op}-unitary and satisfies Property $(\mathcal{S})$, then X is a Hilbert space (see again Theorem 6.4). It turns out a reasonable conjecture that a real Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is w_{op}-unitary. We prove that a real Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is w_{op}-unitary, fulfils Property $(\mathcal{S})$, and the involution (say $\bullet$) given by such a property satisfies $T^\bullet \circ T \neq 0$ for some one-dimensional operator $T \in \mathcal{L}(X)$ (Theorem 6.7). It is shown in [**8**] that a complex Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is unitary and, for Y equal to X, X^* or X^{**}, there exists a biholomorphic automorphism of the open unit ball of Y which cannot be extended to a surjective linear isometry on Y. We note that the existence of such a biholomorphic automorphism of the open unit ball of a complex Banach space Y is plenty guaranteed in the case that Y is a (complex) JB^*-triple [**31**]. We also note that complex Hilbert spaces are JB^*-triples. Keeping in mind these ideas, we extend to the setting of real spaces the results of [**8**] quoted above. Indeed, we prove the following facts:

(1) A real Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is w'_{op}-unitary (where w'_{op} means the dual weak-operator topology [**29**]) and X or X^{**} is a real JB^*-triple in the sense of [**22**] (Theorem 6.12).
(2) A real Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is w''_{op}-unitary (where w''_{op} means the second dual weak-operator topology) and X^* is a real JB^*-triple (Theorem 6.15).

By the way, some of the new techniques developed in this section allows us also to complement the results of [**8**] in their original complex setting (see Theorem 6.18).

In the last two sections of the paper, we are concerned with the generalization of the theory of unitary normed algebras to the non-associative setting. Such a generalization is mainly motivated by the Russo-Dye-type theorem for unital JB^*-algebras, proved by J.D.M. Wright and M.A. Youngson [**52**]. Although non-commutative JB^*-algebras are "nearly" associative (they are in fact non-commutative Jordan algebras in the sense of [**34**]), in a very precise sense they become the largest non-associative generalizations of (associative) C^*-algebras. Indeed, it proved in [**26**] that an associative (respectively, non-associative) normed complex algebra is a C^*-algebra (respectively, a non-commutative JB^*-algebra) if and only if it has an approximate unit bounded by one, and its open unit ball is a bounded symmetric domain (equivalently, the normed space of the algebra is linearly isometric to a JB^*-triple). In view of the above comments, and due to the fact that the setting of unital non-commutative Jordan algebras becomes the largest non-associative one where a notion of invertible element works reasonably [**35**], we restrict our attention to norm-unital normed non-commutative Jordan algebras. Unitary elements of such an algebra are defined verbatim as in the associative case, and the the notions of unitarity, maximality, strong maximality, unique maximality, and strong unique maximality are translated literally from the associative setting to the more general one. Since the set of all unitary elements of a norm-unital normed non-commutative Jordan algebra need not be multiplicatively closed, we introduce weakly unitary normed non-commutative Jordan algebras as those norm-unital normed non-commutative Jordan algebras A such that the convex multiplicatively closed hull of U_A is dense in the closed unit ball of A. Replacing unitarity with weak unitarity, most results obtained in Section 2 remain true in the new setting (see Proposition 7.3 and Corollary 7.5). As a consequence, unital non-commutative JB^*-algebras turn out to be strongly uniquely maximal (Proposition 7.10). Moreover, weakly unitary norm-unital closed subalgebras of non-commutative JB^*-algebras are non-commutative JB^*-algebras (Theorem 7.11). The result just quoted becomes a non-associative generalization of [**20**, Theorem 4]. Alternative algebras (respectively, alternative C^*-algebras) are very particular examples of non-commutative Jordan algebras (respectively, non-commutative JB^*-algebras). It is worth mentioning that, as in the particular associative case, for a norm-unital normed alternative algebra A, the set U_A is multiplicatively closed, and hence the concepts of unitarity and weak unitarity are equivalent for A. We prove that every finite-dimensional maximal unitary normed alternative complex algebra is isometrically isomorphic to an alternative C^*-algebra (Theorem 7.12). This generalizes [**20**, Theorem 6] (see also [**2**, Corollary 2.7]). Moreover, we prove a variant of Theorem 5.8 (reviewed some paragraphs ago) in the case of complex alternative algebras (see Theorem 7.16). Due to the lack of associativity, the proof of such a variant becomes the hardest one in the paper.

In the last section (Section 8) we introduce real non-commutative JB^*-algebras and real alternative C^*-algebras, and extend to the real case some results of the previous section. Among them, we emphasize the variant of Theorem 7.12 (reviewed in the preceding paragraph) for real algebras (see Theorem 8.10). It is also worth mentioning the fact that every group is good if and only if every unitary semisimple complete normed complex alternative algebra satisfies Property $(\mathcal{S})$, if and only if every unitary semisimple complete normed real alternative algebra satisfies Property $(\mathcal{S})$ (Proposition 8.11).

Notation. Given a vector space X, we denote by $L(X)$ the algebra of all linear operators on X. Now, let X be a real or complex normed space. Then the symbol $\mathcal{L}(X)$ (respectively, $\mathcal{K}(X)$, or $\mathcal{F}(X)$) will stand for the normed algebra of all bounded (respectively, compact, or finte-rank) linear operators on X. We denote by B_X, S_X, and X^* the closed unit ball, the unit sphere, and the (topological) dual, respectively, of X. The normed space X will be regarded without notice as a subspace of its second dual X^{**}. For a bounded linear mapping T from X to another normed space Y, we denote by $T^* : Y^* \to X^*$ the transpose of T.

2. Basic definitions and facts

Let A be a norm-unital normed associative algebra. We say that A is $\left\{\begin{array}{c}\text{maximal}\\ \text{strongly maximal}\\ \text{uniquely maximal}\\ \text{strongly uniquely maximal}\end{array}\right\}$ if, whenever $|||\cdot|||$ is an $\left\{\begin{array}{c}\text{equivalent}\\ \text{continuous}\\ \text{equivalent}\\ \text{continuous}\end{array}\right\}$ norm on A converting A into a norm-unital normed algebra and satisfying $U_A \subseteq U_{(A,|||\cdot|||)}$, we have that $\left\{\begin{array}{c}U_A = U_{(A,|||\cdot|||)}\\ U_A = U_{(A,|||\cdot|||)}\\ \|\cdot\| = |||\cdot|||\\ \|\cdot\| = |||\cdot|||\end{array}\right\}$. The implications

$$\text{strong unique maximality} \begin{array}{c}\nearrow\!\!\!\!\nearrow \quad \text{unique maximality} \quad \searrow\!\!\!\!\searrow\\ \searrow\!\!\!\!\searrow \quad \text{strong maximality} \quad \nearrow\!\!\!\!\nearrow\end{array} \text{maximality} \tag{2.1}$$

are clear.

The relation between the notions just introduced and that of unitarity is being clarified by means of the following proposition.

Proposition 2.1. *Let A be a norm-unital normed algebra. Then the following conditions are equivalent:*

(1) *A is unitary.*
(2) *For every continuous norm $|||\cdot|||$ on A satisfying*
  (a) *$(A, |||\cdot|||)$ is a norm-unital normed algebra, and*
  (b) *$U_A \subseteq U_{(A,|||\cdot|||)}$,*
  we have $|||\cdot||| \leq \|\cdot\|$.
(3) *For every equivalent norm $|||\cdot|||$ on A satisfying (a) and (b) above, we have $|||\cdot||| \leq \|\cdot\|$.*
(4) *For every continuous norm $|||\cdot|||$ on A satisfying (a), (b) above, and*
  (c) *$\|\cdot\| \leq |||\cdot|||$,*
  we have $|||\cdot||| = \|\cdot\|$.

Proof. (1) $\Rightarrow$ (2).- Let $|||\cdot|||$ be a continuous norm on A satisfying (a) and (b). Then $B_{(A,|||\cdot|||)}$ is $\|\cdot\|$-closed, and hence, by the assumption (1), we have

$$B_A = \overline{co}(U_A) \subseteq \overline{co}U_{(A,|||\cdot|||)} \subseteq B_{(A,|||\cdot|||)},$$

which implies $|||\cdot||| \leq \|\cdot\|$.

(2) $\Rightarrow$ (3).- This is clear.

(3) $\Rightarrow$ (4).- This is also clear by noticing that continuous norms $|||\cdot|||$ on A satisfying (c) are equivalent to $\|\cdot\|$.

(4) $\Rightarrow$ (1).- This is the implication $(vii) \Rightarrow (vi)$ in [**2**, Theorem 3.8]. ∎

Let A be a (possibly nonassociative) normed algebra. We say that A has minimality of the $\left\{\begin{array}{c}\text{norm}\\ \text{equivalent norm}\end{array}\right\}$ if, for every $\left\{\begin{array}{c}\text{algebra}\\ \text{equivalent algebra}\end{array}\right\}$ norm $|||\cdot|||$ on A satisfying $|||\cdot||| \leq \|\cdot\|$, we have $|||\cdot||| = \|\cdot\|$. The implication

$$\text{minimality of the norm} \Rightarrow \text{minimality of the equivalent norm} \tag{2.2}$$

is clear.

Corollary 2.2. *Let A be a norm-unital associative normed algebra. Then we have:*

(i) *A is uniquely maximal if and only if it is unitary and has minimality of the equivalent norm.*

(ii) *A is strongly uniquely maximal if and only if it is unitary and has minimality of the norm.*

Proof. Assume that A is uniquely maximal. Then, by the implication (3) $\Rightarrow$ (1) in Proposition 2.1, A is unitary. Let $|||\cdot|||$ be an equivalent algebra norm on A with $|||\cdot||| \leq \|\cdot\|$. Then $|||\cdot|||$ is an equivalent norm on A converting A into a norm-unital normed algebra and satisfying $U_A \subseteq U_{(A,|||\cdot|||)}$. Since A is uniquely maximal, we have $|||\cdot||| = \|\cdot\|$. Thus A has minimality of the equivalent norm. Now, assume that A is unitary and has minimality of the equivalent norm. Then, by the implication (1) $\Rightarrow$ (3) in Proposition 2.1, A is uniquely maximal.

The above paragraph proves assertion (i) in the present corollary. By invoking the equivalence (1) $\Longleftrightarrow$ (2) in Proposition 2.1 instead of that (1) $\Longleftrightarrow$ (3), the proof of assertion (ii) is similar. ∎

Since unital C^*-algebras are unitary (by the Russo-Dye theorem [**10**, Theorem 38.12]) and have minimality of the norm (see for example [**45**, Lemma 1]), Corollary 2.2 leads to the following.

Corollary 2.3. *Let A be a unital C^*-algebra. Then A is strongly uniquely maximal.*

The following lemma will be useful for later discussion.

Lemma 2.4. *Let A be a unital normed algebra. If A has minimality of the equivalent norm, then A is norm-unital. If A has minimality of the norm, and if it is associative and maximal, then it is strongly maximal.*

Proof. The Minkowski functional of the absolutely convex hull of $B_A \cup \{\mathbf{1}\}$ is an equivalent norm (say $|||\cdot|||$) on A converting A into a norm-unital normed algebra, and satisfying $|||\cdot||| \leq \|\cdot\|$. Therefore, if A has minimality of the equivalent norm, then we have $|\!|\!|\cdot|\!|\!| = \|\cdot\|$, and hence A is norm-unital. Assume that A is associative and has minimality of the norm. Then continuous algebra norms on A are equivalent to $\|\cdot\|$ [**45**, Proposition 1]. Therefore, if in addition A is maximal, then it is strongly maximal. ■

Most notions and facts collected in this section arise previously in the literature. Thus, minimality of the norm is a very early concept, due to F. F. Bonsall [**9**]. Unique maximality is introduced by E. R. Cowie [**14**] (see also [**15**] and [**51**]). Both unitarity and maximality are independently introduced by Cowie [**14**] and Hansen-Kadison [**20**]. Maximality of normed algebras of the form $\mathcal{L}(X)$, for some normed space X, can be expressed intrinsically in terms of X [**51**, Definitions 1 and 7, Lemma 1, and Theorem 1], thus becoming a much earlier classical notion [**43**, Section 9.6]. The equivalence (1) $\Longleftrightarrow$ (4) in Proposition 2.1 is proved in [**2**]. Its consequence, that unique maximality implies unitarity, is an earlier result [**14**]. Both the fact that minimality of the norm plus unitarity implies unique maximality, and the one that C^*-algebras are uniquely maximal, are also known in [**14**].

Despite the above comments, the notions of minimality of the equivalent norm, strong maximality, and strong unique maximality seem to be new. We have introduced them by the sake of usefulness. Indeed, without the introduction of these concepts, none of the two conclusions in Corollary 2.2 could have been formulated. On the other hand, strong maximality has shown to be useful to characterize C^*-algebras (see Corollary 3.7 and Proposition 4.9 below). Perhaps because the property of minimality of the equivalent norm has not been previously introduced, the fact that unique maximality implies such a property seems to have been not noticed. This has its own interest, since some known results, originally proved under the requirement of unique maximality, actually holds under the weaker requirement of minimality of the equivalent norm. For instance, this is the case of [**14**, Corollary 8.15] (see also [**51**, Theorem 11]). We give here the actual formulation of the result of [**14**] just quoted, and include its proof by the sake of completeness, and to be referred later (see the proof of Theorem 7.8).

THEOREM 2.5. [**14**] *Let A be a norm-unital normed associative algebra with minimality of the equivalent norm, and let M be a closed ideal of A. Then, for every $u \in M$ we have $\|u\| = \sup\{\|uv\| : v \in B_M\}$.*

PROOF. Let $\pi : A \to A/M$ be the natural quotient homomorphism, and consider the equivalent vector space norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on A defined by $\|x\|_1 := \|x\| + \|\pi(x)\|$ and $\|x\|_2 := \|L_x\|_1$. Then $(A, \|\cdot\|_2)$ is a norm-unital normed algebra. Moreover, for $x, y \in A$, we have

$$\|xy\|_1 = \|xy\| + \|\pi(x)\pi(y)\| \leq \|x\|\|y\| + \|\pi(x)\|\|\pi(y)\| \leq \|x\|\|y\|_1,$$

and hence $\|x\|_2 \leq \|x\|$. Since A has minimality of the equivalent norm, we deduce that $\|\cdot\|_2 = \|\cdot\|$ on A, i.e., the equality

$$\|x\| = \sup\{\|xy\|_1 : y \in S_{(A,\|\cdot\|_1)}\} \tag{2.3}$$

holds for every $x \in A$.

Let u be in S_M, and let $\varepsilon > 0$. Then, by (2.3), there exists $y \in S_{(A,\|\cdot\|_1)}$ such that $\|uy\|_1 > 1 - \varepsilon$. Since uy lies in M, and $\|\cdot\|_1 = \|\cdot\|$ on M, we have $\|y\| = \|u\|\|y\| \geq \|uy\| > 1 - \varepsilon$, and hence $\|\pi(y)\| < \varepsilon$ because $\|y\| + \|\pi(y)\| = 1$. Therefore, there exists $w \in M$ such that $\|y + w\| < \varepsilon$, and, for such an w, we have $\|u(y + w)\| < \varepsilon$ and, consequently,

$$\|uw\| = \|u(y + w) - uy\| \geq \|uy\| - \varepsilon > 1 - 2\varepsilon.$$

On the other hand, we have

$$\|w\| \leq \|y\| + \|y + w\| < 1 + \varepsilon.$$

It follows that, putting $v := \frac{w}{1+\varepsilon} \in M$, we have $\|v\| \leq 1$ and $\|uv\| \geq \frac{1-2\varepsilon}{1+\varepsilon}$. By the arbitraryness of $\varepsilon > 0$, we deduce $1 \leq \sup\{\|uv\| : v \in B_M\}$. The converse inequality is clear. ∎

Remark 2.6. (*a*) Clearly, for finite-dimensional norm-unital Banach algebras, maximality (respectively, unique maximality, or minimality of the equivalent norm) is equivalent to strong maximality (respectively, strong unique maximality, or minimality of the norm). Moreover, by (2.1), Corollaries 2.2 and 2.3, [**20**, Theorem 6], and [**51**, Theorem 10], for a finite-dimensional norm-unital complex Banach algebra A, the following conditions are equivalent:

(1) A is a C^*-algebra (for some involution).
(2) A is uniquely maximal.
(3) A is maximal and unitary.
(4) A is semisimple and maximal.

For finite-dimensional norm-unital real Banach algebras, a similar situation happens (see Section 5 below). The subalgebra A of the C^*-algebra $M_2(\mathbb{C})$ given by $A := \{\begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} : \lambda, \mu \in \mathbb{C}\}$ becomes an example of a maximal norm-unital commutative complex Banach algebra which is not unitary [**51**, Example 3].

(*b*) It follows from (2.1) and Corollaries 2.3, 3.7, and 3.11 that, for a norm-unital commutative complex Banach algebra A, the following conditions are equivalent:

(1) A is a C^*-algebra (for some involution).
(2) A is strongly uniquely maximal.
(3) A is uniquely maximal.
(4) A is semisimple, unitary, and strongly maximal.

(*c*) Let G be a nontrivial group, and put $A := \ell_1(G)$. Then A is semisimple and unitary, but is not strongly maximal (see Remark 4.10 below). As a consequence, A does not have minimality of the norm (by (2.1) and Corollary 2.2). By choosing G finite (respectively, abelian), the fact just formulated concludes (respectively, complement) the discussion begun in Part (a) (respectively, (b)) of the present remark. On the other hand, according to [**51**, Theorem 18], we can chose G abelian and such that A is maximal. Therefore, keeping in mind Part (b) of the present remark, we are provided with examples of maximal unitary norm-unital semisimple commutative complex Banach algebras which are neither strongly maximal nor uniquely maximal. This concludes the discussion begun in Part (b). Avoiding the abelian case, we can keep in mind the main result in [**15**] to realize that there are also choices of G such that A becomes uniquely maximal. Therefore, unique maximality does not imply strong maximality (much less, strong unique maximality), and minimality of the equivalent norm does not imply minimality of the norm (by Corollary 2.2), even in the unitary complete semisimple complex case.

(d) Let X be a normed space, and put $A := \mathcal{L}(X)$. Then A has minimality of the norm [**9**]. Therefore, by (2.1) and Corollary 2.2, the following conditions are equivalent:

(1) A is strongly uniquely maximal.
(2) A is uniquely maximal.
(3) A is unitary.

Moreover, by Lemma 2.4, A is strongly maximal if (and only if) it is maximal. On the other hand, we can choose X complete and complex, and such that A is maximal but not unitary. Indeed, by [**43**, Corollary 9.8.6] and Theorem 6.18 below, this is the case of the complex Banach spaces c_0 or ℓ_1. It follows that strong maximality does not imply unitarity (much less, unique maximality), even in the primitive complete complex case.

(e) We do not know whether strong maximality plus unique maximality implies strong unique maximality, nor whether every strongly uniquely maximal norm-unital complex Banach algebra is a C^*-algebra. If this last question had an affirmative answer, then every complex Banach space X such that $\mathcal{L}(X)$ is unitary would be a Hilbert space. Indeed, this would follow from Part (d) of the present remark, and the fact that complex Banach spaces such that $\mathcal{L}(X)$ is a C^*-algebra are Hilbert spaces [**18**] (see also Theorem 6.4 below).

3. Unitary commutative Banach algebras and semisimplicity

By a complex Banach star algebra we mean a complex Banach algebra endowed with a conjugate-linear algebra involution. Such an algebra is said to be hermitian if all its self-adjoint elements have real spectrum.

The following lemma is of straightforward verification.

LEMMA 3.1. *Let A be a Banach star algebra, and let M be a $*$-invariant closed ideal of A. Then A/M, endowed with the quotient involution, becomes a Banach star algebra. Moreover, if the involution of A is continuous* *(respectively, isometric), then so is the quotient involution, and, if A is hermitian, then so is A/M.*

Theorem 3.2. *Let A be a unitary semisimple commutative complex Banach algebra. Then there exists an isometric conjugate-linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$. Moreover, the Banach star algebra $(A, *)$ is hermitian.*

Proof. By [**2**, Theorem 2.3] and its proof, there exists an abelian group G and a closed ideal M of the complex group algebra $\ell_1(G)$ such that $A = \ell_1(G)/M$ and $\pi(G) = U_A$, where $\pi : \ell_1(G) \to A$ denotes the natural quotient homomorphism. We note that, since G is abelian, the complex Banach algebra $\ell_1(G)$, endowed with its natural involution $*$, is hermitian [**37**, 3.6.2 and 9.8.14]. Let y be in M, and let ϕ be a character of A. Then $\phi \circ \pi$ is a character of $\ell_1(G)$, and y belongs to $\ker(\phi \circ \pi)$. But, since $\ker(\phi \circ \pi)$ is a closed finite-codimensional ideal of $\ell_1(G)$, it is $*$-invariant [**16**, Corollary 3.3.27], and hence $\phi(\pi(y^*)) = 0$. Since ϕ is an arbitrary character of A, and A is semisimple, we deduce $\pi(y^*) = 0$, i.e., y^* lies in M. Since y is an arbitrary element of M, we obtain that M is a $*$-invariant subset of $\ell_1(G)$. Now, since $\ell_1(G)$ is hermitian, Lemma 3.1 applies, so that the quotient involution (also denoted by $*$) is isometric, and A, endowed with such an involution, becomes a hermitian complex Banach star algebra. To conclude the proof, let us show that $u^* = u^{-1}$ for every $u \in A$. If u is in U_A, then there exists g in G such that $\pi(g) = u$, and hence $u^* = (\pi(g))^* = \pi(g^*) = \pi(g^{-1}) = (\pi(g))^{-1} = u^{-1}$. ■

Corollary 3.3. *There exists a unitary commutative complex Banach algebra which is not semisimple.*

Proof. By [**2**, Remark 2.9.b], there exists a unitary commutative complex Banach algebra A such that no continuous involution on A takes each unitary element to its inverse. By Theorem 3.2, such an algebra A cannot be semisimple. ■

Corollary 3.4. *Let A be a unitary semisimple commutative real Banach algebra. Then there exists an isometric linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$.*

Proof. By [**2**, Corollary 2.5], the normed complexification $\mathbb{C} \otimes_\pi A$ is a unitary commutative Banach algebra. Since $\mathbb{C} \otimes A$ is semisimple [**32**, Lemma 5.16, page 80], the proof is concluded by noticing that A is invariant under the involution on $\mathbb{C} \otimes A$ given by Theorem 3.2. ■

Let A be a normed associative algebra. As usual, we denote by $r_A(\cdot)$ the spectral radius, i.e., $r_A(x) := \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}$ for $x \in A$.

The following corollary refines [**20**, Theorem 19]. Its proof involves the folklore fact that, if A is a hermitian commutative complex Banach star algebra, then $r_A(\cdot)$ is a C^*-seminorm. Indeed, since A is hermitian, the Gelfand transform $\mathcal{G}$ is a $*$-homomorphism, and hence for $x \in A$ we have

$$r_A(x)^2 = \|\mathcal{G}x\|^2 = \|(\mathcal{G}x)^*(\mathcal{G}x)\| = \|\mathcal{G}(x^*x)\| = r_A(x^*x).$$

Corollary 3.5. *Let A be a norm-unital complex Banach algebra. Then the following assertions are equivalent:*

(1) *A is isometrically isomorphic to a commutative C^*-algebra.*
(2) *A is maximal and unitary, and there exists $k > 0$ such that $\|\cdot\| \leq k r_A(\cdot)$.*

Proof. In view of (2.1) and Corollary 2.3, only the implication (2) $\Rightarrow$ (1) merits a proof. Assume that (2) holds. Then A is commutative [**10**, Corollary 15.7] and semisimple, and hence, by Theorem 3.2, it is hermitian for some involution. It follows that $r_A(\cdot)$ is an equivalent C^*-norm on A. Since $U_A \subseteq U_{(A,r_A(\cdot))}$, and A is maximal, we have $U_A = U_{(A,r_A(\cdot))}$. Finally, since both A and $(A, r_A(\cdot))$ are unitary (the later, by Corollary 2.3), we deduce $\|\cdot\| = r_A(\cdot)$. ■

Remark 3.6. According to [**51**, Theorem 15], the assertion that "maximal unitary commutative complex Banach algebras are C^*-algebras" would be true, and its proof could be found in [**14**] and/or [**20**]. Nevertheless, we have looked at those works without finding in them such an assertion, nor an implicit proof of it. Actually, the statement of [**51**, Theorem 15] must contain a severe misprint, because in Section 4 of the paper [**51**] itself, it is proved that the unitary complex Banach algebra $\ell_1(\mathbb{Z})$ is maximal, thus showing that the assertion we are considering is false, even in the semisimple case. This gives Corollary 3.5 its own interest.

The following corollary provides us with two variants of Corollary 3.5.

Corollary 3.7. *Let A be a semisimple strongly maximal norm-unital complex Banach algebra. Then the following conditions are equivalent:*

(1) *A is commutative and unitary.*
(2) *There exists a conjugate-linear algebra involution $*$ on A satisfying $r(x^*x) = r(x)^2$ for every $x \in A$.*
(3) *A is isometrically isomorphic to a commutative C^*-algebra.*

Proof. (1) $\Rightarrow$ (2).- By Theorem 3.2.

(2) $\Rightarrow$ (3).- Since A is semisimple, the assumption (2) implies that $(A, *)$ is a commutative hermitian complex Banach star algebra [**1**, Corollaire 4.2.1]. As a consequence, $r(\cdot)$ is a continuous norm on A converting A into a norm-unital normed algebra, and satisfying $U_A \subseteq U_{(A,r_A(\cdot))}$. Since A is strongly maximal, we have $U_A = U_{(A,r_A(\cdot))}$. On the other hand, since $(A, *)$ is hermitian, $\exp(ix)$ lies in $U_{(A,r_A(\cdot))}$ whenever x is a self-adjoint element of A. It follows that $\|\exp(irx)\| = 1$ for such an element x and every $r \in \mathbb{R}$, and hence that A is a C^*-algebra (by the Vidav-Palmer theorem [**10**, Theorem 38.14]).

(3) $\Rightarrow$ (1).- This is clear. ■

Let A be an algebra. For x in A, we denote by L_x (respectively, R_x) the operator of left (respectively, right) multiplication by x on A. If A is normed, then we measure the closeness of A to the commutativity by the number $c(A) := \sup\{\|xy - yx\| : x, y \in B_A\}$.

LEMMA 3.8. *Let A be a norm-unital normed associative algebra with minimality of the equivalent norm, and such that $c(A) < \frac{1}{4}$, and let S be a bounded subsemigroup of A. Then we have $\|s\| \leq 1 + 4c(A)$ for every $s \in S$.*

PROOF. We may assume that $\mathbf{1}$ lies in S, so that we have

$$M := \sup\{\|s\| : s \in S\} \geq 1.$$

Then, according to the proof of [**10**, Theorem 4.1], the mapping

$$x \to \|x\|_1 := \sup\{\|sx\| : s \in S\}$$

becomes an equivalent norm on the vector space of A such that, denoting also by $\|\cdot\|_1$ the corresponding operator norm on $\mathcal{L}(A)$, we have $\|L_s\|_1 \leq 1$ for every $s \in S$. On the other hand, for $x \in A$ we have

$$\|x\| \leq \|x\|_1 = \|R_x(\mathbf{1})\|_1 \leq \|R_x\|_1\|\mathbf{1}\|_1 = M\|R_x\|_1,$$

and the definition itself of the norm $\|\cdot\|_1$ on A yields the inequality $\|yx\|_1 \leq \|x\|\|y\|_1$ for all $x, y \in A$ or, equivalently, $\|R_x\|_1 \leq \|x\|$ for every $x \in A$. Since the mapping $x \to \|R_x\|_1$ is an equivalent algebra norm on A, and A has minimality of the equivalent norm, we have $\|R_x\|_1 = \|x\|$ for every $x \in A$. Now, note that, since

$$\|x\| \leq \|x\|_1 \leq M\|x\|$$

for all $x \in A$, we have $\|T\|_1 \leq M\|T\|$ for every $T \in \mathcal{L}(A)$. It follows that, for $s \in S$, we have

$$\|s\| = \|R_s\|_1 \leq \|L_s\|_1 + \|R_s - L_s\|_1 \leq 1 + M\|R_s - L_s\| \leq 1 + M^2 c(A),$$

so $M = \sup\{\|s\| : s \in S\} \leq 1 + M^2 c(A)$, and so $\frac{M-1}{M^2} \leq c(A)$. Now, consider the real-valued function $f : t \to \frac{t-1}{t^2}$ on $[1, \infty[$, and compute its derivative to realize that f is strictly increasing on $[1, 2]$ and strictly decreasing on $[2, \infty[$. Then, since $f(M) \leq c(A)$, and $c(A) < \frac{1}{4}$ (by assumption), and $\frac{1}{4} = f(2)$, we deduce that $M \neq 2$.

Assume that $M < 2$. Then we have $\frac{M-1}{4} \leq \frac{M-1}{M^2} \leq c(A)$, and hence $M \leq 1 + 4c(A)$, as required.

To conclude the proof, it is enough to show that the possibility $M > 2$ leads to a contradiction. Indeed, if $M > 2$, then the set

$$S' := \{\mathbf{1}\} \cup \{\lambda s : 0 \leq \lambda \leq \frac{2}{M}, s \in S\}$$

is a bounded subsemigroup of A with $\mathbf{1} \in S'$ and

$$M' := \sup\{\|s'\| : s' \in S'\} = 2,$$

which is imposible by the first paragraph of the proof. ■

Corollary 3.9. *Let A be a norm-unital normed associative algebra with minimality of the equivalent norm, and such that $c(A) < \frac{1}{4}$. Then we have $\|\cdot\| \leq (1 + 4c(A))r_A(\cdot)$.*

Proof. It is enough to show that $\|x\| \leq 1 + 4c(A)$ whenever x is an element of A with $r_A(x) < 1$. Let x be such an element. Then the set $S := \{x^n : n \in \mathbb{N}\}$ is a bounded subsemigroup of A. Since x belongs to S, Lemma 3.8 yields that $\|x\| \leq 1 + 4c(A)$. ■

The following theorem follows straightforwardly from implications (2.1) and (2.2), and Corollaries 2.2, 2.3, 3.5, and 3.9.

Theorem 3.10. *Let A be a norm-unital complex Banach algebra. Then the following assertions are equivalent:*

(1) *A is isometrically isomorphic to a commutative C^*-algebra.*
(2) *A is uniquely maximal and $c(A) < \frac{1}{4}$.*

Corollary 3.11. [**14**] *Let A be a norm-unital commutative complex Banach algebra. Then A is uniquely maximal if and only if it is isometrically isomorphic to a commutative C^*-algebra.*

4. Noncommutative unitary Banach algebras

The proof of the following proposition involves only minor changes to that of Theorem 3.2, and hence it is left to the reader.

Proposition 4.1. *Let A be a unitary Banach algebra having a faithful family of finite-dimensional irreducible representations. Then there exists an isometric conjugate-linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$.*

We say that a group G is hermitian (respectively, good) if the complex Banach $*$-algebra $\ell_1(G)$ is hermitian (respectively, if every primitive ideal of $\ell_1(G)$ is $*$-invariant). We already know that abelian groups are hermitian. Moreover, hermitian groups are good [**37**, 9.8.2].

Theorem 4.2. *Let A be a unitary semisimple complex Banach algebra such that the group U_A is good. Then there exists an isometric conjugate-linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$. Moreover, if the group U_A is actually hermitian, then the Banach star algebra $(A, *)$ is hermitian.*

Proof. By [**2**, Theorem 2.3] and its proof, there exists a closed ideal M of the complex Banach algebra $\ell_1(U_A)$ such that $A = \ell_1(U_A)/M$ and $\pi(U_A) = U_A$, where $\pi : \ell_1(U_A) \to A$ denotes the natural quotient homomorphism. Noticing that $\phi \circ \pi$ is an irreducible representation of $\ell_1(U_A)$ whenever ϕ is an irreducible representation of A, and that, since the group U_A

is good, the kernels of irreducible representations of $\ell_1(U_A)$ are $*$-invariant, the proof is concluded by repeating that of Theorem 3.2, with irreducible representations instead of characters. ∎

Since abelian groups are hermitian, Theorem 4.2 contains Theorem 3.2.

REMARK 4.3. Let A be a norm-unital Banach algebra, and let G be a subgroup of U_A such that $\overline{co}(G) = B_A$. Then, looking at the proof of [**2**, Theorem 2.3], we realize that there exists a closed ideal M of $\ell_1(G)$ such that $A = \ell_1(G)/M$ and $\pi(G) = G$, where $\pi : \ell_1(G) \to A$ denotes the natural quotient homomorphism. Therefore, if in addition A is complex and semisimple, and if G is good, then we can argue as in the proof of Theorem 4.2 to obtain that there exists an isometric conjugate-linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in G$.

Let X, Y, U, and V be complex Banach spaces, and let $F : X \to U$ and $G : Y \to V$ be bounded linear (respectively, conjugate-linear) operators. Then there exists a unique bounded linear (respectively, conjugate-linear) operator $F\widehat{\otimes}G : X\widehat{\otimes}_\pi Y \to U\widehat{\otimes}_\pi V$ satisfying $(F\widehat{\otimes}G)(x\otimes y) = F(x)\otimes G(y)$ for every $(x, y) \in X \times Y$ (Here, $\widehat{\otimes}_\pi$ denotes complete projective tensor product.) As a consequence, if A and B are complex Banach $*$-algebras whose involutions are continuous, then $A\widehat{\otimes}_\pi B$ becomes canonically a complex Banach $*$-algebra.

LEMMA 4.4. *Let A and B be unital complex Banach algebras, with A commutative, let X be a complex vector space, and let ϕ be an irreducible representation of $A\widehat{\otimes}_\pi B$ on X. Then there exists a character θ of A, and an irreducible representation ψ of B on X, satisfying*

$$\phi(x \otimes y) = \theta(x)\psi(y)$$

for every $(x, y) \in A \times B$. Moreover, if booth A and B are endowed with continuous conjugate-linear algebra involutions, if $(A, *)$ is hermitian, and if $\ker(\psi)$ *is $*$-invariant, then* $\ker(\phi)$ *is $*$-invariant (relative to the canonical involution on $A\widehat{\otimes}_\pi B$).*

PROOF. According to [**42**, Theorem 2.2.6], X can be converted into a Banach space in such a way that the range of ϕ is contained in $\mathcal{L}(X)$ and that, regarded as mapping into $\mathcal{L}(X)$, ϕ is continuous. Define mappings $\theta : A \to \mathcal{L}(X)$ and $\psi : B \to \mathcal{L}(X)$ by $\theta(x) := \phi(x \otimes \mathbf{1})$ and $\psi(y) := \phi(\mathbf{1} \otimes y)$. Then θ and ψ are nonzero continuous homomorphisms, and we have

$$\phi(x \otimes y) = \phi((x \otimes \mathbf{1})(\mathbf{1} \otimes y)) = \phi(x \otimes \mathbf{1})\phi(\mathbf{1} \otimes y) = \theta(x)\psi(y)$$

for every $(x, y) \in A \times B$. Moreover, since $A \otimes \mathbf{1}$ is contained in the center of $A\widehat{\otimes}_\pi B$, $\theta(A)$ consists of complex miltiples of the identity operator on X [**10**, Corollary 25.5], and hence θ can be seen as a character of A. Let z be in $A\widehat{\otimes}_\pi B$. Then there exist sequences x_n and y_n in A and B, respectively, satisfying $\sum_{n=1}^{\infty} \|x_i\|\|y_i\| < \infty$ and $z = \sum_{n=1}^{\infty} x_i \otimes y_i$. Therefore we have

$\sum_{n=1}^{\infty} |\theta(x_i)| \|y_i\| < \infty$, which allows us to define $y := \sum_{n=1}^{\infty} \theta(x_i) y_i \in B$ such that

$$\psi(y) = \sum_{n=1}^{\infty} \theta(x_i)\psi(y_i) = \sum_{n=1}^{\infty} \phi(x_i \otimes y_i) = \phi(z).$$

Since z is arbitrary in $A\widehat{\otimes}_\pi B$, the above shows that the range of ϕ is contained in the range of ψ. This implies that ψ is an irreducible representation of B.

Assume that booth A and B are endowed with continuous conjugate-linear algebra involutions, that $(A, *)$ is hermitian, and that $\ker(\psi)$ is $*$-invariant. Let $z = \sum_{n=1}^{\infty} x_i \otimes y_i$ be in $\ker(\phi)$ (with $(x_i, y_i) \in A \times B$ and $\sum_{n=1}^{\infty} \|x_i\| \|y_i\| < \infty$). Then $\sum_{n=1}^{\infty} \theta(x_i) y_i$ lies in $\ker(\psi)$, and hence we have

$$\phi(z^*) = \sum_{n=1}^{\infty} \theta(x_i^*)\psi(y_i^*) = \sum_{n=1}^{\infty} \overline{\theta(x_i)}\psi(y_i^*) = \psi((\sum_{n=1}^{\infty} \theta(x_i) y_i)^*) = 0.$$

Thus $\ker(\phi)$ is $*$-invariant. ■

Remark 4.5. Let A, B, and X be as in Lemma 4.4 above. Then in fact we are provided with a bijective correspondence between the set of all irreducible representations of $A\widehat{\otimes}_\pi B$ on X, and the set of all couples of the form (θ, ψ), where θ is a character of A, and ψ is an irreducible representation of B on X. Indeed, if (θ, ψ) is such a couple, and if we endow X with a complete norm in such a way that $\psi(B) \subseteq \mathcal{L}(X)$ and that $\psi : B \to \mathcal{L}(X)$ becomes continuous, then

$$\theta\widehat{\otimes}\psi : A\widehat{\otimes}_\pi B \to \mathbb{C}\widehat{\otimes}_\pi \mathcal{L}(X) = \mathcal{L}(X)$$

becomes an irreducible representation of $A\widehat{\otimes}_\pi B$.

Proposition 4.6. *Homomorphic images of good groups are good groups. The direct product of an abelian group and a good group is a good group.*

Proof. Let G be a good group, and let G' be a homomorphic image of G. By [**37**, 1.9.12], there exists a $*$-invariant closed ideal M of $\ell_1(G)$ such that $\ell_1(G') = \ell_1(G)/M$ as Banach $*$-algebras. Let $\pi : \ell_1(G) \to \ell_1(G')$ be the natural quotient homomorphism. Since primitive ideals of $\ell_1(G')$ are ranges under π of primitive ideals of $\ell_1(G)$, and G is a good group, and π is a $*$-homomorphism, it follows that G' is a good group.

Now, let G be a good group, and let G' be an abelian group. By [**37**, 1.10.14], we have $\ell_1(G' \times G) = \ell_1(G')\widehat{\otimes}_\pi \ell_1(G)$ as Banach $*$-algebras. On the other hand, since G' is abelian, the complex Banach $*$-algebra $\ell_1(G')$ is hermitian and commutative. It follows from the goodness of G and Lemma 4.4 that $G' \times G$ is a good group. ■

Corollary 4.7. *Let A be a unitary semisimple real Banach algebra such that the group U_A is good. Then there exists an isometric linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$.*

Proof. Put $G := \{\lambda \otimes u : (\lambda, u) \in S_{\mathbb{C}} \times U_A\} \subseteq \mathbb{C} \otimes_\pi A$. Then, since A is unitary, G is a subgroup of $U_{\mathbb{C}\otimes_\pi A}$ such that $\overline{co}(G) = B_{\mathbb{C}\otimes_\pi A}$. On the other hand, G is a homomorphic image of $S_{\mathbb{C}} \times U_A$. Since U_A is good, and $\mathbb{C} \otimes A$ is semisimple, it follows from Proposition 4.6 and Remark 4.3 that there exists an isometric linear algebra involution $*$ on $\mathbb{C} \otimes_\pi A$ satisfying $u^* = u^{-1}$ for every $u \in U_{\mathbb{C}\otimes_\pi A}$. Finally note that, since A is unitary, it must be invariant under such an involution. ∎

There are non-hermitian groups [**37**, 12.6.24], as well as non-abelian hermitian groups [**37**, 12.6.22 and 12.1.19]. However, it seems to be an open problem whether every group is good. This problem can be equivalently formulated as follows.

Proposition 4.8. *The following assertions are equivalent:*

(1) *Every group is a good group.*
(2) *Every unitary semisimple real Banach algebra has an isometric linear algebra involution sending unitary elements to their inverses.*
(3) *The same as* (2)*, with primitive instead of semisimple.*
(4) *Every unitary semisimple complex Banach algebra has an isometric conjugate-linear algebra involution sending unitary elements to their inverses.*
(5) *The same as* (4)*, with primitive instead of semisimple.*

Proof. The implications (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are clear.

(1) $\Rightarrow$ (2).- By Corollary 4.7.

(2) $\Rightarrow$ (4) (respectively, (3) $\Rightarrow$ (5)).- Let A be a unitary semisimple (respectively, primitive) complex Banach algebra. By the assumption (2) (respectively, (3)), there exists an isometric real-linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$. But, keeping in mind the inclusion $S_{\mathbb{C}} U_A \subseteq U_A$, we realize that $*$ is actually conjugate-linear.

(5) $\Rightarrow$ (1).- Assume that Assertion (1) does not hold. Then there exists a group G, and a primitive ideal M of $\ell_1(G)$ which is not $*$-invariant. Put $A := \ell_1(G)/M$. Then, since quotients of unitary Banach algebras are unitary [**2**, Proposition 2.1], A is a unitary primitive complex Banach algebra. Moreover, since M is not $*$-invariant, we can argue as in [**2**, Remark 2.9.(b)] to realize that there is no continuous conjugate-linear algebra involution on A sending unitary elements to their inverses. Therefore Assertion (4) fails. ∎

Theorem 4.2, together with Proposition 4.9 immediately below, provides us with sufficient conditions for a complex Banach algebra to be a C^*-algebra.

Proposition 4.9. *Let A be a strongly maximal norm-unital complex Banach star algebra whose involution is isometric. Then the following assertions are equivalent:*

(1) *A is hermitian and semisimple.*
(2) *There exists an injective $*$-homomorphism from A to a C^*-algebra.*
(3) *A is a C^*-algebra.*

Proof. (1) $\Rightarrow$ (2).- By the assumption (1), the mapping $x \to s(x) := \sqrt{r(x^*x)}$ is a C^*-algebra norm on A [**10**, Corollary 41.8 and Theorem 41.9]. Therefore the inclusion of A into the completion of $(A, s(\cdot))$ is an injective $*$-homomorphism from A to a C^*-algebra.

(2) $\Rightarrow$ (3).- Let ϕ be the injective $*$-homomorphism ϕ from A to a C^*-algebra, whose existence is assumed. Then $\|\phi(\cdot)\|$ is a C^*-algebra norm on A satisfying $\|\phi(\cdot)\| \leq \|\cdot\|$ (by [**10**, Lemma 39.2.(ii)] and the fact that $*$ is isometric). This implies that $\|\phi(\cdot)\|$ is continuous and that $U_A \subseteq U_{(A,\|\phi(\cdot)\|)}$. Since A is strongly maximal, we have $U_A = U_{(A,\|\phi(\cdot)\|)}$. This equality implies that $\|\exp(irx)\| = 1$ whenever r is in $\mathbb{R}$ and x is a self-adjoint element of A, and hence that A is a C^*-algebra (by the Vidav-Palmer theorem).

(3) $\Rightarrow$ (1).- This is clear. ∎

Remark 4.10. *Let G be a group.* It is easy to verify that, if an element of the complex Banach star algebra $\ell_1(G)$ has its numerical range contained in $\mathbb{R}$, then it is a real multiple of $\mathbf{1}$. Therefore $\ell_1(G)$ cannot be a C^*-algebra unless G is reduced to its unit element. On the other hand, there exists an injective $*$-homomorphism from $\ell_1(G)$ to a C^*-algebra [**16**, Theorems 3.3.34 and 3.3.36]. It follows from Proposition 4.9 that, *if G is nontrivial, then $\ell_1(G)$ is not strongly maximal.*

5. Characterizing real C^*-algebras

Despite real C^*-algebras can be defined by different systems of intrinsic axioms (see [**23**] for a summary), we prefer to introduce them as the norm-closed self-adjoint real subalgebras of (complex) C^*-algebras. The following proposition becomes a partial generalization of Corollary 3.11 to the real setting.

Proposition 5.1. *Let A be a uniquely maximal norm-unital commutative real Banach algebra. Then A is isometrically isomorphic to a real C^*-algebra.*

Proof. By Corollaries 2.2 and 3.9, we have

$$r_A(\cdot) = \|\cdot\|. \tag{5.1}$$

On the other hand, by [**2**, Corollary 2.5], the normed complexification $\mathbb{C}\otimes_\pi A$ is a unitary commutative Banach algebra. Since $\mathbb{C} \otimes_\pi A$ is semisimple (by (5.1)), Theorem 3.2 provides us with an involution $*$ converting $\mathbb{C} \otimes_\pi A$ into a hermitian complex Banach algebra in such a way that A becomes $*$-invariant (because A is unitary). Let K stand for the carrier space of $\mathbb{C} \otimes_\pi A$, and let $\mathcal{G} : \mathbb{C} \otimes_\pi A \to C^{\mathbb{C}}(K)$ be the Gelfand transform. Then $\mathcal{G}$ is a $*$-homomorphism (because $\mathbb{C} \otimes_\pi A$ is hermitian), and hence $\mathcal{G}(A)$ is a $*$-invariant real subalgebra of the C^*-algebra $C^{\mathbb{C}}(K)$. Since $\mathcal{G}_{|A}$ is an isometry (by (5.1)), $\mathcal{G}(A)$ is closed in $C^{\mathbb{C}}(K)$ (and therefore it is a real C^*-algebra), and A is isometrically isomorphic $\mathcal{G}(A)$. ∎

A theorem of S. B. Cleveland [**13**] (see also [**45**] for an alternative proof) asserts that the topology of any algebra norm on a C^*-algebra is stronger than that of the C^*-norm. This result can be easily generalized to real C^*-algebras, as follows.

LEMMA 5.2. *Let A be a real C^*-algebra, and let $\|\cdot\|_1$ be an arbitrary algebra norm on A. Then the topology of $\|\cdot\|_1$ is stronger than that of the natural norm $\|\cdot\|$.*

PROOF. By [**42**, 4.1.13] and [**19**, 15.4], the complexification $\mathbb{C}\otimes A$ can be endowed with an algebra norm $\|\cdot\|_2$ extending $\|\cdot\|$ and converting $\mathbb{C}\otimes A$ into a C^*-algebra. On the other hand the projective tensor product $\mathbb{C}\otimes_\pi(A,\|\cdot\|_1)$ becomes a normed complex algebra [**10**, Proposition 13.3]. By Cleveland's theorem, the topology of $\mathbb{C}\otimes_\pi(A,\|\cdot\|_1)$ is stronger than that of $\|\cdot\|_2$. Since the norm of $\mathbb{C}\otimes_\pi(A,\|\cdot\|_1)$ extends $\|\cdot\|_1$, and $\|\cdot\|_2$ extends $\|\cdot\|$, the result follows. ∎

COROLLARY 5.3. *Let A be a real C^*-algebra. Then A has minimality of the norm.*

PROOF. Let $\|\cdot\|_1$ be an algebra norm on A such that $\|\cdot\|_1\leq\|\cdot\|$. By Lemma 5.2, $\|\cdot\|_1$ is equivalent to $\|\cdot\|$. Therefore, for $x\in A$ we have

$$\begin{aligned}\|x\|^2=\|x^*x\|=r_A(x^*x)&=r_{(A,\|\cdot\|_1)}(x^*x)\\ \leq\|x^*x\|_1\leq\|x^*\|_1\|x\|_1&\leq\|x^*\|\|x\|_1=\|x\|\|x\|_1,\end{aligned}$$

so $\|x\|\leq\|x\|_1$, and so $\|x\|=\|x\|_1$. ∎

Every finite-dimensional real C^*-algebra is unitary [**2**, Remark 2.9]. This fact, together with Corollaries 2.2 and 5.3, yields the following.

PROPOSITION 5.4. *Every finite-dimensional real C^*-algebra is uniquely maximal.*

REMARK 5.5. (a) In view of Corollaries 2.2 and 5.3, unital real C^*-algebras are strongly uniquely maximal if (and only if) they are unitary. Nevertheless, in general, real C^*-algebras need not be unitary, even if they are commutative. Indeed, if A denotes the real C^*-algebra $C^{\mathbb{R}}([0,1])$ (with involution equal to the identity mapping), then we have $U_A=\{\mathbf{1},-\mathbf{1}\}$. Thus, a determination of unitary commutative real C^*-algebras would be interesting in order to be provided with a reasonable converse to Proposition 5.1.

(b) The proof of Corollary 3.5 actually shows that, *if A is a unitary complex Banach algebra, and if there exits $k>0$ such that $\|\cdot\|\leq kr_A(\cdot)$, then A is bicontinuously isomorphic to a commutative C^*-algebra.* Nevertheless, such a result cannot remain true in the real setting, since the non-commutative algebra of Hamilton's quaternions is a finite-dimensional real C^*-algebra (and hence, by Corollary 2.2 and Proposition 5.4, a unitary real Banach algebra) with $\|\cdot\| = r(\cdot)$. Anyway, we are able to prove that *if A is a unitary commutative real Banach algebra, and if there exits $k > 0$ such that $\|\cdot\| \leq kr_A(\cdot)$, then A is bicontinuously isomorphic to a real C^*-algebra.* Indeed, noticing that, in the proof of Proposition 5.1, the stronger assumption that A is uniquely maximal is only applied to show that $r_A(\cdot) = \|\cdot\|$, it is enough to mimic such a proof with $r_A(\cdot) \leq \|\cdot\| \leq kr_A(\cdot)$ instead of $r_A(\cdot) = \|\cdot\|$.

(c) In view of paragraphs (a) and (b) of the present remark, the unique plausible conjecture, concerning a generalization of Corollary 3.5 to the real case, is that maximal unitary commutative real Banach algebras satisfying $\|\cdot\| \leq kr(\cdot)$, for some $k > 0$, are isometrically isomorphic to real C^*-algebras. However, unfortunately, we have been unable to prove or disprove it.

By a quaternionic normed space we mean a left vector space X over the noncommutative field $\mathbb{H}$ of Hamilton's quaternions, such that the real vector space underlying X is a normed space under a norm $\|\cdot\|$ satisfying $\|\lambda x\| = |\lambda|\|x\|$ for every $(\lambda, x) \in \mathbb{H} \times X$. By a quaternionic pre-Hilbert space we mean a left vector space X over $\mathbb{H}$ endowed with a mapping $(\cdot|\cdot) : X \times X \to \mathbb{H}$ (called the $\mathbb{H}$-valued inner product of X) which is linear in its first variable and satisfies for $x, y \in X$ the following:

(1) $(x|y)^* = (y|x)$, where $*$ stands for the standard involution of $\mathbb{H}$.
(2) $(x, x) > 0$ whenever $x \neq 0$.

Every quaternionic pre-Hilbert space becomes both a quaternionic normed space (under the norm $\|x\| := \sqrt{(x|x)}$) and a real pre-Hilbert space (under the real-valued inner product $< x|y >:= \Re e(x|y)$, where $\Re e\lambda := \frac{\lambda+\lambda^*}{2}$ for every $\lambda \in \mathbb{H}$). As we show in Lemma 5.6 immediately below, the converse is also true. Such a lemma is surely well-known, but we have not found an appropriate reference.

Lemma 5.6. *Let X be a quaternionic normed space such that the underlying real normed space is a pre-Hilbert space. Then X is a quaternionic pre-Hilbert space.*

Proof. Let $< \cdot|\cdot >$ be the real-valued inner product on X satisfying $< x|x >= \|x\|^2$ for every $x \in X$, whose existence is assumed. Since the multiplication by an element $\lambda \in S_{\mathbb{H}}$ is a surjective $\mathbb{R}$-linear isometry with inverse equal to the multiplication by λ^*, for such a λ and all $x, y \in X$ we have

$$< \lambda x|y >=< x|\lambda^* y > . \tag{5.2}$$

Now, let $\{1, i, j, k\}$ be a canonical basis of $\mathbb{H}$ (see for example [**10**, Definition 14.3]), so that we have

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j, \tag{5.3}$$

and define $(\cdot|\cdot) : X \times X \to \mathbb{H}$ by

$$(x|y) :=< x|y > -i < ix|y > -j < jx|y > -k < kx|y > .$$

Applying (5.2) (with $\lambda = i, j, k$) and (5.3), we straightforwardly realize that $(\cdot|\cdot)$ is an $\mathbb{H}$-valued inner product on X satisfying $(x|x) = \|x\|^2$ for every $x \in X$. ■

Lemma 5.6 just proved allows us to generalize to quaternionic spaces the celebrated Auerbach theorem:

COROLLARY 5.7. *Let $\mathbb{F}$ stand for either $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, let $(X, \|\cdot\|)$ be a finite-dimensional normed space over $\mathbb{F}$, and let G be a bounded subgroup of the group of all bijective linear operators on X. Then there exists an $\mathbb{F}$-valued inner product $(\cdot|\cdot)$ on X such that all elements of G become isometries on $(X, \|\cdot\|_1)$, where $\|x\|_1 := \sqrt{(x|x)}$.*

PROOF. For x in X, put $\|x\|_2 := \sup\{\|T(x)\| : T \in G\}$. Then $(X, \|\cdot\|_2)$ becomes a normed space over $\mathbb{F}$, on which all elements of G are isometries. By Auerbach's theorem [**48**, Theorem 9.5.1], there is a real-valued inner product $< \cdot|\cdot >$ on X such that all $\mathbb{R}$-linear isometries on $(X, \|\cdot\|_2)$ become also isometries on $(X, \|\cdot\|_1)$, where $\|x\|_1 := \sqrt{< x|x >}$. As a first consequence, all elements of G are isometries on $(X, \|\cdot\|_1)$, which concludes the proof in the case $\mathbb{F} = \mathbb{R}$. In the case that $\mathbb{F}$ is equal to $\mathbb{C}$ or $\mathbb{H}$, note that, since multiplications by elements of $S_{\mathbb{F}}$ are $\mathbb{R}$-linear isometries on $(X, \|\cdot\|_2)$, they are also isometries on $(X, \|\cdot\|_1)$, and therefore $(X, \|\cdot\|_1)$ is a normed space over $\mathbb{F}$. Then the proof is concluded by recalling that elements of G are isometries on $(X, \|\cdot\|_1)$, and applying Lemma 5.6 or its well-known variant for complex spaces. ■

The following theorem generalizes to the real case the corresponding result for complex algebras, first proved in [**14**] (see also [**51**]). Our proof can be also useful to clarify the original proof in the complex case.

THEOREM 5.8. *Let A be a semisimple finite-dimensional maximal norm-unital real Banach algebra. Then A is (isometrically isomorphic to) a real C^*-algebra.*

PROOF. By Wedderburn's theory, we have $A = \oplus_{i=1}^n L(X_i)$, where, for $i = 1, ..., n$, X_i is a left vector space over $\mathbb{F}_i$ ($= \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$), and $L(X_i)$ denotes the algebra of all linear operators on X_i. Fix $i = 1, ..., n$, let $\|\cdot\|_i$ be any norm on X_i converting X_i into a normed space over $\mathbb{F}_i$, and let π_i stand for the projection from A onto $L(X_i)$ corresponding to the decomposition $A = \oplus_{i=1}^n L(X_i)$. Then $\pi_i(U_A)$ is a bounded subgroup of the group of all bijective linear operators on $(X_i, \|\cdot\|_i)$. By Corollary 5.7, there exists an $\mathbb{F}_i$-valued inner product $(\cdot|\cdot)_i$ on X_i such that all elements of $\pi_i(U_A)$ become isometries on $(X_i, \|\!|\cdot\|\!|_i)$, where $\|\!|x_i\|\!|_i := \sqrt{(x_i|x_i)_i}$. Now $L(X_i)$, endowed with the operator norm corresponding to $\|\!|\cdot\|\!|_i$ (also denoted by $\|\!|\cdot\|\!|_i$), becomes a real C^*-algebra. For $a = \sum_{i=1}^{n} a_i \in A$ with $a_i \in L(X_i)$ for all i, put $|||a||| := \max\{|||a_i|||_i : i = 1, ..., n\}$. It follows that $(A, ||| \cdot |||)$ is a real C^*-algebra, and that $U_A \subseteq U_{(A,|||\cdot|||)}$. Since A is maximal, we have in fact $U_A = U_{(A,|||\cdot|||)}$. Finally, since $(A, ||| \cdot |||)$ is uniquely maximal (by Proposition 5.4), we deduce $\| \cdot \| = ||| \cdot |||$ on A. ■

It follows from Proposition 5.4, Theorem 5.8, and [**2**, Remark 2.9] that, for a finite-dimensional norm-unital real Banach algebra A, the following conditions are equivalent:

(1) A is a real C^*-algebra (for some involution).
(2) A is uniquely maximal.
(3) A is maximal and unitary.
(4) A is semisimple and maximal.

6. Banach spaces whose algebras of operators are unitary

Let X be a Banach space, and let x and f be in X and X^*, respectively. We denote by $x \otimes f$ the bounded linear operator on X defined by $(x \otimes f)(y) := f(y)x$ for every $y \in X$.

Lemma 6.1. *Let X be a Banach space, and let α be in X^{**} such that $h \otimes \alpha = T^*$ for some $h \in X^* \setminus \{0\}$ and $T \in \mathcal{L}(X)$. Then α lies in X.*

Proof. Take $x \in X$ such that $h(x) = 1$. Then, for every $g \in X^*$ we have

$$g(T(x)) = T^*(g)(x) = [(h \otimes \alpha)(g)](x) = \alpha(g)h(x) = \alpha(g).$$

Therefore, $\alpha = T(x) \in X$. ■

We recall that an algebra A of linear operator on a vector space X is said to be strictly dense if for every $k \in \mathbb{N}$ and arbitrary vectors $x_1, ..., x_k$ and $y_1, ..., y_k$ where $x_1, ..., x_k$ are linearly independent, there exists $T \in A$ such that $T(x_i) = y_i$ for all $i = 1, ..., k$. The following lemma is proved in [**42**, Theorem 2.5.19] for complex spaces and linear algebra isomorphisms, but it proof works without changes in the case of real spaces, as well as in that of complex spaces and conjugate-linear algebra isomorphisms. Indeed, [**42**, Theorem 2.5.19] is nothing other than an analytic specialization of [**24**, IV.9 and IV.11].

Lemma 6.2. *Let X and Y be real (respectively, complex) Banach spaces, let A and B strictly dense Banach algebras of bounded linear operators on X and Y, respectively, containing finite-rank operators, and let ϕ be a linear (respectively, conjugate-linear) algebra isomorphism from A onto B. Then there exists a bicontinuous linear (respectively, conjugate-linear) bijection $\psi : X \to Y$ such that $\phi(T) = \psi \circ T \circ \psi^{-1}$ for every $T \in A$.*

Let X and Y be complex Banach spaces, and let $\psi : X \to Y$ be a continuous conjugate-linear mapping. The transpose ψ^* of ψ is defined as the continuous conjugate-linear mapping from Y^* to X^* defined by $\psi^*(g)(x) := \overline{g(\psi(x))}$ for every $(g, x) \in Y^* \times X$. We note that, if ψ is bijective, then the equality

$$\psi \circ (x \otimes f) \circ \psi^{-1} = \psi(x) \otimes (\psi^{-1})^*(f) \tag{6.1}$$

holds for every $(x, f) \in X \times X^*$.

The next proposition has a forerunner in [**30**, Lemma 3]. Indeed, it is proved there that, if X is a complex Banach space, and if there exists a linear anti-automorphism ϕ of $\mathcal{L}(X)$, then X is reflexive, and there is a bicontinuous linear bijection $\psi : X \to X^*$ such that $\phi(T) = \psi^{-1} \circ T^* \circ \psi$ for every $T \in \mathcal{L}(X)$.

Proposition 6.3. *Let X be a real (respectively, complex) Banach space. Then the following conditions are equivalent:*

1. *There exists a linear (respectively, conjugate-linear) algebra involution $\bullet$ on $\mathcal{L}(X)$.*
2. *X is reflexive, and there is a bicontinuous linear (respectively, conjugate-linear) bijection $\psi : X \to X^*$ such that $\psi^* = \pm\psi$ (respectively, $\psi^* = \psi$).*

When the above conditions are fulfilled, then the mappings $\bullet$ and ψ above are related by means of the equality $T^\bullet = \psi^{-1} \circ T^* \circ \psi$ for every $T \in \mathcal{L}(X)$.

Proof. (1) $\Rightarrow$ (2).- Let $\bullet$ be the linear (respectively, conjugate-linear) algebra involution on $\mathcal{L}(X)$ whose existence is assumed. Consider the algebras A and B of bounded linear operators on X and X^*, respectively, given by $A := \mathcal{L}(X)$ and $B := \{T^* : T \in \mathcal{L}(X)\}$, both endowed with their natural operator norms, and the linear (respectively, conjugate-linear) algebra isomorphism ϕ from A onto B defined by $\phi(T) := (T^\bullet)^*$. By Lemma 6.2, there exists a bicontinuous linear (respectively, conjugate-linear) bijection $\psi : X \to X^*$ such that

$$\phi(T) = \psi \circ T \circ \psi^{-1} \tag{6.2}$$

for every $T \in A$. Let x and f be in X and X^*, respectively. By (6.1) and (6.2), we have

$$\phi(x \otimes f) = \psi(x) \otimes (\psi^{-1})^*(f). \tag{6.3}$$

Since $\phi(x \otimes f)$ belongs to B, it follows from Lemma 6.1 that $(\psi^{-1})^*(f)$ lies in X. Since f is arbitrary in X^*, and the range of $(\psi^{-1})^*$ is X^{**}, we realize that X is reflexive. Now, from (6.3) and the definition of ϕ we derive $(x \otimes f)^\bullet = (\psi^{-1})^*(f) \otimes \psi(x)$, and hence $x \otimes f = (\psi^{-1})^*(\psi(x)) \otimes \psi((\psi^{-1})^*(f))$ (because the mapping $\bullet$ is involutive). Since x and f are arbitrary in X and X^*, respectively, this implies that all elements in X are eigenvectors of $(\psi^{-1})^* \circ \psi$, and that all elements of X^* are eigenvectors of $\psi \circ (\psi^{-1})^*$, so that there exists in fact a nonzero real (respectively, complex) number λ satisfying $(\psi^{-1})^* \circ \psi = \lambda I_X$ and $\psi \circ (\psi^{-1})^* = \lambda^{-1} I_{X^*}$, where I_X and I_{X^*} stand for the identity mapping on X and X^*, respectively. Then, in the real case we have $\lambda^{-1}(\psi^{-1})^* = \lambda(\psi^{-1})^* = \psi^{-1}$, and hence $\psi^* = \pm\psi$. To conclude the proof of the present implication, let us consider the complex case. Then we have $\lambda^{-1}(\psi^{-1})^* = \overline{\lambda}(\psi^{-1})^* = \psi^{-1}$, and hence $|\lambda| = 1$ and $\psi^* = \overline{\lambda}\psi$. Taking $\mu \in \mathbb{C}$ with $\mu^2 = \overline{\lambda}$, we have $(\mu\psi)^* = \psi^*\mu = \overline{\lambda}\psi\mu = \overline{\lambda\mu}\psi = \mu\psi$. Since (6.2) determines ψ up to a nonzero complex multiple, the proof is concluded by replacing ψ with $\mu\psi$.

(2) $\Rightarrow$ (1).- Assume that Condition (2) is fulfilled. Then we straightforwardly realize that the mapping $T \to T^\bullet := \psi^{-1} \circ T^* \circ \psi$ from $\mathcal{L}(X)$ to itself becomes a linear (respectively, conjugate-linear) algebra involution. ∎

Let X be a Banach space. We put $\mathcal{G}_X := U_{\mathcal{L}(X)}$, and note that the elements of $\mathcal{G}_X$ are precisely the surjective linear isometries on X. We say that X is almost transitive if, for every $x \in S_X$, $\mathcal{G}_X(x)$ is dense in S_X. We say that X is convex-transitive if, for every $x \in S_X$, the convex hull of $\mathcal{G}_X(x)$ is dense in B_X. The weak-operator topology on $\mathcal{L}(X)$ (denoted by w_{op}) is defined as the initial topology on $\mathcal{L}(X)$ relative to the family of functionals

$$W := \{T \to f(T(x)) : (x, f) \in X \times X^*\}. \tag{6.4}$$

Now, let τ be a vector space topology on $\mathcal{L}(X)$ stronger than w_{op}. Then, since $B_{\mathcal{L}(X)}$ is w_{op}-closed, it is τ-closed, and hence contains the τ-closed convex hull of G_X. We say that $\mathcal{L}(X)$ is τ-unitary if the containment just pointed out becomes an equality.

Theorem 6.4. *Let X be a complex Banach space such that there exists a conjugate-linear algebra involution $\bullet$ on $\mathcal{L}(X)$ satisfying $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$. Then the following conditions are equivalent:*

(1) *$\mathcal{L}(X)$ is unitary.*
(2) *$\mathcal{L}(X)$ is w_{op}-unitary.*
(3) *X is convex-transitive.*
(4) *X is almost transitive.*
(5) *X is a Hilbert space.*

Proof. (1) $\Rightarrow$ (2).- Since the weak-operator topology is weaker than the norm topology.

(2) $\Rightarrow$ (3).- By the right part of [**51**, Theorem 5] (see Remark 6.17.(b) below).

(3) $\Rightarrow$ (4).- Since X is reflexive (by Proposition 6.3), and reflexive Banach spaces are Asplund spaces, it follows from the assumption (3) and [**5**, Corollary 3.3] that X is almost transitive.

(4) $\Rightarrow$ (5).- By Proposition 6.3, X is reflexive, and there is a bicontinuous conjugate-linear bijection $\psi : X \to X^*$ satisfying $\psi^* = \psi$ and $\psi^{-1} \circ T^* \circ \psi = T^{-1}$ for every $T \in \mathcal{G}_X$. For $x, y \in X$, put $(x|y) := \psi(y)(x)$. It follows that $(\cdot|\cdot)$ is a continuous nondegenerate hermitian sexquilinear form on X satisfying

$$(T(x)|T(x)) = (x|x) \tag{6.5}$$

for every $x \in X$. By multiplying $(\cdot|\cdot)$ by a suitable real number if necessary, we may assume that the continuous nondegenerate hermitian sexquilinear form $(\cdot|\cdot)$ satisfies $(x_0|x_0) = 1$ for some $x_0 \in S_X$. Then, applying (6.5) and the assumption (4), we derive $\|x\|^2 = (x|x)$ for every $x \in X$. Therefore X is a Hilbert space.

(5) $\Rightarrow$ (1).- By Corollary 2.3. ∎

It follows from Proposition 4.8 and Theorem 6.4 that, if every group is good, then every complex Banach space X such that $\mathcal{L}(X)$ is unitary actually is a Hilbert space. It is also worth mentioning that Theorem 6.4 contains the fact, already commented in Remark 2.6.(e), that complex Banach spaces X such that $\mathcal{L}(X)$ is a C^*-algebra (for some involution) are Hilbert spaces.

Proposition 6.5. *Let H be a real Hilbert space. Then $B_{\mathcal{K}(H)}$ is contained in the norm-closed convex hull of $\mathcal{G}_H$.*

Proof. It is enough to show that $B_{\mathcal{F}(H)}$ is contained in $\overline{co}(\mathcal{G}_H)$. Let $T = \sum_{i=1}^n x_i \otimes y_i$ be in $B_{\mathcal{F}(H)}$ (where, for $x, y \in H$, $x \otimes y$ denotes the operator $z \to (z|y)x$). Let H_1 stand for the linear hull of $\{x_1, ..., x_n, y_1, ..., y_n\}$, and let H_2 be the ortogonal of H_1 in H. Then T is diagonal relative to the decomposition $H = H_1 \oplus H_2$, and the restriction of T to H_2 is zero. Now, let A denote the set of those elements in $\mathcal{L}(H)$ which are diagonal relative to the decomposition $H = H_1 \oplus H_2$, and whose restrictions to H_2 are real multiples of the identity operator on H_2. Then A is a subalgebra of $\mathcal{L}(H)$ isometrically isomorphic to $\mathcal{L}(H_1) \oplus_\infty \mathbb{R}$. Since $\mathcal{L}(H_1)$ is unitary (by Proposition 5.4), it follows from [**2**, Proposition 2.8] that A is unitary. Since T lies in B_A, we deduce that $T \in \overline{co}(U_A)$. Finally, note that, since the identity mapping on H belongs to A, we have $U_A \subseteq \mathcal{G}_H$. ∎

Let X be a Banach space. The ultraweak-operator topology on $\mathcal{L}(X)$ (denoted by $\overline{w_{op}}$) is defined as the initial topology on $\mathcal{L}(X)$ relative to the family of all functionals in the norm-closed linear hull in $(\mathcal{L}(X))^*$ of the set W defined by (6.4). It is well known that, if X is reflexive, then the Banach space $\mathcal{L}(X)$ can be naturally identified with $(X \widehat{\otimes}_\pi X^*)^*$ (where $\widehat{\otimes}_\pi$ denotes the complete projective tensor product) in such a way that $\overline{w_{op}}$ becomes the natural weak* topology (i.e., the weak topology on $\mathcal{L}(X)$ relative to the duality with its predual $X \widehat{\otimes}_\pi X^*$) [**10**, Proposition 42.13].

Corollary 6.6. *Let H be a real Hilbert space. Then $\mathcal{L}(H)$ is $\overline{w_{op}}$-unitary.*

Proof. Keeping in mind Proposition 6.5, and the fact that the ultraweak-operator topology on $\mathcal{L}(H)$ is weaker that the norm topology, we deduce that the $\overline{w_{op}}$-closed convex hull of $\mathcal{G}_H$ contains $B_{\mathcal{K}(H)}$. On the other hand, since $(\mathcal{K}(H))^* = H \widehat{\otimes}_\pi H$, and $(H \widehat{\otimes}_\pi H)^* = \mathcal{L}(H)$, and the weak* topology on $\mathcal{L}(H)$ coincides with $\overline{w_{op}}$, we have that $B_{\mathcal{K}(H)}$ is $\overline{w_{op}}$-dense in $B_{\mathcal{L}(H)}$ (by Goldstine's theorem). ∎

Theorem 6.7. *Let X be a real Banach space such that there exists a linear algebra involution $\bullet$ on $\mathcal{L}(X)$ satisfying $T_0^\bullet \circ T_0 \neq 0$ for some one-dimensional operator $T_0 = x_0 \otimes f_0 \in \mathcal{L}(X)$, and $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$. Then the following conditions are equivalent:*

(1) *$\mathcal{L}(X)$ is $\overline{w_{op}}$-unitary.*
(2) *$\mathcal{L}(X)$ is w_{op}-unitary.*
(3) *X is convex-transitive.*
(4) *X is almost transitive.*
(5) *X is a Hilbert space.*

Proof. (1) $\Rightarrow$ (2).- Since the weak-operator topology is weaker than the ultraweak-operator topology.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) in the present theorem are the same as the corresponding ones in Theorem 6.4, and are proved in the same way.

(4) $\Rightarrow$ (5).- By Proposition 6.3, X is reflexive, and there is a bicontinuous linear bijection $\psi : X \to X^*$ satisfying $\psi^* = \pm\psi$ and $T^\bullet = \psi^{-1} \circ T^* \circ \psi$ for every $T \in \mathcal{L}(X)$. Assume that $\psi^* = -\psi$. Then, for every $x \in X$ we have $\psi(x)(x) = 0$, and hence

$$T_0^\bullet \circ T_0 = \psi^{-1} \circ (x_0 \otimes f_0)^* \circ \psi \circ (x_0 \otimes f_0) = \psi^{-1} \circ (f_0 \otimes x_0) \circ \psi \circ (x_0 \otimes f_0)$$
$$= (\psi^{-1}(f_0) \otimes \psi^*(x_0)) \circ (x_0 \otimes f_0) = \psi(x_0)(x_0)(\psi^{-1}(f_0) \otimes f_0) = 0,$$

which is not possible. Now assume that $\psi^* = \psi$. For $x, y \in X$, put $(x|y) := \psi(y)(x)$. It follows that $(\cdot|\cdot)$ is a continuous nondegenerate symmetric bilinear form on X satisfying $(T(x)|T(x)) = (x|x)$ for every $x \in X$. Then, that X is a Hilbert space follows from the assumption (4) as in the proof of the implication (4) $\Rightarrow$ (5) in Theorem 6.4.

(5) $\Rightarrow$ (1).- By Corollary 6.6. ∎

Corollary 6.8. *Let X be a real Banach space such that there exists a linear algebra involution $\bullet$ on $\mathcal{L}(X)$ satisfying $T_0^\bullet \circ T_0 \neq 0$ for some one-dimensional operator $T_0 = x_0 \otimes f_0 \in \mathcal{L}(X)$, and $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$. If $\mathcal{L}(X)$ is unitary, then X is a Hilbert space.*

An involution $*$ on an algebra A is said to be proper if $x^*x \neq 0$ for every $x \in A \setminus \{0\}$. A joint variant of Theorems 6.4 and 6.7 is the following result in the spirit of [**28**].

Theorem 6.9. *Let X be a real (respectively, complex) Banach space. Then the following assertions are equivalent:*

(1) *$\mathcal{L}(X)$ is maximal, and there exists a proper linear (respectively, conjugate-linear) algebra involution $\bullet$ on $\mathcal{L}(X)$ such that $T^\bullet = T^{-1}$ for every $T \in \mathcal{G}_X$.*
(2) *X is a Hilbert space.*

Proof. (1) $\Rightarrow$ (2).- Let ψ be the linear (respectively, conjugate-linear) bijection from X to X^* given by Proposition 6.3 because of the existence of the involution $\bullet$ on $\mathcal{L}(X)$, and for $x,y\in X$ put $(x|y):=\psi(y)(x)$. We know that $(\cdot|\cdot)$ is a symmetric or antisymmetric bilinear form (respectively, a hermitian sexquilinear form) on X satisfying

$$(T(x)|y)=(x|T^{\bullet}(y)) \tag{6.6}$$

for every $T\in\mathcal{L}(X)$ and all $x,y\in X$, and that for $(x,f)\in X\times X^*$ we have

$$(x\otimes f)^{\bullet}\circ(x\otimes f)=(x|x)(\psi^{-1}(f)\otimes f). \tag{6.7}$$

Now, the assumption that the involution $\bullet$ is proper, together with (6.7), gives $(x|x)\neq 0$ for every $x\in X\setminus\{0\}$ (which implies in the real case that $(\cdot|\cdot)$ cannot be antisymmetric). It follows from the connectedness of $X\setminus\{0\}$ (the case $X=\mathbb{R}$ is trivial) and the continuity of the mapping $x\to(x|x)$ from X to $\mathbb{R}$ that, by multiplying $(\cdot|\cdot)$ by a suitable real number if necessary, there is no loss of generality in assuming that $(\cdot|\cdot)$ is an inner product on X satisfying $(x_0|x_0)=1$ for some prefixed $x_0\in S_X$. Let $|\cdot|$ denote the pre-Hilbertian norm associated to $(\cdot|\cdot)$. We claim that $|\cdot|$ and $\|\cdot\|$ are equivalent norms on X. Indeed, for every $x\in X$ we have

$$|x|^2=\psi(x)(x)\leq\|\psi(x)\|\|x\|\leq\|\psi\|\|x\|^2,$$

and hence $|\cdot|\leq\sqrt{\|\psi\|}\|\cdot\|$ on X. Moreover, for $x\in X$ we can find $f\in S_{X^*}$ with $f(x)=\|x\|$, so that

$$\begin{aligned}\|x\|=f(x)=(x|\psi^{-1}(f))&\leq|x||\psi^{-1}(f)|\leq|x|\sqrt{\|\psi\|}\|\psi^{-1}(f)\|\\&\leq|x|\sqrt{\|\psi\|}\|\psi^{-1}\|\|f\|=|x|\sqrt{\|\psi\|}\|\psi^{-1}\|,\end{aligned}$$

and therefore $\|\cdot\|\leq\sqrt{\|\psi\|}\|\psi^{-1}\||\cdot|$ on X. Now that the claim has been proved, we invoke the assumption that $T^{\bullet}=T^{-1}$ for every $T\in\mathcal{G}_X$, together with (6.6), to realize that $\mathcal{G}_X\subseteq\mathcal{G}_{(X,|\cdot|)}$. In this way, denoting by $|\!|\!|\cdot|\!|\!|$ the operator norm on $\mathcal{L}(X)$ corresponding to the norm $|\cdot|$ on X, it turn out that $|\!|\!|\cdot|\!|\!|$ is an equivalent algebra norm on $\mathcal{L}(X)$ converting $\mathcal{L}(X)$ into a norm-unital normed algebra and satisfying $U_{\mathcal{L}(X)}\subseteq U_{(\mathcal{L}(X),|\!|\!|\cdot|\!|\!|)}$. It follows from the assumption that $\mathcal{L}(X)$ is maximal that $U_{\mathcal{L}(X)}=U_{(\mathcal{L}(X),|\!|\!|\cdot|\!|\!|)}$, or equivalently $\mathcal{G}_X=\mathcal{G}_{(X,|\cdot|)}$. Since $(X,|\cdot|)$ is almost transitive, and x_0 belongs to $S_X\cap S_{(X,|\cdot|)}$, it follows that $|\cdot|=\|\cdot\|$ on X.

(2) $\Rightarrow$ (1).- This is well-known. Indeed, the maximality of $\mathcal{L}(X)$, for a Hilbert space X, follows from the almost transitivity of X, together with [**48**, Theorem 9.6.3] and [**51**, Lemma 1 and Theorem 1]. ■

It follows from the above proof that a real (respectively, complex) Banach space is isomorphic to a Hilbert space if (and only if) there exists a proper linear (respectively, conjugate-linear) algebra involution $\bullet$ on $\mathcal{L}(X)$. The real case of this fact is one of the main results in [**28**].

Let X be a Banach space. Following [**29**], we define the dual weak-operator topology on $\mathcal{L}(X)$ as the initial topology on $\mathcal{L}(X)$ relative to the family of functionals

$$W':=\{T\to\alpha(T^*(f)):(f,\alpha)\in X^*\times X^{**}\},$$

and we denote it by w'_{op}. We also consider the topology $\overline{w'_{op}}$ on $\mathcal{L}(X)$, defined as the initial topology on $\mathcal{L}(X)$ relative to the family of all functionals in the norm-closed linear hull of W' in $(\mathcal{L}(X))^*$. Since $W \subseteq W'$, we have $w_{op} \leq w'_{op}$ and $\overline{w_{op}} \leq \overline{w'_{op}}$. Moreover, the two inequalities above become equalities whenever X is reflexive.

LEMMA 6.10. *Let X be a Banach space such that $\mathcal{L}(X)$ is w'_{op}-unitary. Then, for every f in S_{X^*}, we have*

$$\overline{co}\{T^*(f) : T \in \mathcal{G}_X\} = B_{X^*}.$$

PROOF. Let f be in S_{X^*}, let g be in B_{X^*}, and let $-1 < \delta < 1$. Choose $x \in B_X$ with $f(x) = \delta$, and denote by F the operator on X defined by $F(y) := g(y)x$. Then there exists a net $\{F_\lambda\}$ in the convex hull of $U_{\mathcal{L}(X)}$ converging to F in the dual weak-operator topology. Therefore, $\{\alpha(F_\lambda^*(f))\}$ converges to $\alpha(F^*(f)) = \alpha(\delta g)$ for every $\alpha \in X^{**}$. In other words, $\{F_\lambda^*(f)\}$ converges to δg in the weak topology of X^*, and hence δg belongs to the weak-closed convex hull of $\{T^*(f) : T \in \mathcal{G}_X\}$. Letting $\delta \to 1$, and keeping in mind that weakly closed convex subsets of X^* are norm-closed, the arbitraryness of g in B_{X^*} yields that

$$B_{X^*} \subseteq \overline{co}\{T^*(f) : T \in \mathcal{G}_X\}.$$

■

We recall that a complex JB^*-triple is a complex Banach space X with a continuous triple product $\{\cdot\cdot\cdot\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all x in X, the mapping $y \to \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
(2) The main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.
(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning Condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be hermitian if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$. Following [**22**], we define real JB^*-triples as norm-closed real subtriples of complex JB^*-triples. Here, by a subtriple we mean a subspace which is closed under triple products of its elements. An element e of a real JB^*-triple is said to be a tripotent if $\{eee\} = e$. Real JBW^*-triples where first introduced as those real JB^*-triples which are dual Banach spaces in such a way that the triple product becomes separately weak*-continuous (see [**22**, Definition 4.1 and Theorem 4.4]). Later, it has been shown in [**33**] that the requirement of separate w^*-continuity of the triple product is superabundant.

The following lemma becomes a generalization of [**6**, Corollary 2.6] to the real setting.

LEMMA 6.11. *Let X be an almost transitive real JBW^*-triple. Then X is a Hilbert space.*

PROOF. Keeping in mind that extreme points of the closed unit ball of a real JB^*-triple are tripotents [**22**, Lemma 3.3], the Krein-Milman theorem and the almost transitivity of X give us that the set of all nonzero tripotents of X is dense in S_X. Since the set of tripotents of X is closed, we derive that $\{xxx\} = \|x\|^2 x$ for every $x \in X$. Finally, arguing as in the proof of [**47**, Lemma 1], we realize that X is a Hilbert space. ∎

THEOREM 6.12. *Let X be a real Banach space. Then the following assertions are equivalent:*

(1) *X is a real JB^*-triple, and $\mathcal{L}(X)$ is $\overline{w'_{op}}$-unitary.*
(2) *X is a real JB^*-triple, and $\mathcal{L}(X)$ is w'_{op}-unitary.*
(3) *X^{**} is a real JB^*-triple, and $\mathcal{L}(X)$ is $\overline{w'_{op}}$-unitary.*
(4) *X^{**} is a real JB^*-triple, and $\mathcal{L}(X)$ is w'_{op}-unitary.*
(5) *X is a Hilbert space.*

PROOF. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) hold because $w'_{op} \leq \overline{w'_{op}}$, whereas the ones (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow from the fact that the bidual of every real JB^*-triple is a real JB^*-triple [**22**, Lemma 4.2].

(4) $\Rightarrow$ (5).- Since X^{**} is a real JBW^*-triple (by assumption), and B_{X^*} has extreme points (by the Krein-Milman theorem), it follows from [**40**, Corollary 2.1] that X^{**} has a "minimal tripotent" (see [**40**] for a definition), which is a point of Fréchet-differentiability of the norm [**4**, Lemma 3.1]. This implies that the norm of X^{**} is "non rough" (see [**5**] for a definition). On the other hand, since $\mathcal{L}(X)$ is w'_{op}-unitary (by assumption), Lemma 6.10 applies, giving that X^* is convex-transitive. It follows from the implication (4) $\Rightarrow$ (1) in Theorem 3.2 of [**5**] and [**5**, Remark 4.6] that X is reflexive and almost transitive. By Lemma 6.11, X is a Hilbert space.

(5) $\Rightarrow$ (1).- Keeping in mind that Hilbert spaces are reflexive, it follows from the assumption (5) and Corollary 6.6 that $\mathcal{L}(X)$ is $\overline{w'_{op}}$-unitary. On the other hand, the fact that real Hilbert spaces are real JB^*-triples is well-known. Indeed, a possible choice of the triple product $\{\cdots\}$ is the one given by $\{xyz\} := \frac{(x|y)z+(z|y)x}{2}$. ∎

Let X be a Banach space. We denote by w''_{op} the initial topology on $\mathcal{L}(X)$ relative to the family of functionals

$$W'' := \{T \to \Lambda(T^{**}(\alpha)) : (\alpha, \Lambda) \in X^{**} \times X^{***}\},$$

and by $\overline{w''_{op}}$ the initial topology on $\mathcal{L}(X)$ relative to the family of all functionals in the norm-closed linear hull of W'' in $(\mathcal{L}(X))^*$. We have

$w_{op} \leq w'_{op} \leq w''_{op}$ and $\overline{w_{op}} \leq \overline{w'_{op}} \leq \overline{w''_{op}}$, with equalities instead of inequalities if X is reflexive.

LEMMA 6.13. *Let X be a Banach space such that $\mathcal{L}(X)$ is w''_{op}-unitary. Then, for every α in $S_{X^{**}}$, we have*

$$\overline{co}\{T^{**}(\alpha) : T \in \mathcal{G}_X\} \supseteq B_X.$$

PROOF. Let α be in $S_{X^{**}}$, let x be in B_X, and let $-1 < \delta < 1$. Choose $f \in B_{X^*}$ with $\alpha(f) = \delta$, and denote by F the operator on X defined by $F(y) := f(y)x$. Then there exists a net $\{F_\lambda\}$ in the convex hull of $U_{\mathcal{L}(X)}$ converging to F in the topology w''_{op}, and hence $\{\Lambda(F_\lambda^{**}(\alpha))\}$ converges to $\Lambda(F^{**}(\alpha)) = \Lambda(\delta x)$ for every $\Lambda \in X^{***}$. Therefore, $\{F_\lambda^{**}(\alpha)\}$ converges to δx in the weak topology of X^{**}, and hence δx belongs to the weak-closed convex hull of $\{T^{**}(\alpha) : T \in \mathcal{G}_X\}$. Letting $\delta \to 1$, and keeping in mind the arbitraryness of x in B_X, we obtain that

$$B_X \subseteq \overline{co}\{T^{**}(\alpha) : T \in \mathcal{G}_X\}.$$

■

Let X be a Banach space. We say that X is L-embedded if there exists a linear projection p from X^{**} onto X satisfying

$$\|\alpha\| = \|p(\alpha)\| + \|\alpha - p(\alpha)\|$$

for every $\alpha \in X^{**}$. We note that, in such a case, $1 - 2p$ is an isometry on X^{**}. It is known that, if X satisfies the conclusion in Lemma 6.13, and if there exists a linear projection p from X^{**} onto X such that $1 - 2p$ is an isometry, then both X and X^* are superreflexive and almost transitive (see the proof of Proposition 2.3 in [**8**], and [**8**, Remark 2.6]). Therefore we have the following.

COROLLARY 6.14. *Let X be an L-embedded Banach space over $\mathbb{K}$ such that $\mathcal{L}(X)$ is w''_{op}-unitary. Then both X and X^* are superreflexive and almost transitive.*

THEOREM 6.15. *Let X be a real Banach space. Then the following assertions are equivalent:*

(1) *X is the predual of a real JBW^*-triple, and $\mathcal{L}(X)$ is $\overline{w''_{op}}$-unitary.*
(2) *X is the predual of a real JBW^*-triple, and $\mathcal{L}(X)$ is w''_{op}-unitary.*
(3) *X is a Hilbert space.*

PROOF. The implication (1) ⇒ (2) is clear, whereas the one (3) ⇒ (1) follows from Corollary 6.6 and the already commented fact that real Hilbert spaces are real JB^*-triples.

(2) ⇒ (3).- Since preduals of real JBW^*-triples are L-embedded [**3**, Proposition 2.2], the assumption (2), together with Corollary 6.14, yields that X^* is almost transitive. Then, since X^* is a JBW^*-triple (by assumption), Lemma 6.11 applies, so that X^* (and hence X) is a Hilbert space. ■

The following corollary follows straightforwardly from Theorems 6.12 and 6.15.

COROLLARY 6.16. *Let X be a real Banach space such that $\mathcal{L}(X)$ is unitary. If X, X^*, or X^{**} is a real JB^*-triple, then X is a Hilbert space.*

REMARK 6.17. (*a*) For a Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, consider the following conditions:

(1) X is a Hilbert space.
(2) $\mathcal{L}(X)$ is unitary.

We already know that, if $\mathbb{K} = \mathbb{C}$ or X is finite-dimensional, then (1) implies (2) (cf. Corollary 2.3 and Proposition 5.4). It is also known that, if X is finite-dimensional, then (2) implies (1) (see Part (*b*) of the present remark), so that (1) is actually equivalent to (2) in the finite-dimensional setting. However, the following problems seem to remain still open:

(P1) Does (1) imply (2) when $\mathbb{K} = \mathbb{R}$ and X is infinite-dimensional?
(P2) Does (2) imply (1) when X is infinite-dimensional?

Partial affirmative answers to (P2) are those given by Corollaries 6.8 and 6.16 (for $\mathbb{K} = \mathbb{R}$) and Theorems 6.4 and 6.18 (for $\mathbb{K} = \mathbb{C}$). Nevertheless, if the answer to (P1) were completely negative, then Corollaries 6.8 and 6.16 would become only characterizations of finite-dimensional real Hilbert spaces, and the following problem would merit a special consideration:

(P3) Is there an infinite-dimensional real Banach space X such that $\mathcal{L}(X)$ is unitary?

(*b*) A well-known consequence of Auerbach's Corollary 5.7 is that

($\natural$) Convex-transitive finite-dimensional real or complex Banach spaces are Hilbert spaces

(see [**48**, Theorem 9.7.1 and Proposition 9.6.1]). It follows from ($\natural$) and Lemma 6.10 that, *if X is a finite-dimensional real or complex Banach space such that $\mathcal{L}(X)$ is unitary, then X is a Hilbert space.* The result just formulated seems to have been stated first in [**36**]. Our favorite proof consists of putting together ($\natural$) and the general fact that, *if X is a real or complex Banach space such that $\mathcal{L}(X)$ is unitary (or merely w_{op}-unitary), then X is convex transitive* [**14**, Theorem 6.4] (see also [**51**, Theorem 5]). By the way, in both [**14**] and [**51**] it is claimed that, conversely, if X is a convex-transitive Banach space, then $\mathcal{L}(X)$ is w_{op}-unitary. However, the proof of such a claim contains a gap which seems to us difficult to overcome. Indeed, w_{op}-continuous linear functionals on $\mathcal{L}(X)$ need not be of the form $T \to f(T(x))$ for some $(x, f) \in X \times X^*$. Since Hilbert spaces are convex-transitive, our criticism above gives special interest to Corollary 6.6.

(*c*) Looking at the proof of Lemma 6.10 (respectively, Lemma 6.13), we realize that its conclusion remains true if the assumption that $\mathcal{L}(X)$ is w'_{op}-unitary (respectively w''_{op}-unitary) is relaxed to the one that $B_{\mathcal{K}(X)}$ is contained in the w'_{op}- (respectively w''_{op}-) closed convex hull of $\mathcal{G}_X$. Then, keeping in mind Proposition 6.5, and arguing as in the proof of Theorem 6.12 (respectively, Theorem 6.15), we realize that, *for a real Banach space X, the following assertions are equivalent:*

(1) *X is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the norm-closed convex hull of $\mathcal{G}_X$.*
(2) *X is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the w'_{op}-closed convex hull of $\mathcal{G}_X$.*
(3) *X^* a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the norm-closed convex hull of $\mathcal{G}_X$.*
(4) *X^* is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the w''_{op}-closed convex hull of $\mathcal{G}_X$.*
(5) *X^{**} is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the norm-closed convex hull of $\mathcal{G}_X$.*
(6) *X^{**} is a real JB^*-triple, and $B_{\mathcal{K}(X)}$ is contained in the w'_{op}-closed convex hull of $\mathcal{G}_X$.*
(7) *X is a Hilbert space.*

Some of the new techniques introduced in the present section allow us to complement the main results of [**8**]. Indeed, when the arguments of [**8**] involve the assumption on a Banach space X that $\mathcal{L}(X)$ is unitary, in fact they only use that such an assumption implies the conclusion in Lemma 6.13, that such a conclusion implies that of Lemma 6.10, and that the conclusion of Lemma 6.10 implies that X is convex-transitive [**8**, Lemma 2.1]. Therefore, keeping in mind Lemmas 6.10 and 6.13, and looking carefully at the arguments in [**8**], we obtain Theorem 6.18 immediately below. For a Banach space X, we denote by Δ_X the open unit ball of X.

THEOREM 6.18. *Let X be a complex Banach space X. Then the following assertions are equivalent:*

(1) *X is a JB^*-triple, and $\mathcal{L}(X)$ is unitary.*
(2) *X is a JB^*-triple, and $\mathcal{L}(X)$ is w'_{op}-unitary.*
(3) *There exists a nonlinear biholomorphic automorphism of Δ_X, and $\mathcal{L}(X)$ is unitary.*
(4) *There exists a nonlinear biholomorphic automorphism of Δ_X, and $\mathcal{L}(X)$ is w'_{op}-unitary.*
(5) *X^* is a JB^*-triple, and $\mathcal{L}(X)$ is unitary.*
(6) *X^* is a JB^*-triple, and $\mathcal{L}(X)$ is w''_{op}-unitary.*
(7) *There exists a nonlinear biholomorphic automorphism of Δ_{X^*}, and $\mathcal{L}(X)$ is unitary.*
(8) *There exists a nonlinear biholomorphic automorphism of Δ_{X^*}, and $\mathcal{L}(X)$ is w''_{op}-unitary.*
(9) *X^{**} is a JB^*-triple, and $\mathcal{L}(X)$ is unitary.*
(10) *X^{**} is a JB^*-triple, and $\mathcal{L}(X)$ is w'_{op}-unitary.*
(11) *There exists a nonlinear biholomorphic automorphism of $\Delta_{X^{**}}$, and $\mathcal{L}(X)$ is unitary.*

(12) *There exists a nonlinear biholomorphic automorphism of* $\Delta_{X^{**}}$*, and* $\mathcal{L}(X)$ *is* w''_{op}*-unitary.*
(13) X *is a Hilbert space.*

7. Nonassociative unitary Banach algebras

Given an algebra A, we denote by A^+ the algebra consisting of the vector space of A and the product $x \cdot y := \frac{xy+yx}{2}$. Following [**49**, p. 141], we define non-commutative Jordan algebras as those algebras satisfying the "Jordan identity" $(xy)x^2 = x(yx^2)$ and the "flexibility" condition $(xy)x = x(yx)$. Non-commutative Jordan algebras which are commutative are called simply Jordan algebras. An algebra A is a non-commutative Jordan algebra if and only if it is flexible and A^+ is a Jordan algebra (see again [**49**, p. 141]). Let A be a unital non-commutative Jordan algebra, and let x be an element of A. Following [**35**], we say that x is invertible in A if there exists y in A such that the equalities $xy = yx = \mathbf{1}$ and $x^2y = yx^2 = x$ hold. If x is invertible in A, then the element y above is unique, is called the inverse of x, and is denoted by x^{-1}. Moreover x is invertible in A if and only if it is invertible in the Jordan algebra A^+. This reduces most questions and results on inverses in non-commutative Jordan algebras to the commutative case. For this particular case, the reader is referred to [**25**, Section I.11].

Let A be a normed non-commutative Jordan algebra. Then A is power-associative [**49**, p. 141] (i.e., all single-generated subalgebras are associative), and hence we can consider, as in the associative case, the spectral radius mapping $r_A(\cdot)$ defined by $r_A(x) := \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ for every $x \in A$. Assume that A is norm-unital. Then unitary elements of A are defined verbatim as in the associative case, and the symbol U_A will remain to denote the set of all unitary elements of A. Similarly, the meanings of "unitary", "maximal", "strongly maximal", "uniquely maximal", or "strongly uniquely maximal" for A are translated verbatim from the particular associative case to the new one, and the implications (2.1) remain true.

Proposition 7.1. *Let A be a norm-unital complete normed non-commutative Jordan complex algebra. Then the following assertions are equivalent:*

(1) A *is isometrically isomorphic to a commutative* C^**-algebra.*
(2) A *is maximal and unitary, and there exists* $k > 0$ *such that* $\|\cdot\| \leq kr_A(\cdot)$.

Proof. According to [**43**, Proposition 31], the requirement $\|\cdot\| \leq kr_A(\cdot)$ for some $k > 0$ implies that A is associative. Now apply Corollary 3.5. ∎

The development of a theory of unitary normed non-commutative Jordan algebras, similar to that we know in the particular associative case, stumble on severe handicaps. Indeed, the set of all unitary elements of such an algebra need not be multiplicatively closed, and multiplications by unitary elements need not be isometries. These pathologies arise even in the nontrivial simplest cases, as is the one of the unitary complete normed Jordan algebra $A := B^+$, where B stands for the C^*-algebra of all 2×2 complex matrices. Here we have used the convention that, if B is a normed algebra, then B^+ is considered without notice as a new normed algebra under the norm of B. We have also kept in mind that, if B is a norm-unital normed non-commutative Jordan algebra, then we have clearly $U_B = U_{B^+}$, and hence B is unitary if and only if so is B^+.

Let A be a norm-unital normed non-commutative Jordan algebra. We denote by V_A the multiplicatively closed subset of A generated by U_A, so that, since B_A is multiplicatively closed, we have $V_A \subseteq B_A$. We say that A is weakly unitary if $\overline{co}(V_A) = B_A$.

Proposition 7.2. *Let A be a norm-unital normed non-commutative Jordan algebra. Then we have $co(V_{A^+}) \subseteq co(V_A)$. Therefore, if A^+ is weakly unitary, then A is weakly unitary.*

Proof. Since $co(V_A)$ is a convex multiplicatively closed subset of A, for $x, y \in co(V_A)$ we have $x \cdot y = \frac{1}{2}(xy + yx) \in co(V_A)$. Thus $co(V_A)$ is a multiplicatively closed subset of A^+ containing U_A. Since $U_A = U_{A^+}$, it follows that $co(V_A)$ contains V_{A^+}. ■

Proposition 7.3. *Let A be a norm-unital normed non-commutative Jordan algebra. Then the following conditions are equivalent:*

(1) *A is weakly unitary.*
(2) *For every continuous norm $|||\cdot|||$ on A satisfying*
    (a) *$(A, |||\cdot|||)$ is a norm-unital normed algebra, and*
    (b) *$U_A \subseteq U_{(A,|||\cdot|||)}$,*
    we have $|||\cdot||| \leq \|\cdot\|$.
(3) *For every equivalent norm $|||\cdot|||$ on A satisfying (a) and (b) above, we have $|||\cdot||| \leq \|\cdot\|$.*
(4) *For every continuous norm $|||\cdot|||$ on A satisfying (a), (b) above, and*
    (c) *$\|\cdot\| \leq |||\cdot|||$,*
    we have $|||\cdot||| = \|\cdot\|$.

Proof. (1) $\Rightarrow$ (2).- Let $|||\cdot|||$ be a continuous norm on A satisfying (a) and (b). Then $B_{(A,|||\cdot|||)}$ is $\|\cdot\|$-closed and multiplicatively closed, and hence, by the assumption (1), we have

$$B_A = \overline{co}(V_A) \subseteq \overline{co} V_{(A,|||\cdot|||)} \subseteq B_{(A,|||\cdot|||)},$$

which implies $|||\cdot||| \leq \|\cdot\|$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4).- These implications are clear.

(4) $\Rightarrow$ (1).- We follow with minor changes the proof of the implication $(vii) \Rightarrow (vi)$ in [**2**, Theorem 3.8]. Let $0 < \varepsilon \leq 1$. Since $co\,[(\varepsilon B_A) \cup V_A]$ is an absolutely convex subset of A contained in B_A and containing εB_A, the

Minkowski functional of $co\,[(\varepsilon B_A)\cup V_A]$ (say $|\!|\!|\cdot|\!|\!|_\varepsilon$) is a norm on A satisfying

$$\varepsilon|\!|\!|\cdot|\!|\!|_\varepsilon \leq \|\cdot\| \leq |\!|\!|\cdot|\!|\!|_\varepsilon \tag{7.1}$$

and

$$\{a\in A : |\!|\!|a|\!|\!|_\varepsilon < 1\} \subseteq co\,[(\varepsilon B_A)\cup V_A] \subseteq \{a\in A : |\!|\!|a|\!|\!|_\varepsilon \leq 1\}\,. \tag{7.2}$$

On the other hand, since $(\varepsilon B_A)\cup V_A$ is multiplicatively closed, and the convex hull of a multiplicatively closed subset is multiplicatively closed, we deduce that $|\!|\!|\cdot|\!|\!|_\varepsilon$ actually becomes an algebra norm on A (argue as in [**10**, Proposition 1.9]), and hence the inequality $1 \leq |\!|\!|\mathbf{1}|\!|\!|_\varepsilon$ holds. Now, if u is in U_A, then, by the right inclusion in (7.2), we have

$$1 \leq |\!|\!|\mathbf{1}|\!|\!|_\varepsilon = |\!|\!|uu^{-1}|\!|\!|_\varepsilon \leq |\!|\!|u|\!|\!|_\varepsilon|\!|\!|u^{-1}|\!|\!|_\varepsilon \leq 1.1 = 1\,,$$

and hence $|\!|\!|\mathbf{1}|\!|\!|_\varepsilon = |\!|\!|u|\!|\!|_\varepsilon = |\!|\!|u^{-1}|\!|\!|_\varepsilon = 1$. Therefore the normed algebra $(A,|\!|\!|\cdot|\!|\!|_\varepsilon)$ is norm-unital, and the inclusion $U_A \subseteq U_{(A,|\!|\!|\cdot|\!|\!|_\varepsilon)}$ holds. Since $|\!|\!|\cdot|\!|\!|_\varepsilon$ is a continuous norm on A with $\|\cdot\| \leq |\!|\!|\cdot|\!|\!|_\varepsilon$ (by (7.1)), it follows from the assumption (4) that $|\!|\!|\cdot|\!|\!|_\varepsilon = \|\cdot\|$. Let x be in A with $\|x\| < 1$. If follows from the left inclusion in (7.2) that x belongs to $co\,[(\varepsilon B_A)\cup V_A]$. Since $co\,[(\varepsilon B_A)\cup V_A]$ is contained in $\varepsilon B_A + co\,(V_A)$, there exists y in $co\,(V_A)$ such that $\|x-y\| \leq \varepsilon$. The arbitrariness of $\varepsilon \in]0,1]$ and $x \in \mathrm{int}(B_A)$, yields $\mathrm{int}(B_A) \subseteq \overline{co}\,(V_A)$. Therefore we have $\overline{co}\,V_A = B_A$, that is A is weakly unitary. ∎

REMARK 7.4. *Let A be a norm-unital normed non-commutative Jordan algebra.* It follows from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 7.3 (respectively, from the definitions of maximality, strong maximality, unique maximality, or strong unique maximality) that, *if A^+ is weakly unitary (respectively, maximal, strongly maximal, uniquely maximal, or strongly uniquely maximal), then A is weakly unitary (respectively, maximal, strongly maximal, uniquely maximal, or strongly uniquely maximal).* Note that the part of the above assertion concerning unitarity has been previously proved in Proposition 7.2 without involving Proposition 7.3.

The following corollary follows from Proposition 7.3 in the same way as Corollary 2.2 follows from Proposition 2.1.

COROLLARY 7.5. *Let A be a norm-unital normed non-commutative Jordan algebra. Then we have:*

(1) *A is uniquely maximal if and only if it is weakly unitary and has minimality of the equivalent norm.*
(2) *A is strongly uniquely maximal if and only if it is weakly unitary and has minimality of the norm.*

Alternative algebras are defined as those algebras A satisfying the "left alternative law" $x^2y = x(xy)$ and the "right alternative law" $yx^2 = (yx)x$. We note for later reference that the left alternative law can be written as

$$L_{x^2} = L_x^2, \tag{7.3}$$

and hence, by linearization, as

$$L_{x\cdot y} = L_x \cdot L_y. \tag{7.4}$$

By Artin's theorem [**49**, p. 29], an algebra A is alternative (if and) only if, for all $x, y \in A$, the subalgebra of A generated by $\{x, y\}$ is associative. Artin's theorem implies that alternative algebras are non-commutative Jordan algebras, and that the inverse y of an invertible element x in a unital alternative algebra is characterized by the familiar condition $xy = yx = \mathbf{1}$. Moreover, if A is a unital alternative algebra, and if x, y are invertible elements of A, then xy is invertible with

$$(xy)^{-1} = y^{-1}x^{-1}, \tag{7.5}$$

and L_x (respectively, R_x) is a bijective operator on A with $L_x^{-1} = L_{x^{-1}}$ (respectively, $R_x^{-1} = R_{x^{-1}}$) [**54**, pp. 204-205]. These facts lead straightforwardly to the following.

Lemma 7.6. *Let A be a norm-unital normed alternative algebra. Then U_A is a multiplicative closed subset of A. Moreover, for every $u \in U_A$, the operators L_u and R_u are surjective isometries on A.*

It follows from the first conclusion in Lemma 7.6 that a normed alternative algebra is unitary if (and only if) it is weakly unitary. Therefore, keeping in mind the last conclusion in Proposition 7.2, we deduce the following.

Corollary 7.7. *Let A be a norm-unital normed alternative algebra. Then A is unitary (equivalently, A^+ is unitary) if and only if A^+ is weakly unitary.*

The following theorem is a variant of Theorem 2.5 in the setting of alternative algebras.

Theorem 7.8. *Let A be a norm-unital normed alternative algebra such that A^+ has minimality of the equivalent norm, and let M be a closed ideal of A. Then, for every $u \in M$ we have $\|u\| = \sup\{\|uv\| : v \in B_M\}$.*

Proof. Let $\pi : A \to A/M$ be the natural quotient homomorphism, and consider the equivalent vector space norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on A defined by $\|x\|_1 := \|x\| + \|\pi(x)\|$ and $\|x\|_2 := \|L_x\|_1$. It follows from (7.4), that $(A^+, \|\cdot\|_2)$ is a norm-unital normed algebra. Moreover, as in the proof of Theorem 2.5, we have $\|\cdot\|_2 \leq \|\cdot\|$. Since A^+ has minimality of the equivalent norm, we deduce that $\|\cdot\|_2 = \|\cdot\|$. Now the proof is concluded by repeating verbatim the corresponding part of the argument in the associative case (see again the proof of Theorem 2.5). ∎

By a non-commutative JB^*-algebra we mean a complete normed non-commutative Jordan complex algebra (say A) endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in A. Here, for $x \in A$, U_x stands for the mapping $y \to x(xy + yx) - x^2y$ from A to A. Non-commutative JB^*-algebras which are commutative are simply called JB^*-algebras. If A is a non-commutative JB^*-algebra, then it follows from the equality $U_x(y) = 2x \cdot (x \cdot y) - x^2 \cdot y$ (which is true for all $x, y \in A$) that A^+ becomes naturally a JB^*-algebra. This fact allows us to reduce much questions and results concerning non-commutative JB^*-algebras to the commutative case.

Lemma 7.9. [**11**, Proposition 4.3] *Let A be a unital non-commutative JB^*-algebra. Then unitary elements of A are precisely those invertible elements u in A satisfying $u^{-1} = u^*$.*

Proposition 7.10. *Let A be a unital non-commutative JB^*-algebra. Then A is unitary and strongly uniquely maximal.*

Proof. That A is unitary follows from Lemma 7.9 and [**52**]. Then, that A is strongly uniquely maximal follows from Corollary 7.5 and the fact that A has minimality of the norm [**41**, Proposition 11]. ∎

Theorem 7.11. *Every weakly unitary norm-unital closed subalgebra of a non-commutative JB^*-algebra is a non-commutative JB^*-algebra.*

Proof. Let A be a non-commutative JB^*-algebra, and let B be a weakly unitary norm-unital closed subalgebra of A. It is enough to show that B is $*$-invariant. To this end, note that the unit (say $\mathbf{1}$) of B is a norm-one idempotent of A, and hence, by [**26**, Lemma 2.2], we have $\mathbf{1}^* = \mathbf{1}$. Therefore, keeping in mind [**34**, p. 188], the set $C := \{x \in A : x\mathbf{1} = \mathbf{1}x = x\}$ becomes a closed $*$-invariant subalgebra of A which contains B, and whose unit is $\mathbf{1}$. Thus, replacing A with C if necessary, we may assume that $\mathbf{1}$ is in fact a unit for A. Then we have $U_B \subseteq U_A$, so U_B is a a $*$-invariant subset of A (by Lemma 7.9), and so V_B is also $*$-invariant. Since $*$ is continuous, and $B_B = \overline{co}(V_B)$, we deduce that B is $*$-invariant, as required. ∎

Note that, in the above proof, the assumption that B is weakly unitary can be relaxed to the one that B is equal to the closed linear hull of V_B.

By an alternative C^*-algebra we mean a complete normed alternative complex algebra (say A) with a conjugate-linear algebra-involution $*$ satisfying $\|x^*x\| = \|x\|^2$ for every x in A. Since, for elements x, y in an alternative algebra, the equality $U_x(y) = xyx$ holds, it is not difficult to realize that alternative C^*-algebras become particular examples of non-commutative JB^*-algebras. In fact alternative C^*-algebras are precisely those non-commutative JB^*-algebras which are alternative [**38**, Proposition 1.3]. The following theorem generalizes and refines [**20**, Theorem 6].

Theorem 7.12. *Let A be a norm-unital normed finite-dimensional alternative complex algebra such that A is equal to the linear hull of U_A. Then there exists a conjugate-linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$. Moreover, endowed with such an involution, A is $*$-isomorphic to an alternative C^*-algebra. If in addition A is maximal, then A is in fact an alternative C^*-algebra.*

Proof. By Lemma 7.6 and Corollary 5.7, there exists an inner product $(\cdot|\cdot)$ on A such that L_u belongs to $\mathcal{G}_{(A,(\cdot|\cdot))}$ whenever u is in U_A. Then, for $u \in U_A$, we have $L_u^* = L_u^{-1} = L_{u^{-1}}$, where, for $T \in L(A)$, T^* denotes the adjoint of T relative to $(\cdot|\cdot)$. Therefore, if $\lambda_1, ..., \lambda_n \in \mathbb{C}$ and $u_1, ..., u_n \in U_A$ are such that $\sum_{k=1}^n \lambda_k u_k = 0$, then we have

$$L_{\sum_{k=1}^n \overline{\lambda_k} u_k^{-1}} = \sum_{k=1}^n \overline{\lambda_k} L_{u_k^{-1}} = \sum_{k=1}^n \overline{\lambda_k} L_{u_k}^*$$

$$= (\sum_{k=1}^n \lambda_k L_{u_k})^* = (L_{\sum_{k=1}^n \lambda_k u_k})^* = 0,$$

and hence $\sum_{k=1}^n \overline{\lambda_k} u_k^{-1} = 0$. It follows that

$$x = \sum_{k=1}^n \lambda_k u_k \to x^* := \sum_{k=1}^n \overline{\lambda_k} u_k^{-1}$$

(with $\lambda_1, ..., \lambda_n \in \mathbb{C}$ and $u_1, ..., u_n \in U_A$) is a well-defined mapping from A to A, which, in view of (7.5), becomes a conjugate-linear algebra involution on A satisfying

$$L_x^* = L_{x^*} \tag{7.6}$$

for every $x \in A$, and

$$u^* = u^{-1} \tag{7.7}$$

whenever u lies in U_A. Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be the vector space norms on A defined by $\|x\|_1 := \sqrt{(x|x)}$ and $\|x\|_2 := \|L_x\|_1$. It follows from (7.4) and (7.6) that the mapping $x \to L_x$ is an isometric $*$-homomorphism from $(A^+, *, \| \cdot \|_2)$ to $\mathcal{L}((A, \| \cdot \|_1))^+$. Therefore, since $\mathcal{L}((A, \| \cdot \|_1))$ is a C^*-algebra, $(A^+, *, \| \cdot \|_2)$ is a JB^*-algebra. Since $*$ is an algebra involution on A, it follows from [**44**, Theorem 1] that $(A, *, \| \cdot \|_2)$ is an alternative C^*-algebra.

Assume that A is maximal. Then, since $U_A \subseteq U_{(A,\|\cdot\|_2)}$ (by (7.7) and Lemma 7.9), we have $U_A = U_{(A,\|\cdot\|_2)}$. Since $(A, \| \cdot \|_2)$ is uniquely maximal (by Proposition 7.10), we have in fact $\| \cdot \| = \| \cdot \|_2$. ■

The following lemma is a byproduct of the proof of [**12**, Theorem 2.11].

Lemma 7.13. *Let X be a finite-dimensional complex vector space, and let g be a nondegenerate symmetric bilinear form on X. Then, given an arbitrary inner product $< \cdot|\cdot >$ on X, we have:*

1. *There exists a unique bijective conjugate-linear mapping $\sigma : X \to X$ satisfying $g(x, y) =< x|\sigma(y) >$ for all $x, y \in X$.*
2. *The bijective linear operator $F := \sigma^2$ on $(X, < \cdot|\cdot >)$ is positive, and hence the mapping $(\cdot|\cdot) : X \times X \to \mathbb{C}$, defined by*

$$(x|y) :=< F^{\frac{1}{2}}(x)|y >,$$

is an inner product on X*.*

(3) *The mapping* $* := F^{-\frac{1}{2}} \circ \sigma$ *is an isometric conjugate-linear involution on* $(X, (\cdot|\cdot))$ *satisfying* $g(x, y) = (x|y^*)$ *for all* $x, y \in X$*.*

Let X be a finite-dimensional vector space, and let g be a nondegenerate symmetric bilinear form g on X. For T in the algebra $L(X)$ of all linear operators on X, we denote by $T^\sharp$ the unique element in $L(X)$ satisfying $g(T(x), y) = g(x, T^\sharp(y))$ for all $x, y \in X$, and we recall that the mapping $T \to T^\sharp$ is a linear algebra involution on $L(X)$.

Corollary 7.14. *Let* X *be a finite-dimensional complex Banach space, and let* g *be a nondegenerate symmetric bilinear form on* X*. Then there exists an inner product* $(\cdot|\cdot)$ *on* X*, and an isometric conjugate-linear involution* $*$ *on* $(X, (\cdot|\cdot))$ *satisfying:*

(1) $g(x, y) = (x|y^*)$ *for all* $x, y \in X$*.*

(2) $\mathcal{G}_X \cap \mathcal{G}_X^\sharp \subseteq \mathcal{G}_{(X,(\cdot|\cdot))}$*, where* $\mathcal{G}_X^\sharp := \{T^\sharp : T \in \mathcal{G}_X\}$*.*

Proof. By Corollary 5.7, there exists an inner product $< \cdot|\cdot >$ on X such that $\mathcal{G}_X \subseteq \mathcal{G}_{(X,<\cdot|\cdot>)}$. Let σ, F, $(\cdot|\cdot)$, and $*$ be the mappings corresponding to $< \cdot|\cdot >$ via Lemma 7.13. Then, by that lemma, $(\cdot|\cdot)$ is an inner product on X, and $*$ is an isometric conjugate-linear involution on $(X, (\cdot|\cdot))$ satisfying condition (1) in the present corollary. Let T be in $\mathcal{G}_X \cap \mathcal{G}_X^\sharp$. Then, since T belongs to $\mathcal{G}_{(X,<\cdot|\cdot>)}$, we have

$$< x|T^{-1}(\sigma(y)) >=< T(x)|\sigma(y) >= g(T(x), y)$$
$$= g(x, T^\sharp(y)) =< x|\sigma(T^\sharp(y)) >$$

for all $x, y \in X$, and hence $T^{-1} \circ \sigma = \sigma \circ T^\sharp$. Since $\mathcal{G}_X \cap \mathcal{G}_X^\sharp$ is a $\sharp$-invariant group of bijective operators on X, it follows

$$T \circ F = T \circ \sigma^2 = \sigma \circ (T^{-1})^\sharp \circ \sigma = \sigma \circ (T^\sharp)^{-1} \circ \sigma = \sigma^2 \circ (T^\sharp)^\sharp = F \circ T,$$

and hence $T \circ F^{\frac{1}{2}} = F^{\frac{1}{2}} \circ T$ (because $F^{\frac{1}{2}}$ is a limit of polynomials in F). Finally, applying again that T belongs to $\mathcal{G}_{(X,<\cdot|\cdot>)}$, we have

$$(T(x)|T(x)) =< F^{\frac{1}{2}}(T(x))|T(x) >$$
$$=< T(F^{\frac{1}{2}}(x))|T(x) >=< F^{\frac{1}{2}}(x)|x >= (x|x)$$

for every $x \in X$, and hence T belongs to $\mathcal{G}_{(X,(\cdot,|\cdot))}$. ■

Up to isomorphisms, there exists a unique simple finite-dimensional alternative nonassociative complex algebra, which will be denoted by $\mathcal{C}$. We refer the reader to [**49**] for the fact just quoted, as well as for the remaining properties of $\mathcal{C}$ needed in our argument.

Proposition 7.15. *Let* $\|\cdot\|$ *be a vector space norm on* $\mathcal{C}$*. Then there exists a vector space norm* $|||\cdot|||$ *on* $\mathcal{C}$*, and a vector space involution* $*$ *on* $\mathcal{C}$*, satisfying:*

(1) $(\mathcal{C}^+, |||\cdot|||, *)$ *is a* JB^**-algebra.*

(2) $u^* = u^{-1}$ *whenever* u *is in* $\mathcal{C}$ *such that* L_u *and* R_u *are isometries on* $(\mathcal{C}, \|\cdot\|)$.

Proof. There exists a unique linear algebra involution τ on $\mathcal{C}$ such that $x+\tau(x) \in \mathbb{C}\mathbf{1}$ and $x\tau(x) = \tau(x)x \in \mathbb{C}\mathbf{1}$ for every $x \in \mathcal{C}$. Therefore, if for $x \in \mathcal{C}$ we put $x + \tau(x) = 2t(x)\mathbf{1}$ and $x\tau(x) = n(x)\mathbf{1}$, with $t(x)$ and $n(x)$ in $\mathbb{C}$, then the mappings $t(\cdot)$ and $n(\cdot)$ are linear and quadratic, respectively, and we have

$$x^2 - 2t(x)x + n(x)\mathbf{1} = 0. \tag{7.8}$$

On the other hand, an element $x \in \mathcal{C}$ is invertible if and only if $n(x) \neq 0$, and, if this is the case, then

$$x^{-1} = n(x)^{-1}\tau(x). \tag{7.9}$$

Moreover, the mapping $g : (x, y) \to t(xy)$ becomes a nondegenerate symmetric bilinear form on $\mathcal{C}$ satisfying

$$g(xy, z) = g(x, yz) \tag{7.10}$$

and

$$g(\tau(x), y) = g(x, \tau(y)) \tag{7.11}$$

for all $x, y, z \in \mathcal{C}$.

Define a vector space norm $\|\cdot\|_1$ on $\mathcal{C}$ by

$$\|x\|_1 := \|x\| + \|\tau(x)\|. \tag{7.12}$$

Then, applying Corollary 7.14, we find an inner product $(\cdot|\cdot)$ on $\mathcal{C}$, and an isometric conjugate-linear vector space involution $*$ on $(\mathcal{C}, (\cdot|\cdot))$ satisfying:

$$g(x, y) = (x|y^*) \tag{7.13}$$

for all $x, y \in \mathcal{C}$, and

$$\mathcal{G}_{(\mathcal{C},\|\cdot\|_1)} \cap \mathcal{G}^{\sharp}_{(\mathcal{C},\|\cdot\|_1)} \subseteq \mathcal{G}_{(\mathcal{C},(\cdot|\cdot))}. \tag{7.14}$$

We note that, by (7.13), for T in $L(\mathcal{C})$, we have

$$T^\bullet = * \circ T^\sharp \circ *, \tag{7.15}$$

where $T^\bullet$ denotes the adjoint of T relative to $(\cdot|\cdot)$. Now, applying (7.11), (7.12), (7.14), and (7.15), we obtain that τ commutes with $*$, and hence that $\mathbb{C}\mathbf{1}$ (equal to the range of $1+\tau$) is $*$-invariant. Therefore, since $*$ is isometric on $(\mathcal{C}, (\cdot|\cdot))$, we have $\mathbf{1}^* = \gamma\mathbf{1}$ for some $\gamma \in S_{\mathbb{C}}$. But, since

$$1 = t(\mathbf{1}) = g(\mathbf{1}, \mathbf{1}) = (\mathbf{1}|\mathbf{1}^*) = \overline{\gamma}(\mathbf{1}|\mathbf{1})$$

(by (7.13)), we have in fact $\gamma = 1$, and hence

$$\mathbf{1}^* = \mathbf{1}. \tag{7.16}$$

Put $U := \{u \in \mathcal{C} : \{L_u, R_u\} \subseteq \mathcal{G}_{(\mathcal{C},\|\cdot\|)}\}$. We claim that U is τ-invariant. To prove the claim, let us take u in U. Then L_u is a surjective linear isometry on a suitable complex Banach space, and satisfies $L_u^2 - 2t(u)L_u + n(u) = 0$ (by (7.8) and (7.3)). This implies that $|n(u)| = 1$ (because $n(u)$ is the product of the elements in the spectrum of L_u). Therefore, since $\tau(u)=n(u)u^{-1}$ (by (7.9)), and U is closed by passing to inverses and by multiplication of its elements by unimodular numbers, it follows that $\tau(u)$ lies in U. Now that the claim has been proved, it follows from (7.12) that L_u and R_u lie in $\mathcal{G}_{(\mathcal{C},|\!|\!|\cdot|\!|\!|_1)}$ whenever u belongs to U. Then, applying (7.10), (7.14), and (7.15), we obtain that $L_u^{-1}=*\circ R_u\circ *$ whenever u belongs to U, and hence, by (7.16), that $u^{-1}=L_{u^{-1}}(\mathbf{1})=L_u^{-1}(\mathbf{1})=(*\circ R_u\circ *)(\mathbf{1})=u^*$. This proves condition (2) in the statement.

Let x and y be in $\mathcal{C}$. Then, since $x\tau(x)=n(x)\mathbf{1}$, we have $t(x\tau(x))=n(x)$. This allows us to linearize (7.8) to obtain

$$x\cdot y-t(x)y-t(y)x+\frac{t(x\tau(y))+t(y\tau(x))}{2}\mathbf{1}=0.$$

Keeping in mind the definition of g, and invoking (7.11), (7.13), and (7.16), the equality above reads as

$$x\cdot y=(x|\mathbf{1})y+(y|\mathbf{1})x-(x|\tau(y)^*)\mathbf{1}. \tag{7.17}$$

Replacing in (7.17) x and y with y^* and x^*, respectively, keeping in mind that $*$ is an isometric conjugate-linear involution on $(\mathcal{C},(\cdot,\cdot))$, and applying (7.16), we realize that $(x\cdot y)^*=y^*\cdot x^*$. Thus, $*$ is an algebra involution on $\mathcal{C}^+$. Put $\sigma:=*\circ\tau$. Since τ is an isometry on $(\mathcal{C},(\cdot|\cdot))$ (by (7.11), (7.12), and (7.14)), and commutes with $*$, σ becomes an isometric conjugate-linear vector space involution on $(\mathcal{C},(\cdot|\cdot))$. Therefore $\mathcal{C}$ becomes a JB^*-triple under the triple product

$$\{xzy\}:=(x|z)y+(y|z)x-(x|\sigma(y))\sigma(z),$$

and a suitable norm $|\!|\!|\cdot|\!|\!|$ satisfying $|\!|\!|xyz|\!|\!|\leq|\!|\!|x|\!|\!||\!|\!|y|\!|\!||\!|\!|z|\!|\!|$ for all $x,y,z\in\mathcal{C}$, and $|\!|\!|x|\!|\!|^2=(x|x)$ whenever x is in $\mathcal{C}$ with $\sigma(x)=x$ [**50**, Example 20.36]. Since $x\cdot y=\{x\mathbf{1}y\}$ (by (7.17)), and $(\mathbf{1}|\mathbf{1})=1$, it follows that $|\!|\!|x\cdot y|\!|\!|\leq|\!|\!|x|\!|\!||\!|\!|y|\!|\!|$ for all $x,y,\in\mathcal{C}$. Thus, $|\!|\!|\cdot|\!|\!|$ is an algebra norm on $\mathcal{C}^+$. On the other hand, a straightforward computation, involving (7.17), shows that $U_x(x^*)=\{xxx\}$, so that we have $|\!|\!|U_x(x^*)|\!|\!|=|\!|\!|x|\!|\!|^3$ for every $x\in\mathcal{C}$. In this way, the proof of condition (1) in the statement is concluded. ■

Theorem 7.16. *Let A be a semisimple finite-dimensional norm-unital normed complex alternative algebra such that A^+ is maximal. Then A is (isometrically isomorphic to) an alternative C^*-algebra.*

Proof. By [**49**, Theorem 3.12], we have $A=\oplus_{i=1}^n A_i$, where, for $i=1,...,n$, either $A_i=L(X_i)$ for some complex vector space X_i, or $A_i=\mathcal{C}$. In the first case, we know that there exists an involution $*_i$ and a norm $|\!|\!|\cdot|\!|\!|_i$ on A_i such that $(A_i,|\!|\!|\cdot|\!|\!|_i,*_i)$ becomes a C^*-algebra in such a way that $\pi_i(U_A)\subseteq U_{(A_i,|\!|\!|\cdot|\!|\!|_i)}$, where π_i stands for the projection from A onto A_i corresponding to the decomposition $A=\oplus_{i=1}^n A_i$ (see the proof of Theorem 5.8). In any case, for $u\in U_A$ and $x_i\in A_i$, we have $\pi_i(u)x_i=ux_i$ and $x_i\pi_i(u)=x_iu$, and hence, by Lemma 7.6,

$\|\pi_i(u)x_i\| = \|x_i\|$ and $\|x_i\pi_i(u)\| = \|x_i\|$. It follows from Proposition 7.15 that, in the second case, there exists an involution $*_i$ and a norm $\|\!| \cdot |\!\|_i$ on A_i such that $(A_i^+, \|\!| \cdot |\!\|_i, *_i)$ becomes a JB^*-algebra in such a way that $\pi_i(U_A) \subseteq U_{(A_i^+, \|\!|\cdot|\!\|_i)}$. For $a = \sum_{i=1}^n a_i \in A$ with $a_i \in A_i$ for all i, put $\|\!|a|\!\| := \max\{\|\!|a_i|\!\|_i : i = 1, ..., n\}$, and $a^* := \sum_{i=1}^n a_i^{*_i}$. It follows that $(A^+, \|\!| \cdot |\!\|, *)$ is a JB^*-algebra, and that $U_{A^+} = U_A \subseteq U_{(A^+, \|\!|\cdot|\!\|)}$. Since A^+ is maximal, we have in fact $U_{A^+} = U_{(A^+, \|\!|\cdot|\!\|)}$. Since $(A^+, \|\!|\cdot|\!\|)$ is uniquely maximal (by Proposition 7.10), we deduce $\|\cdot\| = \|\!|\cdot|\!\|$ on A. Now $\|\cdot\|$ is an algebra norm on A converting A^+ into a JB^*-algebra, so that, by [**46**, Corollary 1.2], A is an alternative C^*-algebra. ∎

8. Real non-commutative JB^*-algebras

By a real non-commutative JB^*-algebra we mean a closed $*$-invariant real subalgebra of a (complex) non-commutative JB^*-algebra. If B is a non-commutative JB^*-algebra, and if τ is an involutive conjugate-linear $*$-automorphism of B, then the set $A := \{x \in B : \tau(x) = x\}$ is a closed $*$-invariant real subalgebra of B, and hence a real non-commutative JB^*-algebra. Note that, in this case, we have $B = A \oplus iA$, and therefore B is algebraically isomorphic to the complexification $\mathbb{C} \otimes A$ of A.

Lemma 8.1. *Let A be a real non-commutative JB^*-algebra. Then there exists a non-commutative JB^*-algebra B, and an involutive conjugate-linear $*$-automorphism τ of B, such that $A = \{x \in B : \tau(x) = x\}$.*

Proof. Let C be a non-commutative JB^*-algebra containing A as a closed $*$-invariant real subalgebra. Let $\overline{C}$ stand for a set-copy of C with operations and norm defined by $\overline{x} + \overline{y} := \overline{x+y}$, $\overline{x}\,\overline{y} := \overline{xy}$, $\lambda\overline{x} := \overline{\overline{\lambda}x}$ (where, for $\lambda \in \mathbb{C}$, $\overline{\lambda}$ means the conjugate of λ), $\overline{x}^* := \overline{x^*}$, and $\|\overline{x}\| := \|x\|$. Then $\overline{C}$ is a non-commutative JB^*-algebra, and hence $D := C \oplus_\infty \overline{C}$ is a non-commutative JB^*-algebra. Moreover, the mapping $\tau : (x, \overline{y}) \to (y, \overline{x})$ becomes an involutive conjugate-linear $*$-automorphism of D, and A can be identified with the closed $*$-invariant real subalgebra of D given by $\{(x, \overline{x}) : x \in A\}$. Now, $B := A + iA$ is a closed $*$- and τ-invariant subalgebra of D, and we have $A = \{x \in B : \tau(x) = x\}$. ∎

For a non-commutative Jordan algebra A, let $(x, y) \to U_{x,y}$ be the unique symmetric bilinear mapping from $A \times A$ to $L(A)$ satisfying $U_{x,x} = U_x$ for every $x \in A$. It is well-known that, if A is a non-commutative JB^*-algebra, then A becomes a JB^*-triple under its own norm and the triple product $\{\cdots\}$ defined by $\{xyz\} := U_{x,z}(y^*)$ (see [**11**] and [**53**]). Therefore, real non-commutative JB^*-algebras are real JB^*-triples in a natural way. These facts will be applied without notice in what follows (mainly, in the proof of Proposition 8.3 below). The following lemma follows from Lemma 8.1 and [**11**, Proposition 4.3 and Lemma 4.1].

LEMMA 8.2. *Let A be a unital real non-commutative JB^*-algebra. Then U_A is closed in A, and every element of U_A is an extreme point of B_A.*

It is well-known that C^*-algebras whose Banach space is reflexive are finite-dimensional. This fact is no longer true when non-commutative JB^*-algebras replace C^*-algebras. Anyway, non-commutative JB^*-algebras whose Banach space is reflexive have a unit, and are in fact Hilbertizable (i.e., their Banach spaces are isomorphic to Hilbert spaces) [**39**, Theorem 3.5]. It follows from Lemma 8.1 that real non-commutative JB^*-algebras whose Banach space is reflexive have a unit, and are Hilbertizable.

PROPOSITION 8.3. *Let A be a Hilbertizable real non-commutative JB^*-algebra. Then the extreme points of B_A are precisely the unitary elements of A.*

PROOF. By Lemma 8.2, unitary elements of A are extreme points of B_A. Le u be an extreme point of B_A. Consider the non-commutative JB^*-algebra B, and the involutive conjugate-linear $*$-automorphism τ of B, given by Lemma 8.1. By the proof of [**22**, Lemma 3.3], u is a "complex extreme point" of B_B, so u is an extreme point of B_B (by [**11**, Lemma 4.1]), and so, since B is Hilbertizable, u is a denting point of B_B (by [**7**, Theorem 4.1]). Since $B_B = \overline{co}(U_B)$ (by Proposition 7.10), it follows from [**2**, Lemma 4.2] and Lemma 8.2 that u belongs to U_B. Therefore u lies in $U_B \cap A = U_A$. ■

COROLLARY 8.4. *Let A be a Hilbertizable real non-commutative JB^*-algebra. Then A is unitary.*

PROOF. By the Krein-Milman theorem and Proposition 8.3, the convex hull of U_A is weak-dense in B_A. But weak-closed convex subsets of A are norm-closed. ■

It is known that the topology of any algebra norm on a JB^*-algebra is stronger than that of the JB^*-norm [**41**, Theorem 10]. Keeping in mind this result and Lemma 8.1, we can argue as in the proof of Lemma 5.2 to obtain the following.

LEMMA 8.5. *Let A be a real non-commutative JB^*-algebra, and let $\|\cdot\|_1$ be an arbitrary algebra norm on A. Then the topology of $\|\cdot\|_1$ is stronger than that of the natural norm $\|\cdot\|$.*

By a real alternative C^*-algebra we mean a closed $*$-invariant real subalgebra of a (complex) alternative C^*-algebra. Since real alternative C^*-algebras are real non-commutative JB^*-algebras, we can argue as in the proof of Corollary 5.3 (applying Lemma 8.5 instead of Lemma 5.2) to obtain the following.

COROLLARY 8.6. *Let A be a real alternative C^*-algebra. Then A has minimality of the norm.*

Now, putting together Corollaries 7.5, 8.4, and 8.6, we derive the following.

Corollary 8.7. *Let A be a finite-dimensional real alternative C^*-algebra. Then A is uniquely maximal.*

In Corollary 8.7 just formulated, we could have relaxed the requirement that A is finite-dimensional to the one that A is Hilbertizable. However, such a relaxing is only apparent. Indeed, it follows from Lemma 8.1 and [**27**, Remark 7.3] that real alternative C^*-algebras whose Banach space is reflexive are finite-dimensional.

The tensor product of two non-commutative Jordan algebras need not be a non-commutative Jordan algebra. Indeed, if $M_2(\mathbb{F}) \otimes A$ is flexible, for an algebra A over a field $\mathbb{F}$, then A is associative. Anyway, the tensor product $B \otimes A$ is an alternative (respectively, non-commutative Jordan) algebra whenever B is an associative commutative algebra, an A is an alternative (respectively, non-commutative Jordan) algebra. Moreover, as in the associative case, we have the following.

Proposition 8.8. *Let B be a norm-unital normed associative commutative algebra, an let A be a norm-unital normed non-commutative Jordan algebra, both over the same fiel $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If A and B are unitary, then the projective tensor product $B \otimes_\pi A$ is a unitary normed non-commutative Jordan algebra. If A is the closed linear hull of U_A, and if B is the closed linear hull of U_B, then $B \otimes_\pi A$ is the closed linear hull of $U_{B\otimes_\pi A}$.*

In particular, we have the following.

Corollary 8.9. *Let A be a norm-unital normed non-commutative Jordan real algebra. If A is unitary, then the normed complexification $\mathbb{C} \otimes_\pi A$ of A is unitary. If A is the closed linear hull of U_A, then $\mathbb{C} \otimes_\pi A$ is the closed linear hull of $U_{\mathbb{C}\otimes_\pi A}$.*

Now we are ready to prove the main result in this section.

Theorem 8.10. *Let A be a norm-unital normed finite-dimensional alternative real algebra such that A is equal to the linear hull of U_A. Then there exists a linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$. Moreover, endowed with such an involution, A is $*$-isomorphic to a real alternative C^*-algebra. If in addition A is maximal, then A is in fact a real alternative C^*-algebra.*

Proof. By Corollary 8.9 and Theorem 7.12, there exists a norm $|\!|\!| \cdot |\!|\!|$ and an involution $*$ on $\mathbb{C} \otimes A$, such that $(\mathbb{C} \otimes A, |\!|\!| \cdot |\!|\!|, *)$ is an alternative C^*-algebra, and $u^* = u^{-1}$ for every $u \in U_{\mathbb{C}\otimes_\pi A}$. This last property of $*$ implies that A is $*$-invariant, and hence that $(A, |\!|\!| \cdot |\!|\!|, *)$ is a real alternative C^*-algebra.

Assume that A is maximal. Then, since $U_A \subseteq U_{(A,|\!|\!|\cdot|\!|\!|)}$, we have $U_A = U_{(A,|\!|\!|\cdot|\!|\!|)}$. Since $(A, |\!|\!| \cdot |\!|\!|)$ is uniquely maximal (by Corollary 8.7), we have in fact $\| \cdot \| = |\!|\!| \cdot |\!|\!|$ on A. ∎

Let A be a nonassociative algebra. Then maximal modular left ideals of A are defined verbatim as in the associative case, and primitive ideals of A are defined as the cores of maximal modular left ideals of A. Here, by the core of a given subspace X of A we mean the largest ideal of A contained in X. According to [**10**, Definition 24.11], the notion of primitive ideal just introduced agrees with the familiar one when A is associative. The radical of A is defined as the intersection of all primitive ideals of A, and A is said to be primitive (respectively, semisimple) if zero is a primitive ideal of A (respectively, if the radical of A is equal to zero). If A is complete normed, then, as in the associative case, maximal modular left ideals of A are closed, and hence primitive ideals of A are closed either.

In the case of non-commutative Jordan algebras, the notions of radical, primitivity, and semisimplicity, introduced above, are not subtle enough to allow the development of a satisfactory structure theory, and therefore they have been suitably refined in the literature (see [**35**], [**21**], and [**17**]). Nevertheless, in the particular case of alternative algebras, such refinements are unnecessary [**54**, Theorem 10.4.5]. Moreover, primitive alternative algebras are either associative or unital simple eight-dimensional over their centers [**54**, Theorem 10.1.1]. It follows from the Gelfand-Mazur theorem that primitive alternative normed algebras are either associative or finite-dimensional.

Proposition 8.11 immediately below complement Proposition 4.8. Among other facts, its proof involves the one that, as in the associative case [**2**, Proposition 2.1], quotients of unitary normed non-commutative Jordan algebras are unitary.

Proposition 8.11. *The following assertions are equivalent:*

(1) *Every group is a good group.*
(2) *Every unitary semisimple complete normed real alternative algebra has an isometric linear algebra involution sending unitary elements to their inverses.*
(3) *The same as* (2)*, with primitive instead of semisimple*
(4) *Every unitary semisimple complete normed complex alternative algebra has an isometric conjugate-linear algebra involution sending unitary elements to their inverses.*
(5) *The same as* (4)*, with primitive instead of semisimple.*

Proof. (1) $\Rightarrow$ (3) (respectively, (1) $\Rightarrow$ (5)) .- Since primitive alternative normed algebras are associative or finite-dimensional, this implication follows from Theorem 8.10 (respectively, Theorem 7.12) and Proposition 4.8. When Theorems 7.12 and 8.10 are applied, note that, as in the associative case [**2**, Remark 2.9.(c)], continuous involutions on unitary normed non-commutative Jordan algebras, sending unitaries to their inverses, are isometries.

(3) $\Rightarrow$ (2) (respectively, (5) $\Rightarrow$ (4)).- Let A be a unitary semisimple complete normed real (respectively, complex) alternative algebra. Let $\{\lambda_u\}_{u\in U_A}$ be a family of real (respectively, complex) numbers satisfying

$\sum_{u\in U_A}|\lambda_u| < \infty$ and $\sum_{u\in U_A}\lambda_u u = 0$. In view of the assumption (3), (respectively, (5)), for each primitive ideal P of A we have

$$\sum_{u\in U_A}\overline{\lambda_u}u^{-1}+P = \sum_{u\in U_A}\overline{\lambda_u}(u+P)^{-1} = \sum_{u\in U_A}\overline{\lambda_u}(u+P)^{*} = (\sum_{u\in U_A}\lambda_u u+P)^{*} = 0,$$

and hence $\sum_{u\in U_A}\overline{\lambda_u}u^{-1} = 0$ by semisimplicity. Now, according to [**2**, Lemma 2.2], given $x \in A$ and $\varepsilon > 0$, there exists a family $\{\lambda_u\}_{u\in U_A}$ of real (respectively, complex) numbers satisfying $\sum_{u\in U_A}|\lambda_u| < \|x\| + \varepsilon$ and $\sum_{u\in U_A}\lambda_u u = x$. It follows that

$$x = \sum_{u\in U_A}\lambda_u u \to x^* := \sum_{u\in U_A}\overline{\lambda_u}u^{-1}$$

is a well-defined mapping from A to A, which actually becomes an isometric linear (respectively, conjugate-linear) algebra involution on A satisfying $u^* = u^{-1}$ for every $u \in U_A$.

(2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1).- These implications follow from Proposition 4.8. ■

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