ABSOLUTE-VALUABLE BANACH SPACES

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Abstract. Absolute-valuable Banach spaces are introduced as those Banach spaces which underlie complete absolute-valued algebras. Examples and counterexamples are given. It is proved that every Banach space can be isometrically enlarged to an absolute-valuable Banach space, which has the same density character as that of the given one, and whose dual space is also absolute-valuable. It is also shown that every weakly countably determined Banach space different from $\mathbb{R}$ can be renormed in such a way that neither it nor its dual are absolute-valuable. Hereditarily indecomposable Banach spaces become examples of Banach spaces which cannot be renormed as absolute-valuable Banach spaces.

1. Introduction

Let $\mathbb{K}$ denote the field of real or complex numbers. By an **absolute-valued algebra** over $\mathbb{K}$ we mean a non-zero algebra A over $\mathbb{K}$ endowed with a norm $\|\cdot\|$ satisfying $\|xy\| = \|x\|\|y\|$ for all x, y in A. The reader is referred to the survey paper [21] for a comprehensive view of the theory of absolute-valued algebras. The aim of the present paper is to study those Banach spaces which underlie complete absolute-valued algebras. Such Banach spaces will be called "absolute-valuable". Since finite-dimensional absolute-valuable Banach spaces are well-understood from the early work of A. Albert [1] (see Proposition 2.1 below for details), we center our attention in the infinite-dimensional case.

We begin Section 2 by providing the reader with several natural examples and counterexamples of absolute-valuable Banach spaces. Concerning examples, it is worth mentioning that, roughly speaking, many classical Banach spaces, including all infinite-dimensional Hilbert spaces, are absolute-valuable (see Theorem 2.3 and Corollary 2.5). On the contrary, it has been difficult for us to find classical Banach spaces which are not absolute-valuable. Nevertheless, the space c of all real or complex convergent sequences becomes an example of such a Banach space (Proposition 2.8). As main result, we prove that every Banach space can be isometrically imbedded into a suitable

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absolute-valuable Banach space, which has the same density character as that of the given one, and whose dual space is also absolute-valuable (Theorem 2.11).

Section 3 is devoted to the isomorphic aspects of the absolute valuability. As main result we prove that every weakly countably determined real Banach space different from $\mathbb{R}$ can be equivalently renormed in such a way that neither it nor its dual are absolute-valuable (Theorem 3.4). We note that a Banach space is weakly countably determined whenever it is either reflexive, separable, or of the form $c_0(\Gamma)$ for any set Γ. We also show that both the separable reflexive Banach space of Gowers-Maurey [12] and the non-separable reflexive one of Shelah-Steprans-Wark (see [23] and [25]) are not isomorphic to any absolute-valuable Banach space (Propositions 3.7 and 3.8).

The concluding Section 4 deals with the relation between the absolute valuability and the Banach-Mazur rotation problem. We provide examples of non-Hilbert absolute-valuable almost transitive separable Banach spaces (Proposition 4.4) as well as of non-Hilbert absolute-valuable transitive non-separable Banach spaces (Corollary 4.5). However, the particular case of the rotation problem whether every absolute-valuable transitive separable Banach space is a Hilbert space, remains open (Problem 4.1).

Most absolute-valuable Banach spaces X arising in this paper have the additional property that X^*, $\mathcal{L}(X)$, and $\mathcal{K}(X)$ are absolute-valuable (see Theorem 2.3, Proposition 2.4, Corollary 2.5, and Theorem 2.11) or, even more, that $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$ are absolute-valuable for every absolute-valuable Banach space Y (Proposition 4.4). As usual, given Banach spaces X and Y over $\mathbb{K}$, we denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X to Y, and by $\mathcal{K}(X,Y)$ the closed subspace of $\mathcal{L}(X,Y)$ consisting of all compact operators from X to Y. Moreover, we write X^*, $\mathcal{L}(X)$, and $\mathcal{K}(X)$ instead of $\mathcal{L}(X,\mathbb{K})$, $\mathcal{L}(X,X)$, and $\mathcal{K}(X,X)$, respectively.

2. Isometric aspects of the absolute valuability.

By a **product** on a vector space X we mean a bilinear mapping from $X \times X$ into X. A Banach space $(X, \|.\|)$ over $\mathbb{K}$ is said to be **absolute-valuable** if there exists a product $(x,y) \to xy$ on X satisfying

$$\|xy\| = \|x\|\|y\|$$

for all x, y in X. Finite-dimensional absolute-valuable Banach spaces are well-understood thank to the next result, which follows from the work of A. A. Albert in [1].

Proposition 2.1. *The real Hilbert spaces of dimension* $1, 2, 4$ *or* 8 *are the unique finite-dimensional absolute-valuable real Banach spaces. The complex field* $\mathbb{C}$ *is the unique finite-dimensional absolute-valuable complex Banach space.*

Despite the above fact, most classical Banach spaces are absolute-valuable. For instance, this is the case for the real or complex spaces c_0 and ℓ_p with $1 \leq p \leq \infty$. More generally, we have the result given by the next proposition. Let Y be a Banach space over $\mathbb{K}$, and let Γ be an infinite set. For $0 \leq p < \infty$, we denote by $\ell_p(\Gamma, Y)$ the Banach space over $\mathbb{K}$ of all mappings $f : \Gamma \to Y$ such that

$$\|f\|^p := \sum_{\gamma \in \Gamma} \|f(\gamma)\|^p < \infty.$$

By $\ell_\infty(\Gamma, Y)$ we mean the Banach space over $\mathbb{K}$ of all bounded mappings from Γ to Y, endowed with the sup norm, and $c_0(\Gamma, Y)$ stands for the subspace of $\ell_\infty(\Gamma, Y)$ consisting of those functions $f : \Gamma \to Y$ such that $\lim_{\gamma \to \infty} f(\gamma) = 0$ (were $\lim_{\gamma \to \infty}$ denotes the limit along the filter of all co-finite subsets of Γ). When $Y = \mathbb{K}$, we simply write $\ell_p(\Gamma)$, $\ell_\infty(\Gamma)$, and $c_0(\Gamma)$, respectively.

Proposition 2.2. *Let Γ be an infinite set, let Y be an absolute-valuable Banach space, and let X stand for either $c_0(\Gamma, Y)$ or $\ell_p(\Gamma, Y)$ $(1 \leq p \leq \infty)$. Then X is absolute-valuable.*

Proof. Choose a product $(y, z) \to yz$ on Y converting Y into an absolute-valued algebra, and an injective mapping $\phi : \Gamma \times \Gamma \to \Gamma$. Given two functions u and v from Γ to Y, we can consider the mapping $u \diamond v : \Gamma \to Y$ defined by

$$(u \diamond v)(\gamma) := \begin{cases} u(i)v(j) & \text{if } \gamma = \phi(i,j) \text{ for some } (i,j) \in \Gamma \times \Gamma \\ 0 & \text{if } \gamma \notin \phi(\Gamma \times \Gamma) \end{cases}.$$

Then it is straightforward that $u \diamond v$ belongs to X whenever u and v are in X, and that X becomes an absolute-valued algebra under the product $\diamond$. ∎

Other examples of absolute-valuable Banach spaces are given in the next theorem.

Theorem 2.3. *Let $1 \leq p \leq \infty$, let Γ_1 be an infinite set, and let X_1 stand for $\ell_p(\Gamma_1)$. Then X_1^* is absolute-valuable. Moreover, if Γ_2 is another infinite set, and if X_2 stands for $\ell_p(\Gamma_2)$, then $\mathcal{L}(X_1, X_2)$, and $\mathcal{K}(X_1, X_2)$ are absolute-valuable.*

Proof. Let $n = 1, 2$. Fix a bijective mapping $\phi_n : \Gamma_n \times \Gamma_n \to \Gamma_n$. Then the mapping $\Psi_n : X_n \to \ell_p(\Gamma_n \times \Gamma_n)$, defined by $\Psi_n(x) := x \circ \phi_n$ for every x in X_n, is a surjective linear isometry. On the other hand, given h in $\ell_p(\Gamma_n \times \Gamma_n)$, we can consider the element $\Phi_n(h)$ of $\ell_p(\Gamma_n, X_n)$ defined by $[\Phi_n(h)(i)](j) := h(i,j)$ for all i, j in Γ_n, so that the mapping $\Phi_n : h \to \Phi_n(h)$ becomes a linear isometry from $\ell_p(\Gamma_n \times \Gamma_n)$ onto $\ell_p(\Gamma_n, X_n)$. Moreover, given $T \in \mathcal{L}(X, Y)$ (where X and Y stand for arbitrary Banach spaces), we can consider the bounded linear operator $T^{[n]}$ from $\ell_p(\Gamma_n, X)$ to $\ell_p(\Gamma_n, Y)$ defined by $[T^{[n]}(g)](i) := T(g(i))$ for every $g \in \ell_p(\Gamma_n, X)$ and every $i \in \Gamma_n$ (so that we have $\|T^{[n]}\| \leq \|T\|$). Finally, we consider the surjective linear isometry $r : \ell_p(\Gamma_1, X_2) \to \ell_p(\Gamma_2, X_1)$ defined by $[(r(f))(i)](j) = [f(j)](i)$ for every $f \in \ell_p(\Gamma_1, X_2)$ and every $(i,j) \in \Gamma_2 \times \Gamma_1$.

Now, given $F, G \in \mathcal{L}(X_1, X_2)$, we put

$$F \Box G := \Psi_2^{-1} \circ \Phi_2^{-1} \circ F^{[2]} \circ r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1 \in \mathcal{L}(X_1, X_2)\,,$$

so that we have $\|F \Box G\| \leq \|F\|\|G\|$. To see the converse inequality, recall from the proof of Proposition 2.2 that X_n becomes an absolute-valued algebra under the product $\diamond_n$ defined by $(x \diamond_n y)(\phi_n(i,j)) := x(i)y(j)$ for all x, y in X_n and all i, j in Γ_n. We claim that, for x, y in X_1 and F, G in $\mathcal{L}(X_1, X_2)$, the equality

$$(F \Box G)(x \diamond_1 y) = F(x) \diamond_2 G(y)$$

holds. Indeed, for i, j in Γ_2, we have

$$\begin{aligned}
[(F \Box G)(x \diamond_1 y)]\phi_2(i,j) &= [\Psi_2^{-1} \circ \Phi_2^{-1} \circ F^{[2]} \circ r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y)]\phi_2(i,j) \\
&= [\Phi_2^{-1} \circ F^{[2]} \circ r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y)](i,j) \\
&= [[F^{[2]} \circ r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y)](j)](i) \\
&= [F[(r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y))(j)]](i).
\end{aligned}$$

But, for k in Γ_1, we have

$$\begin{aligned}
[(r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y))(j)](k) &= [(G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y))(k)](j) \\
&= [G[\Phi_1 \circ \Psi_1(x \diamond_1 y)(k)]](j).
\end{aligned}$$

Also, for t in Γ_1, we have

$$\begin{aligned}
[\Phi_1 \circ \Psi_1(x \diamond_1 y)(k)](t) &= [\Psi_1(x \diamond_1 y)](k,t) \\
&= (x \diamond_1 y)(\phi_1(k,t) \\
&= x(k)y(t) = [x(k)y](t),
\end{aligned}$$

and therefore

$$\begin{aligned}
[(r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y))(j)](k) &= [G[x(k)y]](j) \\
&= x(k)[G(y)](j) \\
&= [G(y)(j)x](k).
\end{aligned}$$

It follows

$$\begin{aligned}
[(F \Box G)(x \diamond_1 y)]\phi_2(i,j) &= [F[G(y)(j)x]](i) \\
&= G(y)(j)F(x)(i) \\
&= [F(x) \diamond_2 G(y)]\phi_2(i,j)\,,
\end{aligned}$$

which proves the claim. Now, for F, G in $\mathcal{L}(X_1, X_2)$ and x, y in the closed unit ball of X_1, we have

$$\|F(x)\|\|G(y)\| = \|F(x) \diamond_2 G(y)\| = \|(F \Box G)(x \diamond_1 y)\| \leq \|F \Box G\|\,,$$

and hence $\|F\|\|G\| \leq \|F\Box G\|$. It follows that $(\mathcal{L}(X_1, X_2), \Box)$ is an absolute-valued algebra, and therefore $\mathcal{L}(X_1, X_2)$ is absolutely valuable.

Since X_2 has the approximation property, to prove that $\mathcal{K}(X_1, X_2)$ is absolutely valuable it is enough to show that, if F and G are rank-one operators from X_1 to X_2, then so is the operator $F\Box G$. Let the elements F and G of $\mathcal{L}(X_1, X_2)$ have one-dimensional range (say $F = f \odot x : z \to f(z)x$ and $G = g \odot y$, for some $x, y \in X_2 \backslash \{0\}$ and $f, g \in X_1^* \backslash \{0\}$). A straightforward but tedious calculation (like the one in the proof of the claim above) shows that the equality $F\Box G = (f \sharp g) \odot (x \diamond_2 y)$ holds, where $f\sharp g$ is the element of X_1^* defined by $f\sharp g := f \circ g^{[1]} \circ \Phi_1 \circ \Psi_1$. Therefore $F\Box G$ has one-dimensional range.

To conclude the proof, let us show that X_1^* becomes an absolute-valued algebra under the product $\sharp$ defined in the above paragraph. But, if f, g are in X_1^*, it is enough to choose a norm-one element x in X_2, to have

$$\begin{aligned} \|f\|\|g\| &= \|f \odot x\|\|g \odot x\| = \|(f \odot x)\Box(g \odot x)\| \\ &= \|(f\sharp g) \odot (x \diamond_2 x)\| = \|f\sharp g\|\|x \diamond_2 x\| = \|f\sharp g\| \,. \end{aligned}$$

■

Repeating almost verbatim the proof of Theorem 2.3, we obtain:

Proposition 2.4. *Let Γ_1 and Γ_2 be infinite sets. Then $\mathcal{L}(c_0(\Gamma_1), c_0(\Gamma_2))$ and $\mathcal{K}(c_0(\Gamma_1), c_0(\Gamma_2))$ are absolute-valuable.*

For later reference, we emphasize the following consequence of Theorem 2.3.

Corollary 2.5. *Every infinite-dimensional Hilbert space over $\mathbb{K}$ is absolute-valuable. Moreover, if H and K are infinite-dimensional Hilbert spaces over $\mathbb{K}$, then $\mathcal{L}(H, K)$ and $\mathcal{K}(H, K)$ are absolute-valuable.*

According to Proposition 2.1, a finite-dimensional Banach space is absolute-valuable if and only if so is its dual. Moreover, by the same proposition, $\mathbb{K}$ is the unique finite-dimensional absolute-valuable Banach space X over $\mathbb{K}$ such that $\mathcal{L}(X)$ is absolute-valuable. In view of Theorem 2.3, in the infinite-dimensional case things are not so clear, so that the following question becomes natural.

Question 2.6. For an infinite-dimensional Banach space X, consider the following conditions:

1. X is absolute-valuable.
2. X^* is absolute-valuable.
3. $\mathcal{L}(X)$ is absolute-valuable.
4. $\mathcal{K}(X)$ is absolute-valuable.

Is there some dependence between the four conditions above?

For the moment, we only know that Condition 2 in Question 2.6 does not imply Condition 1. This will follow from Proposition 2.8 below. Such a proposition will also provide us with the first "natural" examples of infinite-dimensional non absolute-valuable Banach spaces. Given an infinite set Γ, we denote by $c(\Gamma)$ the subspace of $\ell_\infty(\Gamma)$ consisting of those functions $f : \Gamma \to \mathbb{K}$ such that $\lim_{\gamma\to\infty} f(\gamma)$ does exist. Given a Hausdorff compact topological space E, we denote by $C^{\mathbb{K}}(E)$ the Banach space over $\mathbb{K}$ of all $\mathbb{K}$-valued continuous functions on E.

Lemma 2.7. *Let Γ be an infinite set, and let f_1, f_2 be in $c(\Gamma)$ such that there exist linear isometries $T_1, T_2 : c(\Gamma) \to c(\Gamma)$ satisfying $T_1(f_1) = T_2(f_2)$. Then $\lim_{\gamma\to\infty} |f_1(\gamma)| = \lim_{\gamma\to\infty} |f_2(\gamma)|$.*

Proof. After considering the one-point compactification $\Gamma\cup\{\infty\}$ of the discrete space Γ, we identify $c(\Gamma)$ with $C^{\mathbb{K}}(\Gamma\cup\{\infty\})$ by putting $f(\infty) := \lim_{\gamma\to\infty} f(\gamma)$ for every $f \in c(\Gamma)$. According to [15], for each linear isometry $T : c(\Gamma) \to c(\Gamma)$ there exist a closed subset E_T of $\Gamma \cup \{\infty\}$, a surjective continuous mapping $\phi_T : E_T \to \Gamma \cup \{\infty\}$, and an $\alpha_T \in C^{\mathbb{K}}(\Gamma \cup \{\infty\})$ with $|\alpha_T(t)| = 1$ for every $t \in E_T$ satisfying

$$T(f)(t) = f(\phi_T(t))\alpha_T(t)$$

for all $t \in E_T$ and $f \in C^{\mathbb{K}}(\Gamma \cup \{\infty\})$. This implies that ∞ belongs to E_T and that $\phi_T(\infty) = \infty$. Now, for the elements $f_1, f_2 \in c(\Gamma)$ in the statement of the lemma we have

$$\begin{aligned} f_1(\infty)\alpha_{T_1}(\infty) &= f_1(\phi_{T_1}(\infty))\alpha_{T_1}(\infty) = T_1(f_1)(\infty) \\ &= T_2(f_2)(\infty) = f_2(\phi_{T_2}(\infty))\alpha_{T_2}(\infty) = f_2(\infty)\alpha_{T_2}(\infty), \end{aligned}$$

and hence $|f_1(\infty)| = |f_2(\infty)|$. ∎

Proposition 2.8. *Let Γ be an infinite set. Then $c(\Gamma)$ is not absolute-valuable.*

Proof. Assume that $c(\Gamma)$ is an absolute-valued algebra under some product $\diamond$. Let f_1 denote the constant function equal to 1 on Γ, let f_2 be the characteristic function on Γ of a previously chosen singleton, and let T_1 and T_2 stand for the linear isometries from $c(\Gamma)$ to itself defined by $T_1(f) := f_2 \diamond f$ and $T_2(f) := f \diamond f_1$, respectively. Since $T_1(f_1) = T_2(f_2)$, Lemma 2.7 applies giving

$$1 = \lim_{\gamma\to\infty} |f_1(\gamma)| = \lim_{\gamma\to\infty} |f_2(\gamma)| = 0,$$

a contradiction. ∎

Despite Propositions 2.1 and 2.8, every Banach space becomes absolute-valuable up to a suitable enlargement. This will be proved in Theorem 2.11 below. Following [9, 12.1], a **tensor norm** α on the class BAN of all Banach spaces assigns to each pair (X, Y) of Banach spaces a norm $\|\cdot\|_\alpha = \alpha(\cdot\,; X, Y)$ on the algebraic tensor product $X \otimes Y$ such that the following two conditions are satisfied:

(1) For every pair (X, Y) of Banach spaces, we have $\|\cdot\|_\epsilon \leq \|\cdot\|_\alpha \leq \|\cdot\|_\pi$ on $X \otimes Y$, where ϵ and $_\pi$ denote respectively the injective and the projective tensor norm.

(2) If X, Y, Z, and T are Banach spaces, and if $F : X \to Z$ and $G : Y \to T$ are bounded linear operators, then the linear operator $F \otimes G$ from $(X \otimes Y, \|\cdot\|_\alpha)$ to $(Z \otimes T, \|\cdot\|_\alpha)$, determined by

$$(F \otimes G)(x \otimes y) := F(x) \otimes G(y) \tag{2.1}$$

for every $(x, y) \in X \times Y$, is continuous with norm $\leq \|F\|\|G\|$.

Given a tensor norm α on BAN, and Banach spaces X and Y, we denote by $X\widetilde{\otimes}_\alpha Y$ the completion of the algebraic tensor product $X \otimes Y$ under the norm $\|\cdot\|_\alpha$. Given Banach spaces X and Y over $\mathbb{K}$, we denote by $\mathcal{F}(X, Y)$ the space of all finite-rank operators from X to Y, and by $\overline{\mathcal{F}(X, Y)}$ the closure of $\mathcal{F}(X, Y)$ in $\mathcal{L}(X, Y)$. The convention $\mathcal{F}(X) := \mathcal{F}(X, X)$ will be subsumed.

Lemma 2.9. *Let X and Y be Banach spaces over $\mathbb{K}$. Assume that there exists a tensor norm α on BAN such that $X\widetilde{\otimes}_\alpha X$ is linearly isometric to a quotient of X and $Y\widetilde{\otimes}_\alpha Y$ is linearly isometric to a subspace of Y. Then $\mathcal{L}(X, Y)$ and $\overline{\mathcal{F}(X, Y)}$ are absolute-valuable.*

Proof. Let Φ be a linear isometry from $Y\widetilde{\otimes}_\alpha Y$ to Y, and let Ψ be a continuous linear surjection from X to $X\widetilde{\otimes}_\alpha X$ such that the induced bijection $X/\ker(\Psi) \to X\widetilde{\otimes}_\alpha X$ is an isometry. Note that, as a consequence of Condition 1 for tensor norms, for u, v both in either X or Y we have

$$\|u \otimes v\|_\alpha = \|u\|\|v\| \,. \tag{2.2}$$

Then, by (2.2) and Condition 2 for tensor norms, given F and G in $\mathcal{L}(X, Y)$, there exists a unique element $F\widetilde{\otimes}G$ in $\mathcal{L}(X\widetilde{\otimes}_\alpha X, Y\widetilde{\otimes}_\alpha Y)$ which extends $F \otimes G : X \otimes X \to Y \otimes Y$, and we have $\|F\widetilde{\otimes}G\| = \|F\|\|G\|$. It follows that, putting

$$F\Box G := \Phi \circ (F\widetilde{\otimes}G) \circ \Psi \in \mathcal{L}(X, Y)\,, \tag{2.3}$$

$(\mathcal{L}(X, Y), \Box)$ becomes an absolute-valued algebra.

Let F and G be finite-rank operators from X to Y. Then, by (2.1), $F(X) \otimes G(X)$ is a finite-dimensional subspace of $Y\widetilde{\otimes}_\alpha Y$ containing the range of $F\widetilde{\otimes}G$. Therefore, by (2.3), also $F\Box G$ has finite-dimensional range. In this way $\mathcal{F}(X, Y)$ (and hence $\overline{\mathcal{F}(X, Y)}$) becomes a subalgebra of $(\mathcal{L}(X, Y), \Box)$. ■

Corollary 2.10. *For a Banach space Y over $\mathbb{K}$ and a tensor norm α on BAN, consider the following conditions:*

(1) *$Y\widetilde{\otimes}_\alpha Y$ is linearly isometric to a subspace of Y.*

(2) *$Y\widetilde{\otimes}_\alpha Y$ is linearly isometric to a quotient of Y.*

Then we have:

i) *If the Banach space Y satisfies Condition* (1) *for some tensor norm α, then Y is absolute-valuable.*
ii) *If the Banach space Y satisfies Condition* (2) *for some tensor norm α, then Y^* is absolute-valuable.*
iii) *If the Banach space Y satisfies both Conditions* (1) *and* (2) *for the same tensor norm α, then $\mathcal{L}(Y)$ and $\overline{\mathcal{F}(Y)}$ are absolute-valuable.*

Proof. Since $Y = \mathcal{L}(\mathbb{K}, Y)$, and $Y^* = \mathcal{L}(Y, \mathbb{K})$, and $\mathbb{K}\widetilde{\otimes}_\alpha\mathbb{K}$ is linearly isometric to $\mathbb{K}$, the result follows from Lemma 2.9 ■

Given a Banach space X, we denote by $\mathrm{dens}(X)$ the density character of X.

Theorem 2.11. *Every Banach space X over $\mathbb{K}$ can be isometrically regarded as a subspace of a Banach space Y over $\mathbb{K}$ with* $\mathrm{dens}(Y) = \mathrm{dens}(X)$ *and such that Y, Y^*, $\mathcal{L}(Y)$, and $\mathcal{K}(Y)$ are absolute-valuable.*

Proof. Let X be a Banach space over $\mathbb{K}$, and let F denote the Hausdorff compact space consisting of the closed unit ball of X^* and the weak* topology. Then we can see isometrically X as a subspace of $C^{\mathbb{K}}(F)$. For any Hausdorff compact space G, let $*$ denote the identity mapping on $C^{\mathbb{K}}(G)$ or the natural involution on $C^{\mathbb{K}}(G)$ depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let Z stand for the unital closed $*$-invariant subalgebra of $C^{\mathbb{K}}(F)$ generated by X. Then we have $\mathrm{dens}(Z) = \mathrm{dens}(X)$ and $Z = C^{\mathbb{K}}(E)$ for some Hausdorff compact space E. Put $Y := C^{\mathbb{K}}(E^{\mathbb{N}})$. Then Z is linearly isometric to a subspace of Y. Indeed, fixing $n \in \mathbb{N}$ and denoting by π_n the n-coordinate projection from $E^{\mathbb{N}}$ onto E, the mapping $f \mapsto f \circ \pi_n$ is a linear isometry from Z into Y. It follows that X is linearly isometric to a subspace of Y. Moreover the equality $\mathrm{dens}(Y) = \mathrm{dens}(Z)$ holds. Indeed, if $\mathcal{D}$ is a dense subset of Z whose cardinal equals $\mathrm{dens}(Z)$, then, by the Stone-Weierstrass theorem, the unital $*$-invariant subalgebra of Y generated by the set $\{f \circ \pi_n : (f, n) \in \mathcal{D} \times \mathbb{N}\}$ is dense in Y. On the other hand, by [9, Example 4.2.(3)], the complete injective tensor product $Y\widetilde{\otimes}_\epsilon Y$ is linearly isometric to $C^{\mathbb{K}}(E^{\mathbb{N}} \times E^{\mathbb{N}})$. Since $E^{\mathbb{N}} \times E^{\mathbb{N}}$ is homeomorphic to $E^{\mathbb{N}}$, the actual situation is that $Y\widetilde{\otimes}_\epsilon Y$ is linearly isometric to Y. Since ϵ is a tensor norm [9, 4.1], and Y has the approximation property, the proof is concluded by applying Corollary 2.10. ■

As a consequence of Theorem 2.11 above we realize that the absolute valuability cannot imply any hereditary property of isometric or isomorphic type. This situation is not new. Indeed, the same happens with transitivity [5, Corollary 2.21] (see Section 4 below for the definition), Mazur's intersection property, and Mazur's w^*-intersection property of the dual [16]. In any case, we already know that, both in the finite and infinite dimension, the absolute valuability is not isometrically innocuous (by Propositions 2.1 and 2.8), and we will see below that even it is not isomorphically innocuous (a consequence of Proposition 3.7 or 3.8).

3. Isomorphic aspects of the absolute valuability

Let X be a Banach space over $\mathbb{K}$. We denote by B_X and S_X the unit closed ball and the unit sphere, respectively, of X. For x in S_X, we define the set $D(X,x)$ of states of X relative to x by

$$D(X,x) := \{f \in B_{X^*} : f(x) = 1\}.$$

For convenience, we say that X is **almost smooth** if, for every $x \in S_X$ and all $\phi, \psi \in D(X,x)$, we have $\|\phi - \psi\| < 2$.

Lemma 3.1. *Let Y and Z be non-zero almost smooth Banach spaces, and put $X := Y \overset{\ell_1}{\oplus} Z$. Then X is not absolute-valuable.*

Proof. The key observation is that the elements of $S_Y \cup S_Z$ are characterized in X as those elements x of S_X such that there exist ϕ, ψ in $D(X,x)$ satisfying $\|\phi - \psi\| = 2$. To prove this, let us identify X^* with $Y^* \overset{\ell_\infty}{\oplus} Z^*$. For y in S_Y, we can choose $f \in D(Y,y)$ and $g \in S_{Z^*}$, so that $\phi := (f,g)$ and $\psi := (f,-g)$ belong to $D(X,y)$ and satisfy $\|\phi - \psi\| = 2$. Conversely, if $x = (y,z)$ is in S_X and satisfies $y \neq 0 \neq z$, then we realize that

$$D(X,x) = \left\{(f,g) : (f,g) \in D\left(Y, \frac{y}{\|y\|}\right) \times D\left(Z, \frac{z}{\|z\|}\right)\right\},$$

and hence, since Y and Z are almost smooth, for all $\phi, \psi \in D(X,x)$ we have $\|\phi - \psi\| < 2$.

Let $T : X \to X$ be a linear isometry, and let y be in S_Y. By the above paragraph and the Hahn-Banach theorem, there exist ϕ, ψ in $D(X, T(y))$ with $\|\phi - \psi\| = 2$. Again by the above paragraph, this implies $T(y) \in S_Y \cup S_Z$. Now assume that X is an absolute-valued algebra under some product $\diamond$. Let us fix y in S_Y. For x in S_X, the mapping $t \to x \diamond t$ from X to X is a linear isometry, and hence we have $x \diamond y \in S_Y \cup S_Z$. Since $X \diamond y$ is a subspace of X, and $Y \cap Z = 0$, it follows that either $X \diamond y \subseteq Y$ or $X \diamond y \subseteq Z$. But both possibilities are contradictory because, if one of them happened (say $X \diamond y \subseteq Y$), then the mapping $x \to x \diamond y$ would be a linear isometry from the non almost smooth Banach space X to the almost smooth Banach space Y. ∎

Again for convenience, we introduce **almost rotund** Banach spaces as those Banach spaces X such that there is no segment of length 2 contained in S_X.

Lemma 3.2. *Let Y and Z be non-zero almost rotund Banach spaces over $\mathbb{K}$, and put $X := Y \overset{\ell_\infty}{\oplus} Z$. Then X is not absolute-valuable.*

Proof. First note that elements of $S_Y \cup S_Z$ lie in S_X and are mid-points of segments in B_X of length 2. Indeed, if y is in S_Y, then, choosing $z \in S_Z$, we have $y = \frac{1}{2}[(y,z)+(y,-z)]$ with $(y,z),(y,-z) \in B_X$ and $\|(y,z)-(y,-z)\| = 2$. As a partial converse, we claim that, if x belongs to S_X and is a mid point of a segment in B_X of length 2, then $\min\{\|\pi_Y(x)\|, \|\pi_Z(x)\|\} < 1$, where π_Y and π_Z stand for the natural projections from X to Y and Z, respectively. Indeed, if $x = (y,z)$ is in S_X (say for example $y \in S_Y$) and if $x = \frac{1}{2}[(y_1,z_1)+(y_2,z_2)]$ with $(y_1,z_1),(y_2,z_2) \in B_X$ and $\|(y_1,z_1)-(y_2,z_2)\| = 2$, then, by the almost rotundness of Y, we have $\|y_1 - y_2\| < 2$ and hence $\|\frac{z_1-z_2}{2}\| = 1$, so that, by the almost rotundness of Z, we have $\|z_1 + z_2\| < 2$ and therefore $\|z\| < 1$.

It follows from the above paragraph that, if $T : X \to X$ is a linear isometry and if y is in S_Y, then each of the possibilities $\|\pi_Y T(y)\| = 1$ and $\|\pi_Z T(y)\| = 1$ excludes the other. Now assume that X is an absolute-valued algebra under some product $\diamond$. Let us fix y in S_Y. Since for x in S_X, the mapping $t \to x \diamond t$ from X to X is a linear isometry, we deduce that $S_X = A \cup B$, where

$$A := \{x \in S_X : \|\pi_Y(x \diamond y)\| = 1\} \text{ and } B := \{x \in S_X : \|\pi_Z(x \diamond y)\| = 1\}$$

are disjoint closed subsets of S_X. Since S_X is connected, we have that either $A = S_X$ or $B = S_X$. But both possibilities are contradictory because, if one of them happened (say $A = S_X$), then the mapping $x \to \pi_Y(x \diamond y)$ would be a linear isometry from the non almost rotund Banach space X to the almost rotund Banach space Y. ■

Remark 3.3. In the case $\mathbb{K} = \mathbb{R}$, particular cases of Lemmas 3.1 and 3.2 can be reformulated in more classical terms, giving rise to new "natural" examples of non absolute-valuable Banach spaces. To this end, we recall that a **compact convex set** is a convex compact subset of some Hausdorff locally convex space, and that, given a compact convex set K, the real Banach space of all real-valued affine continuous functions on K (with the sup norm) is usually denoted by $\mathcal{A}(K)$. *Let Y be a non-zero real Banach space, and let K_Y stand for the compact convex set (B_{Y^*}, w^*).* It is well-known and easy to see that $\mathcal{A}(K_Y)$ is linearly isometric to $\mathbb{R}\overset{\ell_1}{\oplus}Y$. *Then*, by Lemma 3.1, $\mathcal{A}(K_Y)$ *fails to be absolute-valuable provided Y is almost smooth. Moreover,* noticing that almost rotundness of Y^* implies almost smoothness of Y, *the same holds provided Y^* is almost rotund,* with the additional information that, *in this case $\mathcal{A}(K_Y)^*$ fails also to be absolute-valuable* (thanks to Lemma 3.2).

To emphasize particular outstanding cases of the facts just commented, recall that the Banach space Y is said to be **smooth** if, for every y in S_Y, $D(Y,y)$ is reduced to a singleton, and **rotund** if every element in S_Y is an extreme point of B_Y. It follows from the above paragraph that, *if Y is smooth, then $\mathcal{A}(K_Y)$ is not absolute-valuable, and if actually Y^* is rotund, then $\mathcal{A}(K_Y)^*$ fails also to be absolute-valuable.*

A Banach space X is called **weakly countably determined** if there exists a countable collection $\{K_n\}_{n\in\mathbb{N}}$ of w^*-compact subsets of X^{**} such that for every x in X and every u in $X^{**} \setminus X$ there is n_0 such that $x \in K_{n_0}$ and $u \notin K_{n_0}$. If X is either reflexive, separable, or of the form $c_0(\Gamma)$ for any set Γ, then X is weakly countably determined. In fact, the class of weakly countably determined Banach spaces is hereditary, and contains the non hereditary class of weakly compactly generated Banach spaces (see [10, Example VI.2.2] for details).

Theorem 3.4. *Every weakly countably determined real Banach space, different from $\mathbb{R}$, is isomorphic to a real Banach space X such that both X and X^* are not absolute-valuable.*

Proof. Let Z be a weakly countably determined real Banach space different from $\mathbb{R}$. Take a closed maximal subspace P of Z. Since P is also a weakly countably determined real Banach space, P is isomorphic to a Banach space Y such that Y^* becomes rotund [10, Theorem VII.1.16]. Since such a space Y is smooth, it follows from Lemma 3.1 that the Banach space $X := \mathbb{R}\overset{\ell_1}{\oplus}Y$ is not absolute-valuable. Moreover, clearly, X is isomorphic to Z. Since $X^* = \mathbb{R}\overset{\ell_\infty}{\oplus}Y^*$ and Y^* is rotund, it follows from lemma 3.2 that X^* is not absolute-valuable. ■

A consequence of Proposition 2.1 is that every finite-dimensional Banach space over $\mathbb{K}$, different from $\mathbb{K}$, is isomorphic to a Banach space X such that both X and X^* are not absolute-valuable. We do not know if the same remains true when finite dimensionality is altogether removed. In fact we are unable for the moment to answer the following simpler question.

Question 3.5. Is every infinite-dimensional Banach space over $\mathbb{K}$ isomorphic to a non-absolute-valuable Banach space?

Remark 3.6. Theorem 3.4 becomes a partial affirmative answer to Question 3.5. However, one of the main tools in its proof has been infra-applied. Indeed, Lemma 3.1 actually shows that a Banach space Z over $\mathbb{K}$ is isomorphic to a non-absolute-valuable Banach space whenever Z is different from $\mathbb{K}$ and has an almost smooth equivalent renorming. Thus one could wonder whether every Banach space Z has such an equivalent renorming. As a matter of fact, the answer to this last question is negative even for $\mathbb{K} = \mathbb{R}$ A counterexample is given by the space $Z := \ell_\infty(\Gamma)$ for any uncountable set Γ, since every equivalent renorming of Z has an isometric copy of ℓ_∞ [20] (see also [10, Theorem II.7.12]).

By Proposition 2.1, most finite-dimensional Banach spaces are not isomorphic to any absolute-valuable Banach space. The remaining part of this section is devoted to prove the existence of infinite-dimensional Banach spaces which are not isomorphic to any absolute-valuable Banach space. Such spaces are precisely those constructed by Gowers-Maurey [12] and Shelah-Steprans-Wark (see [23] and [25]) with the common property of having "few" operators. We recall that a Banach space X is said to be **hereditarily indecomposable** if, for every closed subspace Y of X, the unique complemented subspaces of Y are the finite-dimensional ones and the closed finite-codimensional ones.

Proposition 3.7. *There exists an infinite-dimensional separable reflexive Banach space over $\mathbb{K}$ which is not isomorphic to any absolute-valuable Banach space.*

Proof. We are in fact proving that every infinite-dimensional hereditarily indecomposable Banach space over $\mathbb{K}$ fails to be absolute-valuable. The result will follow from the existence of infinite-dimensional hereditarily indecomposable separable reflexive Banach spaces over $\mathbb{K}$ [12, Section 3], and the clear fact that the hereditary indecomposability is preserved under isomorphisms. Let X be an infinite-dimensional hereditarily indecomposable Banach space over $\mathbb{K}$. By [12, Corollary 19 and Theorem 21], X is not isomorphic to any of its proper subspaces. Assume that, for some product $\diamond$ on X, $(X, \diamond)$ becomes an absolute-valued algebra. Then, for every nonzero element x in X, the operators of left and right multiplication by x on the algebra $(X, \diamond)$ are isomorphisms onto their ranges, and hence they are bijective. Therefore $(X, \diamond)$ is an absolute valued division algebra. By [26], X is finite dimensional, a contradiction. ∎

In the above proof we have just applied the theorem of F. B. Wright [26] that absolute-valued division algebras are finite-dimensional. It is worth mentioning that, today, such a theorem is an easy consequence of the celebrated Urbanik-Wright theorem [24] that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued real algebras with a unit (see [17, Proposition 1.2] for details).

Proposition 3.8. *There exists a non-separable reflexive Banach space over $\mathbb{K}$ which is not isomorphic to any absolute-valuable Banach space.*

Proof. The authors of [23] construct a non-separable Banach space Z over $\mathbb{K}$ satisfying Property $\mathcal{P}$ which follows:

- $\mathcal{P}$ Every element F of $\mathcal{L}(Z)$ has the form $S + \rho I_Z$, where I_Z denotes the identity mapping on Z, $\rho = \rho(F)$ belongs to $\mathbb{K}$, and $S = S(F) \in \mathcal{L}(Z)$ has separable range.

Very recently, H. M. Wark [25] refines the construction of [23] to show the existence of a non-separable reflexive Banach space Z satisfying $\mathcal{P}$. Now, since Property $\mathcal{P}$ is preserved under isomorphisms, to prove the result it is enough to show that every non-separable Banach space Z satisfying $\mathcal{P}$ fails to be absolute-valuable. Let Z be such a Banach space. We note that, for F in $\mathcal{L}(Z)$, the couple $(\rho(F), S(F))$ given by $\mathcal{P}$ is uniquely determined, and that the mappings $\rho : F \to \rho(F)$ and $S : F \to S(F)$ from $\mathcal{L}(Z)$ to $\mathbb{K}$ and $\mathcal{L}(Z)$, respectively, are linear. Since $\ker(\rho)$ consists of those elements of $\mathcal{L}(Z)$ which have separable range, we have $\rho(F) \neq 0$ whenever the operator F on Z is an isomorphism onto its range. Assume that Z is an absolute-valued algebra under some product $\diamond$. For z in Z, denote by L_z the operator of left multiplication by z on the algebra $(Z, \diamond)$. Since L_z is an isomorphism onto its range whenever z is nonzero, it follows that the linear mapping $z \to \rho(L_z)$ from Z to $\mathbb{K}$ is injective. Therefore Z is one-dimensional, a contradiction. ∎

Remark 3.9. (a).- By a **normed algebra** over $\mathbb{K}$ we mean a non-zero algebra A over $\mathbb{K}$ endowed with a norm $\|\cdot\|$ satisfying $\|xy\| \leq \|x\| \, \|y\|$ for all x, y in A. Following [18], a **nearly absolute-valued algebra** over $\mathbb{K}$ will be a normed algebra A over $\mathbb{K}$ such that there exists $\delta > 0$ satisfying $\|xy\| \geq \delta \|x\| \|y\|$ for all x, y in A. Of course, one can think about those Banach spaces underlying complete nearly absolute-valued algebras. Such Banach spaces will be called **nearly absolute-valuable**. It is easy to see that the near absolute valuability is preserved under isomorphisms. Consequently, isomorphic copies of absolute-valuable Banach spaces are nearly absolute-valuable. If X is a finite-dimensional nearly absolute-valuable Banach space over $\mathbb{C}$ (respectively, $\mathbb{R}$), then, by [18, Remark 2.8] (respectively, [11, Chapter 11]), X has dimension equal to 1 (respectively, 1, 2, 4, or 8), and therefore, by Proposition 2.1, X is isomorphic to an absolute-valuable Banach space. However, in the infinite-dimensional setting, we do not know if every nearly absolute-valuable Banach space is isomorphic to an absolute-valuable Banach space. We certainly know that, both in the finite- and infinite-dimensional case, nearly absolute-valued algebras need not be algebra-isomorphic to absolute-valued algebras (see [18, Introduction] and [18, Example 1.1], respectively). In any case, the proof of Proposition 3.8 actually shows that the non-separable reflexive Banach space of Wark [25] cannot be isomorphic to any nearly absolute-valuable Banach space.

(b).- Recently, S. A. Argyros, J. López-Abad, and S. Todorcevic [2] have constructed a non-separable reflexive hereditarily indecomposable Banach space. This, together with the fact that every infinite-dimensional hereditarily indecomposable Banach space is not absolute-valuable (see the proof of Proposition 3.7), provides us with a new proof of Proposition 3.8.

4. Transitivity of the norm and absolute valuability

Given a Banach space X, we denote by $\mathcal{G}$ the group of all surjective linear isometries on X. We recall that a Banach space X is said to be **transitive** (respectively, **almost transitive**) if, for every (equivalently, some) element u in S_X we have $\mathcal{G}(u) = S_X$ (respectively, $\overline{\mathcal{G}(u)} = S_X$, where $^-$ means norm closure). The reader is referred to the book of Rolewicz [22] and the survey papers of Cabello [7] and Becerra-Rodríguez [5] for a comprehensive view of known results and fundamental questions in relation to the notions just introduced. Hilbert spaces become the natural motivating examples of transitive Banach spaces, but there are also examples of non-Hilbert almost transitive separable Banach spaces, as well as of non-Hilbert transitive non-separable Banach spaces. However, the Banach-Mazur **rotation problem** [3], if every transitive separable Banach space is a Hilbert space, remains unsolved to date. Since transitive finite-dimensional Banach spaces are indeed Hilbert spaces, the rotation problem is actually interesting only in the infinite-dimensional setting. Then, since infinite-dimensional Hilbert spaces are absolute-valuable (by Corollary 2.5), we feel authorized to raise the following strong form of the Banach-Mazur rotation problem.

Problem 4.1. Let X be an absolute-valuable transitive separable Banach space. Is X a Hilbert space?

Unfortunately, we are unable to provide the reader with an affirmative answer to Problem 4.1, even under the reasonable additional assumption that $\mathcal{L}(X)$ and $\mathcal{K}(X)$ are absolute-valuable (see again Corollary 2.5). On the other hand, a negative answer to Problem 4.1 could not be be reasonably expected, since such a negative answer would solve scandalously the classical Banach-Mazur rotation problem by the negative. Thus we limit ourselves for the moment to discuss the different requirements in Problem 4.1. Indeed, we are going to show that an affirmative answer to Problem 4.1 cannot be expected if either the transitivity of X is relaxed to the almost transitivity or if the separability of X is removed.

Lemma 4.2. *Let X and Y be Banach spaces over $\mathbb{K}$. Assume that the complete projective tensor product $X\widetilde{\otimes}_\pi X$ is linearly isometric to a quotient of X, and that Y is absolute-valuable. Then $\mathcal{L}(X,Y)$ and $\overline{\mathcal{F}(X,Y)}$ are absolute-valuable.*

Proof. By the assumed absolute valuability of Y and [9, 3.2.(1)], there exists a norm-one continuous linear mapping $\Phi : Y\widetilde{\otimes}_\pi Y \to Y$ satisfying $\|\Phi(y_1\otimes y_2)\| = \|y_1\|\|y_2\|$ for all y_1, y_2 in Y. On the other hand, the assumption on X provides us with a continuous linear surjection $\Psi : X \to X\widetilde{\otimes}_\pi X$ such that the induced bijection $X/\ker(\Psi) \to X\widetilde{\otimes}_\pi X$ is an isometry. let F and G be in $\mathcal{L}(X,Y)$. By [9, 3.2.(4)], there exists a unique element $F\widetilde{\otimes}G$ in $\mathcal{L}(X\widetilde{\otimes}_\pi X, Y\widetilde{\otimes}_\pi Y)$ which extends $F\otimes G : X\otimes X \to Y\otimes Y$, and we have $\|F\widetilde{\otimes}G\| \le \|F\|\|G\|$. Then, writing $F\Box G := \Phi\circ(F\widetilde{\otimes}G)\circ\Psi \in \mathcal{L}(X,Y)$, the inequality $\|F\Box G\| \le \|F\|\|G\|$ is clear. To see the converse inequality, note that, by the properties of Ψ, the equality $\|F\Box G\| = \|\Phi\circ(F\widetilde{\otimes}G)\|$ holds, and that, by the properties of Φ and [9, 3.2.(3)], for x_1, x_2 in B_X we have

$$\|F(x_1)\|\|G(x_2)\| = \|\Phi(F(x_1)\otimes G(x_2))\|$$

$$= \|(\Phi \circ (F \widetilde{\otimes} G))(x_1 \otimes x_2)\| \leq \|\Phi \circ (F \widetilde{\otimes} G)\| .$$

Now $(\mathcal{L}(X,Y),\square)$ becomes an absolute-valued algebra. Arguing as in the concluding paragraph of the proof of Lemma 2.9, we realize that $\mathcal{F}(X,Y)$ (and hence $\overline{\mathcal{F}(X,Y)}$) becomes a subalgebra of $(\mathcal{L}(X,Y),\square)$. ■

In relation to the above lemma, the following question arises naturally.

Question 4.3. Let X be a Banach space such that $\mathcal{L}(X,Y)$ and $\overline{\mathcal{F}(X,Y)}$ are absolute-valuable, for every absolute-valuable Banach space Y. Is $X \widetilde{\otimes}_\pi X$ linearly isometric to a quotient of X?

Proposition 4.4. *There exists a non-Hilbert absolute-valuable almost transitive separable Banach space X such that, for every absolute-valuable Banach space Y, $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$ are absolute-valuable.*

Proof. Take X equal to the non-Hilbert separable Banach space $L_1([0,1])$. By [22, Theorem 9.6.4], X is almost transitive. On the other hand, by [6, Example 42.14] and [19, Theorem 2.3], $X \widetilde{\otimes}_\pi X$ is linearly isometric to X. This last property implies that X is absolute-valuable (by Corollary 2.10) as well as the remaining part of the result (by Lemma 4.2 and the fact that X^* has the approximation property [9, 5.2 and 5.3]). ■

To derive from Proposition 4.4 that Problem 4.1 cannot be answered by the affirmative if the separability is removed, let us recall some notions and results from the theory of (Banach) ultraproducts [14]. Let $\mathcal{U}$ be an ultrafilter on a nonempty set I, and $\{X_i\}_{i\in I}$ a family of Banach spaces. We can consider the Banach space $\oplus_{i\in I}^{\ell_\infty} X_i$, together with its closed subspace

$$N_{\mathcal{U}} := \{\{x_i\}_{i\in I} \in \oplus_{i\in I}^{\ell_\infty} X_i : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The quotient space $(\oplus_{i\in I}^{\ell_\infty} X_i)/N_{\mathcal{U}}$ is called the **ultraproduct** of the family $\{X_i\}_{i\in I}$ relative to the ultrafilter $\mathcal{U}$, and is denoted by $(X_i)_{\mathcal{U}}$. Let (x_i) stand for the element of $(X_i)_{\mathcal{U}}$ containing a given family $\{x_i\} \in \oplus_{i\in I}^{\ell_\infty} X_i$. It is easy to check that $\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|$. It follows from this equality the folklore fact that $(X_i)_{\mathcal{U}}$ is absolute-valuable when, for every $i \in I$, X_i is absolute-valuable (see for instance [17, Section 3]). On the other hand, it is also folklore that, if I is countable, if $\mathcal{U}$ is nontrivial, and if X_i is almost transitive for every $i \in I$, then $(X_i)_{\mathcal{U}}$ is transitive (see either [8, Lemma 1.4], [13, Remark p. 479], or [5, Proposition 2.18]). When $X_i = X$ for every $i \in I$ and some prefixed Banach space X, the ultraproduct $(X_i)_{\mathcal{U}}$ is called the **ultrapower** of X relative to $\mathcal{U}$, and is denoted by $X_{\mathcal{U}}$. In this case $X_{\mathcal{U}}$ contains "naturally" an isometric copy of X.

Now, taking a nontrivial ultrafilter $\mathcal{U}$ on the set of all natural numbers, and considering the ultrapower $X_{\mathcal{U}}$, where X stands for the Banach space given by Proposition 4.4, we obtain the next corollary.

Corollary 4.5. *There exists a non-Hilbert absolute-valuable transitive (non-separable) Banach space.*

Remark 4.6. In view of Proposition 2.1, for finite-dimensional Banach spaces, absolute valuability is a condition much stronger than transitivity. In the infinite-dimensional setting things change drastically. Indeed, if H is the infinite-dimensional separable Hilbert space, then $\mathcal{L}(H)$ is absolute-valuable (by Corollary 2.5) but, by [4, Theorem 4.5], it is not transitive (nor even convex-transitive). We recall that a Banach space X is said to be **convex-transitive** if, for every element u in S_X, the convex hull of $\mathcal{G}(u)$ is dense in B_X. On the other hand, to realize that an affirmative answer to Problem 4.1 is actually easier than an affirmative answer to the classical rotation problem, we would have to be sure that infinite-dimensional transitive Banach spaces need not be absolute-valuable, a fact that (although clearly expectable) is not clear for us for the moment.

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