ABSOLUTE-VALUED ALGEBRAIC ALGEBRAS

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ABSTRACT. We review the paper [9] where it is proved that, if A is a non-associative algebra over the field of real numbers, if there is a norm $\|\cdot\|$ on A satisfying $\|xy\| = \|x\|\,\|y\|$ for all x, y in A, and if every one-generated subalgebra of A is finite-dimensional, then A is finite-dimensional. Some variants on the original proof are introduced.

1.- Introduction and previously known results

Let $\mathbb{K}$ denote the field of real or complex numbers. An absolute-valued algebra over $\mathbb{K}$ is a non-zero algebra A over $\mathbb{K}$ endowed with a norm $\|\cdot\|$ satisfying $\|xy\| = \|x\|\,\|y\|$ for all x, y in A. The most natural examples of absolute-valued algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (the algebra of Hamilton quaternions), and $\mathbb{O}$ (the algebra of Cayley numbers), with norms equal to their usual absolute values. The reader is referred to [7] for basis facts and intrinsic characterizations of these classical absolute-valued algebras. From an associative point of view, absolute-valued algebras are not much interesting because, as a consequence of an old result of I. Kaplansky [10], $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ are the unique absolute-valued associative real algebras. Therefore, $\mathbb{C}$ is the unique absolute-valued associative complex algebra.

Even in a "nearly associative" context a similar obstruction happens, since $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued power-associative real algebras [14]. This provided in particular the curious characterization of $\mathbb{C}$ as the unique absolute-valued power-associative complex algebra.

Now, let us focus attention on finite-dimensional absolute-valued algebras. In the complex case there is not much to say: $\mathbb{C}$ is the unique finite-dimensional absolute-valued complex algebra. This fact becomes mathematical folklore, and can also be derived from the results in the paper of A. A. Albert [1]. He proves that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique finite-dimensional absolute-valned real algebras with a unit, and observes that the non-unital case can be reduced in some sense to the unital one, by the method explained in what follows. If A is an absolute-valued algebra, if a is a norm-one element in A, and if the equalities $aA = Aa = A$ are true (a superfluous requirement in the finite-dimensional case), then the normed space of A, with the new product $\bullet$ defined by $x \bullet y := R_a^{-1}(x)L_a^{-1}(y)$ (L_a and R_a denoting left and right, respectively, multiplication by a), becomes an absolute-valued algebra with a^2 as a unit. These results of Albert imply the following proposition.

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PROPOSITION 1.- *Let A be a finite-dimensional absolute-valued real algebra. Then A has dimension 1, 2, 4, or 8, and the norm of A derives from an inner product.*

Without doubt, the most important result in the theory of absolute-valued algebras is the celebrated Urbanik-Wright theorem [21] asserting that $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued real algebras with a unit. With the ideas of Albert commented above, the Urbanik-Wright theorem gives rise to the following corollary.

COROLLARY 1.- *Let A be an absolute-valued algebra, and assume that there exists a in A such that $aA = Aa = A$. Then A is finite-dimensional.*

Following [2], we say that an algebra A is algebraic if, for every x in A, the subalgebra $A(x)$ of A generated by x is finite-dimensional. If in fact there exists a natural number m such that $\dim(A(x)) \leq m$ for all x in A, then the algebraic algebra A is said to be of bounded degree, and the smallest such a number m is called the degree of A and is denoted by $\deg(A)$. It is folklore that there is no absolute-valued algebraic complex algebra other than the complex field (see for instance [9; Section 1] for a proof). Concerning the real setting, Proposition 1 provides some non-trivial information, that we collect in the following remark.

REMARK 1.- *If A is an absolute-valued algebraic real algebra, then A is of bounded degree equal to 1, 2, 4,* or *8.*

The first attempt to obtain that absolute-valued algebraic real algebras are finite-dimensional is due to A. A. Albert [2]. However Albert's result, answering affirmatively the question under the additional assumption of the existence of a unit, becomes at present only an auxiliary tool for the proof of the Urbanik-Wright theorem. Later, M. L. El-Mallah showed that an absolute-valued algebraic real algebra A is finite-dimensional whenever either A has a non-zero idempotent commuting with all elements of A [11] or A has an algebra involution $*$ satisfying $\|x^*\| = \|x\|$ and $xx^* = x^*x$ for all x in A [12] (note that the first requirement implies the second one [13]). Let us recall also the folklore fact that $\mathbb{R}$ is the unique absolute-valued real algebra of degree one, and the result proved in [18] that there are only thirteen absolute-valued real algebras of degree two, and all of them are finite-dimensional.

Actually we have the following theorem.

THEOREM 1.- *Every absolute-valued algebraic real algebra is finite-dimensional.*

The aim of this note is to review the recent paper of the authors [9], where a proof of Theorem 1 is provided by the first time. By the sake of pleasantness, we introduce some minor variants on the original proof.

Besides Proposition 1, Corollary 1, and Remark 1, our proof will need another previously known result, namely Corollary 2 below. It is proved in [5] (see also [9; Proposition 1.3]) that, if A is a normed algebra over $\mathbb{K}$, and if n is a natural number, then the set $\{a \in A : \dim(A(a)) \leq n\}$ is closed in A. As a direct consequence, we have:

COROLLARY 2.- *Let A be a normed algebraic algebra of bounded degree. Then the set $\{a \in A : \dim(A(a)) = \deg(A)\}$ is open in A, and the completion of A is algebraic.*

2.- Reduction to the complete separable case

From now on, all absolute-valued algebras are assumed to be real. The first step in the proof of Theorem 1 given in [9] consists in showing that, if A is an absolute-valued algebraic algebra, then the set of points of smoothness of (the normed space of) A is dense in A [9; Proposition 2.2]. Since the verification of this fact is rather long and laborious, we prefer here to reduce the proof of Theorem 1 to the complete separable case, where the denseness of the set of points of smoothness is guaranteed by a celebrated theorem of S. Mazur. In this way the present section becomes the main variant on the original proof of Theorem 1.

CLAIM 1.- *If every complete separable absolute-valued algebraic algebra is finite-dimensional, then every absolute-valued algebraic algebra is finite-dimensional.*

Proof.- Let A be an absolute-valued algebraic algebra. By Remark 1 and Corollary 2, the completion $\hat{A}$ of A is an absolute-valued algebraic algebra. Now, fix a non-zero element a in $\hat{A}$, let b be in $\hat{A}$, and let B denote the closure in $\hat{A}$ of the subalgebra of $\hat{A}$ generated by $\{a, b\}$. Then B is a complete separable absolute-valued algebraic algebra, hence it is finite-dimensional (by assumption). Therefore there exist c, d in B such that $ca = b$ and $ad = b$. Since b is an arbitrary element in $\hat{A}$, we have $a\hat{A} = \hat{A}a = \hat{A}$, and Corollary 1 applies.

Let X be a normed space over $\mathbb{K}$, and u a norm-one element in X. As usual, we say that X is smooth at u if there exists a unique continuous linear functional φ on X satisfying $\|\varphi\| = \varphi(u) = 1$. According to Mazur's theorem (see for instance [19; Proposition 9.4.3]), if X is complete and separable, then the set

$$\{u \in X : \|u\| = 1 \text{ and } X \text{ is smooth at } u\}$$

is dense in the unit sphere of X. Now, the next corollary follows from Remark 1, and Corollary 2.

COROLLARY 3.- *Let A be a complete separable absolute-valued algebraic algebra. Then there exists a norm-one element a in A such that A is smooth at a and* $\dim(A(a)) = \deg(A)$.

3.- Applying ultraproduct techniques

The theory of (Banach) ultraproducts of Banach spaces and algebras has shown to be powerful in different fields (see for instance [8], [15], [3], and [4]). It is of straightforward verification that, in the setting of complete normed algebras, the class of complete absolute-valued algebras is closed under ultraproducts. However, up to date this observation never was useful to obtain any significant progress in the theory of absolute-valued algebras. On the contrary, concerning the proof of Theorem 1, the above observation becomes crucial. We begin by summarizing those aspects of the theory of ultraproducts that we need for our purpose (cf. [8]).

From now on I will denote a non-empty set, and $\mathcal{U}$ will stand for an ultrafilter on I. Given a family $\{X_i\}_{i\in I}$ of Banach spaces, we may consider the Banach space $\overset{l_\infty}{\bigoplus_{i\in I}} X_i$ ℓ_∞-sum of this family (consisting of all families $\{x_i\} \in \bigoplus_{i\in I} X_i$ such that

$$\| \{x_i\} \| := \sup\{\|x_i\| : i \in I\} < \infty)$$

and the closed subspace $N_{\mathcal{U}}$ of $\overset{l_\infty}{\bigoplus_{i\in I}} X_i$ given by

$$N_{\mathcal{U}} := \{\{x_i\} \in \overset{l_\infty}{\bigoplus_{i\in I}} X_i : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The (Banach) ultraproduct of the family $\{X_i\}_{i\in I}$ (with respect to the ultrafilter $\mathcal{U}$) is defined as the quotient Banach space $(\overset{l_\infty}{\bigoplus_{i\in I}} X_i)/N_{\mathcal{U}}$, and is denoted by $(X_i)_{\mathcal{U}}$.

If we denote by (x_i) the element in $(X_i)_{\mathcal{U}}$ containing a given family $\{x_i\} \in \overset{l_\infty}{\bigoplus_{i\in I}} X_i$, then it is easy to verify that $\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|$. If, for all i in I, X_i is equal to a given Banach space X, then the ultraproduct $(X_i)_{\mathcal{U}}$ is called the ultrapower of X (with respect to $\mathcal{U}$) and is denoted by $X_{\mathcal{U}}$. In this case the mapping $x \to x\hat{}$ from X into $X_{\mathcal{U}}$, where $x\hat{} = (x_i)$ with $x_i = x$ for all i in I, is an isometric linear embedding.

Let A be a complete normed algebra. Then the Banach space $A_{\mathcal{U}}$ is usually considered as a new complete normed algebra under the (well-defined) product

$$(x_i)(y_i) := (x_i y_i).$$

In this way, the natural embedding $A \hookrightarrow A_{\mathcal{U}}$ becomes an algebra homomorphism. If in addition A is an absolute-valued algebra, then the equality $\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|$ shows that $A_{\mathcal{U}}$ is an absolute-valued algebra too. Now, standard techniques of ultraproducts involving Proposition 1 (see [9; Section 3] for details) lead to the following proposition.

PROPOSITION 2.- *Every ultrapower $A_{\mathcal{U}}$ of a complete absolute-valued algebraic algebra A is an absolute-valued algebraic algebra too, with the same degree as A.*

4.- Two lemmas on normed spaces

We devote this section to collect two not difficult lemmas on normed spaces. The reader is referred to [9; Section 4] for the details of the proof.

LEMMA 1.- *Let X be a Banach space, and $F : X \to X$ be a non surjective linear isometry. Then there exists a sequence $\{x_n\}$ of norm-one elements in X such that $\{F(x_n) - x_n\} \to 0$.*

LEMMA 2.- *Let X be a normed space, $F : X \to X$ a linear contraction, and M a finite-dimensional subspace of X. Assume that $F(m) = m$ for all m in M, that the restriction of the norm of X to M derives from an inner product, and that X is smooth at every norm-one element of M. Then there exists a continuous linear projection $\pi : X \to X$ such that $\pi(X) = M$ and $Ker(\pi)$ is invariant under F.*

5.- Conclusion of the proof of Theorem 1

Now we are almost ready to conclude the proof of Theorem 1. Before to formally attack such a proof, let us formulate a last lemma.

LEMMA 3 [9; Lemma 5.1].- *Let A be an absolute-valued algebraic algebra, and a be a norm-one element of A.*

i) If $\dim(A(a)) = \deg(A)$, and if b is in $A\backslash\{0\}$ with $ab = b$, then $A(b) = A(a)$.

ii) If A is smooth at a, then A is smooth at every norm-one element of $A(a)$.

Proof of Theorem 1.- Let A be an absolute-valued algebraic algebra. We must show that A is finite-dimensional. By Claim 1, we can assume that A is complete and separable. Then, by Corollary 3, there exists a norm-one element a in A such that A is smooth at a and $\dim(A(a)) = \deg(A)$. Now assume that A is infinite-dimensional. Then, by Corollary 1, we must have for example $aA \neq A$, so that the mapping $F : x \to ax$ from A to A is a non surjective linear isometry. Let M denote the finite-dimensional subspace of A given by $M := \{x \in A(a) : ax = x\}$. By Proposition 1, the restriction of the norm of A to M derives from an inner product, and, by Lemma 3.ii), A is smooth at every norm-one element of M. Now, we are in a position to apply Lemma 2, so that there exists a continuous linear projection $\pi : A \to A$ such that $\pi(A) = M$ and $Ker(\pi)$ is invariant under F. Keeping in mind that $A = M \oplus Ker(\pi)$, that F is the identity mapping on M, that $Ker(\pi)$ is invariant under F, and that F is a non surjective linear isometry, it follows that the mapping $y \to F(y) = ay$ from $Ker(\pi)$ to $Ker(\pi)$ is a non surjective linear isometry. By Lemma 1, there exists a sequence $\{x_n\}$ of norm-one elements in $Ker(\pi)$ such that $\{ax_n - x_n\} \to 0$. Choose an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ refining the Fréchet filter (of all cofinite subsets of $\mathbb{N}$), let β denote the norm-one element in $A_{\mathcal{U}}$ given by $\beta := (x_n)$, and consider A as a subalgebra of $A_{\mathcal{U}}$ via the canonical embedding. Then we have $a\beta = \beta$. But, by Proposition 2, $A_{\mathcal{U}}$ is an absolute-valued algebraic algebra with $\deg(A_{\mathcal{U}}) = \deg(A)$, hence, since $\dim(A(a)) = \deg(A)$, we have $A_{\mathcal{U}}(a) = A(a)$ and

$\dim(A_{\mathcal{U}}(a)) = \deg(A_{\mathcal{U}})$. It follows from Lemma 3.i) that β lies in $A(a)$ and, more precisely, in M. Now that we know that β is in A, the equality $\beta = (x_n)$ reads as $\lim_{\mathcal{U}} \| x_n - \beta \| = 0$, hence, since x_n is in $Ker(\pi)$ for all n in $\mathbb{N}$, β is in $Ker(\pi)$ as well. Then $\beta \in M \cap Ker(\pi) = \{0\}$, a contradiction.

To conclude this note, let us comment that Theorem 1 would not have interest if examples of infinite-dimensional absolute-valued algebras were not known. As a matter of fact, such examples are abuudant, as the reader can see in [21], [30], [6], [16], and [17].

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