Absolute-valued algebras with involution, and infinite-dimensional Terekhin's trigonometric algebras

Julio Becerra Guerrero and Ángel Rodríguez Palacios

Abstract. We prove that, if A is an absolute-valued $*$-algebra in the sense of [**8**], then the normed space of A becomes a trigonometric algebra (in the meaning of [**7**]) under the product $\wedge$ defined by $x \wedge y := \frac{x^*y - y^*x}{2}$. Moreover, we show that, "essentially", all infinite-dimensional complete trigonometric algebras derive from absolute-valued $*$-algebras by the above construction method.

1. Introduction

Given nonzero elements x, y of a real pre-Hilbert space, we define as usual the angle $\alpha := \alpha(x, y)$ between x and y by the equality $\cos\alpha := \frac{(x|y)}{\|x\|\|y\|}$. By a **trigonometric algebra** we mean a nonzero real pre-Hilbert space B endowed with a (bilinear) product $\wedge : B \times B \to B$ satisfying

$$\|x \wedge y\| = \|x\|\|y\| \sin\alpha$$

for all $x, y \in B \setminus \{0\}$. We note that the above requirement is equivalent to

$$\|x \wedge y\|^2 + (x|y)^2 = \|x\|^2\|y\|^2.$$

The motivating example for trigonometric algebras is the Euclidean tridimensional space endowed with the usual vector product. Since for every x in a trigonometric algebra we have $x \wedge x = 0$, trigonometric algebras are anticommutative.

Trigonometric algebras have been introduced recently by P. A. Terekhin [**7**], who shows that *the dimensions of finite-dimensional trigonometric algebras are precisely 1, 2, 3, 4, 7, and 8.* The existence of complete trigonometric algebras of arbitrary infinite Hilbertian dimension is implicitly known in [**4**]. Indeed, we have the following

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Example 1.1. Let H be any infinite-dimensional real Hilbert space. Take an orthonormal basis U of H, together with an injective mapping $\vartheta : U \times U \to U$. Then the mapping $(u, v) \to \frac{\vartheta(u,v)-\vartheta(v,u)}{\sqrt{2}}$, from $U \times U$ to H, extends to a product $\wedge$ on H converting H into a trigonometric algebra (see Remark 1.6 of [**4**] for details).

The aim of the present paper is to entering the structure of infinite-dimensional trigonometric algebras, by relating them to the so called "absolute-valued $*$-algebras". An **absolute value** on a real or complex algebra A is a norm $\| \cdot \|$ on the vector space of A satisfying

$$\|xy\| = \|x\|\|y\|$$

for all $x, y \in A$. By an **absolute-valued algebra** we mean a nonzero real or complex algebra endowed with an absolute value. **Absolute-valued $*$-algebras** are defined as those absolute-valued real algebras A endowed with an isometric algebra involution $*$ which is different from the identity operator and satisfies $xx^* = x^*x$ for every $x \in A$. Absolute-valued $*$-algebras were introduced in the early paper of K. Urbanik [**8**], and have been reconsidered by B. Gleichgewicht [**3**], Urbanik himself [**9**], M. L. El-Mallah [**1, 2**], and A. Rochdi [**5**]. The reader is referred to the recent survey paper [**6**] for a complete view of the theory of absolute-valued algebras.

To precisely reviewing our results, let us introduce some additional definitions. By a **super-trigonometric algebra** we mean a nonzero real pre-Hilbert space B endowed with a product $\wedge : B \times B \to B$ satisfying

$$(x \wedge y | u \wedge v) = (x|u)(y|v) - (x|v)(y|u)$$

for all $x, y, u, v \in B$. Takin $(u, v) = (x, y)$ in the above equality, we obtain

$$\|x \wedge y\|^2 + (x|y)^2 = \|x\|^2\|y\|^2.$$

Therefore, super-trigonometric algebras are trigonometric. Following Urbanik's pioneering paper [**8**], we say that an absolute-valued $*$-algebra A is **regular** if the equality $\langle(ux, vy)\rangle = \langle(uv^*, x^*y)\rangle$ holds for all $x, y, u, v \in A$, where $\langle(x, y)\rangle := \frac{xy^*+yx^*}{2}$.

We prove that, if A is an absolute-valued $*$-algebra, then the normed space of A becomes a trigonometric algebra (say B) under the product $\wedge$ defined by $x \wedge y := \frac{x^*y-y^*x}{2}$, and that A is regular if and only if B is super-trigonometric (Theorem 4.1). Moreover, up to a natural equivalence on the class of trigonometric algebras (which respects super-trigonometric algebras), all infinite-dimensional complete trigonometric algebras derive from absolute-valued $*$-algebras by the construction method provided in Theorem 4.1 just reviewed (Theorem 4.2).

As far as we know, super-trigonometric algebras have been not previously introduced. They have their own life, so that their structure can be nicely described (see Proposition 2.1 for details). As a consequence, the dimensions of finite-dimensional super-trigonometric algebras are precisely $1, 2$, and 3 (Corollary 2.3).

2. Super-trigonometric algebras

Let X be a real vector space. We define the antisymmetric tensor product $X \otimes_a X$ as the subspace of $X \otimes X$ spanned by the set

$$\{x \otimes y - y \otimes x : x, y \in X\}.$$

For $x, y \in X$, we put $x \otimes_a y := \frac{x \otimes y - y \otimes x}{\sqrt{2}} \in X \otimes_a X$. It is easy to see that, for every real vector space Z and every antisymmetric bilinear mapping $f : X \times X \to Z$, there exists a unique linear mapping $\Phi : X \otimes_a X \to Z$ satisfying $f(x, y) = \Phi(x \otimes_a y)$ for all $x, y \in X$. Now, let H be a real pre-Hilbert space. It is well-known that $H \otimes H$ becomes a real pre-Hilbert space under the inner product $(\cdot|\cdot)$ determined on elementary tensors by

$$(x \otimes y | u \otimes v) := (x|u)(y|v).$$

Therefore $H \otimes_a H$ is also a real pre-Hilbert space under an inner product $(\cdot|\cdot)$ satisfying

$$(x \otimes_a y | u \otimes_a v) := (x|u)(y|v) - (x|v)(y|u)$$

for all $x, y, u, v \in H$. Keeping in mind the above facts, the following result is of straightforward verification.

Proposition 2.1. *Given a real pre-Hilbert space H and a linear isometry Φ from the pre-Hilbertian antisymmetric tensor product $H \otimes_a H$ to H, H becomes a super-trigonometric algebra under the product $\wedge$ defined by $x \wedge y := \Phi(x \otimes_a y)$. Moreover, all super-trigonometric algebras can be obtained by the construction method just described.*

Corollary 2.2. *Every infinite-dimensional real Hilbert space can be converted into a super-trigonometric algebra under a suitable product.*

Proof. Let H be an infinite-dimensional real Hilbert space. Then the completion $H \widetilde{\otimes} H$ of the pre-Hilbert space $H \otimes H$ is a Hilbert space with the same Hilbertian dimension as that of H. Therefore, the closure $H \widetilde{\otimes}_a H$ of $H \otimes_a H$ in $H \widetilde{\otimes} H$ is a Hilbert space whose Hilbertian dimension is less than or equal to that of H. This allows us to find a linear isometry from $H \widetilde{\otimes}_a H$ into H, and to restrict such an isometry to $H \otimes_a H$. Finally, apply Proposition 2.1. ∎

Corollary 2.3. *The dimensions of finite-dimensional super-trigonometric algebras are precisely* 1, 2, *and* 3.

Proof. We note that, if the dimension of a real vector space X is $n \in \mathbb{N}$, then the dimension of $X \otimes_a X$ is $\frac{n(n-1)}{2}$. It follows from Proposition 2.1 that a natural number n is the dimension of a super-trigonometric algebra if and only if $\frac{n(n-1)}{2} \leq n$, if and only if $n \leq 3$. ∎

We conclude this the present section with Lemma 2.4 immediately below. Such a lemma will be useful later.

LEMMA 2.4. *Let H be a real pre-Hilbert space endowed with an anticommutative product $\wedge$. Then $(H, \wedge)$ is a super-trigonimetric algebra if and only if the equality*

$$(x|y)(u|v) + (x \wedge y|u \wedge v) = (v|y)(u|x) + (v \wedge y|u \wedge x) \tag{2.1}$$

holds for all $x, y, u, v \in H$.

PROOF. Let x, y, u, v be in H. Assume that $(H, \wedge)$ is a super-trigonimetric algebra. Then, subtracting the equality $(v \wedge y|u \wedge x) = (v|u)(y|x) - (v|x)(y|u)$ from the one $(x \wedge y|u \wedge v) = (x|u)(y|v) - (x|v)(y|u)$, we obtain (2.1). Conversely, assume that (2.1) holds. Interchanging the rolls of y and v in (2.1), we obtain

$$(x|v)(u|y) + (x \wedge v|u \wedge y) = (y|v)(u|x) + (y \wedge v|u \wedge x), \tag{2.2}$$

and, replacing in (2.1) (x, y, u, v) with (u, v, y, x), we also obtain

$$(u|v)(y|x) + (u \wedge v|y \wedge x) = (x|v)(y|u) + (x \wedge v|y \wedge u). \tag{2.3}$$

Subtracting (2.3) from the equality obtained by summing (2.1) and (2.2), we get

$$(x \wedge y|u \wedge v) = (x|u)(y|v) - (x|v)(y|u),$$

and hence $(H, \wedge)$ is a super-trigonimetric algebra. ■

3. Revisiting absolute-valued $*$-algebras

Throughout this section, A will denote an absolute-valued $*$-algebra.

The following result summarizes Lemmas 1, 2, and 3 of Urbanik's paper [**8**]. The idea of such a summary is taken from Gleichgewicht's note [**3**].

PROPOSITION 3.1. *Self-adjoint elements of A commute with skew elements of A. Moreover, there exists an idempotent $e \in A$ such that the equality $x^*x = \|x\|^2 e$ holds for every $x \in A$.*

The following corollary is also known in [**8**]

COROLLARY 3.2. *The absolute value of A comes from an inner product $(\cdot|\cdot)$. Moreover, if h is a self-adjoint element of A, and if k is a skew element of A, we have $(h|k) = 0$.*

PROOF. Since Proposition 3.2 shows ostensibly that the square of the norm of A is a quadratic function, the first assertion in the corollary seems to us obvious. On the other hand, for elements h and k self-adjoint and skew, respectively, in A, Proposition 3.2 gives

$$\|h + k\|^2 e = (h + k)^*(h + k) = (h - k)(h + k)$$

$$= h^2 - k^2 = h^*h + k^*k = (\|h\|^2 + \|k\|^2)e,$$

so $\|h + k\|^2 = \|h\|^2 + \|k\|^2$, and so $(h|k) = 0$. ■

Corollary 3.3. *Let e be the idempotent in A given by Proposition 3.1. Then we have $(xy|e) = (x|y^*)$ for all $x, y \in A$. Moreover, if for $x \in A$ we put $x^\sigma := 2(x|e)e - x$, then $*$ and σ coincide on $A^2 := \text{lin}\{xy : x, y \in A\}$.*

Proof. Let x, y be in A with $\|y\| = 1$. Since the operator of right multiplication on A by y is a linear isometry, we have $(xy|y^*y) = (x|y^*)$. But, by Proposition 3.1, $y^*y = e$.

Linearizing the equality $xx^* = \|x\|^2 e$ in Proposition 3.1, we get $xy^* + yx^* = 2(x|y)e$ for all $x, y \in A$. Then, replacing y with y^*, we derive $(xy)^* = 2(x|y^*)e - xy$. Finally, since $(x|y^*) = (xy|e)$ (by the first paragraph in the proof), we obtain $(xy)^* = (xy)^\sigma$. ■

The last conclusion in Corollary 3.3 can be also deduced by putting together [**3**, Theorem] and the proof of [**9**, Theorem 5].

Remark 3.4. In [**8**, pp. 249-250], Urbanik introduces the so-called $*$-product of A as the bilinear mapping $\langle(\cdot,\cdot)\rangle : A \times A \to A$ defined by $\langle(x,y)\rangle := \frac{xy^*+yx^*}{2}$, and comments that "it imitates an inner product". It is worth mentioning that, in view of the equality $xy^* + yx^* = 2(x|y)e$ in the proof of Corollary 3.3, the $*$-product of A is essentially the inner product of A. Therefore, the regularity of A (as defined in the introduction) is equivalent to the equality $(ux|vy) = (uv^*|x^*y)$ for all $x, y, u, v \in A$.

It was proved by El-Mallah [**1**] that the commutant of e in A is a subalgebra of A, and that such a subalgebra is infinite-dimensional whenever so is A. The following corollary refines both facts.

Corollary 3.5. *Let C denote the commutant of e in A. Then C contains A^2. Therefore C is an ideal of A, and A is linearly isometric to a subspace of C.*

Proof. Let x be in A^2. Put $y := \frac{x+x^*}{2}$ and $z := \frac{x-x^*}{2}$. By Corollary 3.3, we have $y = \frac{x+x^\sigma}{2} = (x|e)e$. Since $x = y + z$, and z is a skew element of A, and skew elements of A commute with e (by Proposition 3.1), it follows that x lies in C. Now that we know that C contains A^2, the fact that C is an ideal of A becomes obvious. Moreover, the mapping $\phi : A \to A^2 \subseteq C$ defined by $\phi(x) := ex$ is a linear isometry. ■

It follows from Corollary 3.5 that *e commutes with all elements of A whenever A^2 is dense in A*. As a consequence, *if A is finite-dimensional, then e commutes with all elements of A* [**1**, Corollary 4.2].

Remark 3.6. Let C, A_{sa}, and A_{sk} stand for the commutant of e in A, the set of all self-adjoint elements of A, and the set of all skew elements of A, respectively. The argument in the proof of Corollary 3.5 shows that the set $\{x \in A : x^* = x^\sigma\}$ is contained in C. On the other hand, by [**1**, Lemma 3.3], C is contained in $\mathbb{R}e \oplus A_{sk}$. Since the direct sum $A = A_{sa} \oplus A_{sk}$ is ortogonal (by Corollary 3.2), it follows

$$\{x \in A : x^* = x^\sigma\} = C = \mathbb{R}e \oplus A_{sk}.$$

Applying again Proposition 3.1, we derive that $*$ *coincides with* σ (on A) *if and only if* $A = \mathbb{R}e \oplus A_{sk}$*, if and only if* e *commutes with all elements of* A.

In [**2**], El-Mallah proves a remarkable converse to Corollary 3.5. Indeed, *if an absolute-valued algebra* C *has a non-zero idempotent* e *which commutes with all elements of* C*, then the norm of* C *derives from an inner product* $(\cdot|\cdot)$*, and the operator* $*$ *on* C *defined by* $x^* := 2(x|e)e - x$ *becomes an (isometric) algebra involution on* C *satisfying* $xx^* = x^*x$ *for every* $x \in C$.

To conclude the present section, let us emphasize that Urbanik [**8**] completely describes all complete regular absolute-valued $*$-algebras. A consequence of such a description is the following result.

PROPOSITION 3.7. *Every infinite-dimensional real Hilbert space can be endowed with a product and an involution converting it into a regular absolute-valued* $*$*-algebra.*

4. Infinite-dimensional Terekhin's trigonometric algebras

THEOREM 4.1. *Let* A *be an absolute-valued* $*$*-algebra. Then the normed space of* A *becomes a trigonometric algebra (say* B*) under the product*

$$x \bigtriangledown y := \frac{x^*y - y^*x}{2}.$$

Moreover, the absolute-valued $*$*-algebra* A *is regular if and only if the trigonometric algebra* B *is in fact super-trigonometric.*

PROOF. By Corollary 3.2, the absolute value of A derives from an inner product $(\cdot|\cdot)$. Moreover, by Corollary 3.3, for x, y in A we have

$$\begin{aligned} 4\|x \bigtriangledown y\|^2 &= \|x^*y - y^*x\|^2 = \|(y^*x)^* - y^*x\|^2 = \|(y^*x)^\sigma - y^*x\|^2 \\ &= 4\|(y^*x|e)e - y^*x\|^2 = 4(\|y^*x\|^2 - (y^*x|e)^2) \\ &= 4(\|y^*\|^2\|x\|^2 - (y^*|x^*)^2) = 4(\|x\|^2\|y\|^2 - (x|y)^2), \end{aligned}$$

and hence B is a trigonometric algebra.

Let x, y, u, v be in A. Applying again Corollary 3.3, we have

$$\begin{aligned} x^*y &= \frac{x^*y + (x^*y)^*}{2} + \frac{x^*y - y^*x}{2} \\ &= \frac{x^*y + (x^*y)^\sigma}{2} + x \bigtriangledown y = (x^*y|e)e + x \bigtriangledown y, \end{aligned}$$

and hence

$$x^*y = (x|y)e + x \bigtriangledown y. \tag{4.1}$$

Replacing in (4.1) (x, y) with (u^*, v^*), we obtain

$$uv^* = (u|v)e + u^* \bigtriangledown v^*. \tag{4.2}$$

Since $A \bigtriangledown A$ consists of skew elements of A, and self-adjoint elements are ortogonal to skew elements (by Corollary 3.2), it follows from (4.1) and (4.2) that

$$(uv^*|x^*y) = (x|y)(u|v) + (x \bigtriangledown y|u^* \bigtriangledown v^*), \tag{4.3}$$

and, replacing in (4.3) (v, x) with (x^*, v^*), also

$$(ux|vy) = (v^*|y)(u|x^*) + (v^* \bigtriangledown y|u^* \bigtriangledown x). \tag{4.4}$$

Keeping in mind Remark 3.4, it follows from (4.3) and (4.4) that the absolute-valued $*$-algebra A is regular if and only if we have

$$(x|y)(u|v) + (x \bigtriangledown y|u^* \bigtriangledown v^*) = (v^*|y)(u|x^*) + (v^* \bigtriangledown y|u^* \bigtriangledown x),$$

or equivalently (by replacing (u, v) with (u^*, v^*))

$$(x|y)(u|v) + (x \bigtriangledown y|u \bigtriangledown v) = (v|y)(u|x) + (v \bigtriangledown y|u \bigtriangledown x). \tag{4.5}$$

But, by Lemma 2.4, the equality (4.5) is equivalent to the fact that B is a super-trigonometric algebra ∎

In the particular case that A is equal to either $\mathbb{C}$, $\mathbb{H}$ (the algebra of Hamilton's quaternions), or $\mathbb{O}$ (the algebra of Cayley numbers), and $*$ is the standard involution on A, the first assertion in Theorem 4.1 is due to Terekhin (see [**7**, part 2 of the proof of the theorem]). We note that Corollary 2.2 follows from Urbanik's Proposition 3.7 and Theorem 4.1.

Theorem 4.1 provides us with a method to build trigonometric algebras. More trigonometric algebras can be obtained from a given one (say B), by taking any (possibly non surjective) linear isometry φ from

$$B^2 := \text{lin}\{x \wedge y : x, y \in B\}$$

to B, an then by replacing the product of B by the one $\triangle$ defined by $x \triangle y := \varphi(x \wedge y)$. The new trigonometric algebras obtained in this way will be called **isotone** algebras of the given one B. It is easy to see that the isotony just defined becomes an equivalence relation on the class of all trigonometric algebras, and that isotone algebras of a super-trigonometric algebra are super-trigonometric. We also note that every trigonometric (respectively, super-trigonometric) algebra can be seen as a dense subalgebra of a complete trigonometric (respectively, super-trigonometric) algebra.

THEOREM 4.2. *Let B be a complete infinite-dimensional trigonometric algebra. Then there exists an absolute-valued $*$-algebra A such that B is isotone to the trigonometric algebra obtained from A by the construction method given in Theorem 4.1.*

PROOF. Fix a norm-one element $e \in B$. Since B is complete and infinite-dimensional, there exists a linear isometry ϕ from B to the orthogonal complement of $\mathbb{R}e$. Now, consider the isometric involutive linear operator $*$ and the product $(x, y) \to xy$ on B defined by $x^* := 2(x|e)e - x$ and $xy := \phi(x^* \wedge y) + (x^*|y)e$, respectively. We claim that the normed space of B endowed with the involution and product just defined becomes an absolute-valued $*$-algebra (say A). Indeed, for x, y in A we have

$$\begin{aligned}\|xy\|^2 &= \|\phi(x^* \wedge y)\|^2 + (x^*|y)^2 = \|x^* \wedge y\|^2 + (x^*|y)^2 \\ &= \|x^*\|^2\|y\|^2 = \|x\|^2\|y\|^2.\end{aligned}$$

Moreover, since B is an anticommutative algebra, and $*$ is an involutive operator, we get

$$x^*x = \|x\|^2 = \|x^*\|^2 = xx^*$$

for every $x \in A$, and

$$\begin{aligned}(xy)^* &= (\phi(x^* \wedge y) + (x^*|y)e)^* = -\phi(x^* \wedge y) + (x^*|y)e \\ &= \phi(y \wedge x^*) + (y|x^*)e = y^*x^*\end{aligned}$$

for all $x, y \in A$. Now that the claim is proved, consider the trigonometric algebra $(D, \bigtriangledown)$ obtained from A by the construction method given in Theorem 4.1. Then, after a straightforward computation, we obtain $x \bigtriangledown y = \phi(x \wedge y)$ for all $x, y \in B$. It follows that D is an isotone of B. ■

We note that Urbanik's Proposition 3.7 follows from Corollary 2.2 and Theorems 4.2 and 4.1.

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Universidad de Granada, Facultad de Ciencias. Departamento de Matemática Aplicada, 18071-Granada (Spain)

E-mail address: juliobg@ugr.es

Universidad de Granada, Facultad de Ciencias. Departamento de Análisis Matemático, 18071-Granada (Spain)

E-mail address: apalacio@ugr.es