A NOTE ON SPECTRALLY DOMINANT NORMS

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0 Introduction

Throughout this paper A will denote a finite-dimensional associative complex algebra. A vector space norm $\|.\|$ on A is said to be *stable* if there exists a positive constant σ satisfying $\|x^n\| \leq \sigma\|x\|^n$ for every x in A and all natural numbers n. It is well-known that stable norms $\|.\|$ on A are *spectrally dominant* (that is, for every x in A, the inequality $\rho(x) \leq \|x\|$ holds, where $\rho(.)$ denotes the spectral radius). In the case that A is the algebra of all $m \times m$ complex matrices for some m in $\mathbb{N}$, a celebrated theorem of S. Friedland and C. Zenger [FZ] asserts that, conversely, spectrally dominant norms on A are stable. As a matter of fact, the assertion in the Friedland-Zenger theorem does not remain true for arbitrary A, even if A is commutative and has a unit. An elemental counter-example is the following.

EXAMPLE 0.1.- Take A equal to the complex associative algebra with basis $\{u, v\}$ and multiplication table defined by $u^2 = u$, $uv = vu = v$, and $v^2 = 0$. Consider the norm $\|.\|$ on A given by $\|\lambda u + \mu v\| := \max\{|\lambda|, |\mu|\}$. Then $\|.\|$ is spectrally dominant (since $\rho(\lambda u + \mu v) = |\lambda|$). However, for n in $\mathbb{N}$ we have

$$\|(u + v)^n\| = \|u + nv\| = n,$$

so that $\|.\|$ cannot be stable.

We devote Section 1 of this note to study the limits of the Friedland-Zenger theorem. We show that the condition that $Rad(A)$ is a direct summand of A is sufficient to ensure that all spectrally dominant norms on A are stable (Proposition 1.3). Moreover, if either A is commutative or has a unit, then the above condition becomes also necessary (Corollaries 1.6 and 1.7). (Of course, in the case that A has a unit, $Rad(A)$ is a direct summand of A if and only if A is semisimple.)

In Section 2, we obtain a general version of the Friedland-Zenger theorem. We define *almost stable* norms on A as those norms $\|.\|$ on A such that $(1 + \varepsilon)\|.\|$ is stable for every positive number ε, and we prove that a norm on A is spectrally dominant (if and) only if it is almost stable (Theorem 2.2). In terms of a natural topology on the set of all norms on A, the above result means that the set of all stable norms on A is dense in the set of all spectrally dominant norms on A (Remark 2.3).

1 The limits of the Friedland-Zenger theorem

Let x be an element of A. We say that x is quasi-invertible in A if there exists y in A satisfying $xy = yx = x + y$. It is straightforward that x is quasi-invertible in A if an only if $\mathbf{1} - x$ is invertible in the unital hull of A. We define the *spectrum* $sp(A, x)$ and the *spectral radius* $\rho(A, x)$ of x by means of the equalities

$$sp(A, x) := \{0\} \cup \{\lambda \in \mathbb{C}\backslash\{0\} \ : \ \lambda^{-1}x \text{ is not quasi-invertible in } A\} \ ,$$

$$\rho(A, x) := \max\{|\lambda| \ : \ \lambda \in sp(A, x)\} \ .$$

As we pointed out in the introduction, the fact that stable norms are spectrally dominant is folklore. Indeed, if $\|.\|$ is any norm on A, then, by the finite-dimensionality of A, the product of A is $\|.\|$-continuous, and hence there exists $k > 0$ satisfying $\|xy\| \leq k\|x\|\|y\|$ for all x, y in A. It follows that $|||\,.\,||| = k\|.\|$ is an algebra norm on A. By [BD, Theorem 5.8], we have $\rho(A, x) = \lim_{n\to\infty} |||\, x^n \,|||^{1/n}$ for every x in A. Therefore

$$\lim_{n\to\infty} \|x^n\|^{1/n} \ = \ \lim \ k^{-1/n} \, |||\, x^n \,|||^{1/n} \ = \rho(A, x).$$

Now, if $\|.\|$ is stable (say $\|x^n\| \leq \sigma\|x\|^n$ for all x in A and n in $\mathbb{N}$), we obtain

$$\rho(A, x) \ = \lim_{n\to\infty} \|x^n\|^{1/n} \ \leq \lim_{n\to\infty} \sigma^{1/n}\|x\| \ = \ \|x\| \ .$$

LEMMA 1.1.- *Let $\|.\|$ be a spectrally dominant norm on A, and M an ideal of A. Then the natural quotient norm on A/M is spectrally dominant.*

Proof.- Let $\pi : A \to A/M$ be the natural quotient homomorphism. For $x \in \alpha \in A/M$ we have

$$sp(A/M, \alpha) \ = \ sp(A/M, \pi(x)) \ \subseteq \ sp(A, x) \ ,$$

hence $\rho(A/M, \alpha) \leq \rho(A, x) \leq \|x\|$. Therefore, for $\alpha \in A/M$ we obtain

$$\rho(A/M, \alpha) \ \leq \ \inf\{\|x\| \ : \ x \in \alpha\} \ = \ \|\alpha\| \ ,$$

so that the quotient norm on A/M is spectrally dominant. ∎

Since all norms on A are pair-wise equivalent, it is easily seen that, if $|||\,.\,|||$ and $\|.\|$ are norms on A with $|||\,.\,||| \leq \|.\|$, and if $|||\,.\,|||$ is stable, then so is $\|.\|$. This fact will be applied in the proof of the next lemma.

LEMMA 1.2.- *Let M_1,M_2 be ideals of A such that $A = M_1 \oplus M_2$. Assume that, for $i = 1, 2$, all spectrally dominant norms on M_i are stable. Then all spectrally dominant norms on A are stable.*

Proof.- Let $\|.\|$ be a spectrally dominant norm on A. By Lemma 1.2, for $i = 1, 2$, the natural quotient norm on A/M_i is spectrally dominant. Since A/M_1 (respectively, A/M_2) is isomorphic to M_2 (respectively, M_1), it follows from the assumption that, for $i = 1, 2$, the natural quotient norm on A/M_i is stable. Therefore there exists a positive number σ satisfying $\|\alpha^n\| \leq \sigma\|\alpha\|^n$ for all $i = 1, 2$, α in A/M_i, and n in $\mathbb{N}$. For $i = 1, 2$, let $\pi_i : A \to A/M_i$ be the natural quotient homomorphism, and define a norm $|\!|\!| \cdot |\!|\!|$ on A by

$$|\!|\!| x |\!|\!| := \max\{\|\pi_1(x)\|, \|\pi_2(x)\|\} \ .$$

Then, for every x in A and all natural numbers n we have $|\!|\!| x^n |\!|\!| \leq \sigma |\!|\!| x |\!|\!|^n$, and hence $|\!|\!| \cdot |\!|\!|$ is stable. Finally, since $|\!|\!| \cdot |\!|\!| \leq \|.\|$, the given spectrally dominant norm $\|.\|$ on A is stable too. ■

Recall that an ideal M of A is said to be a *direct summand* of A if there exists another ideal M' of A such that $A = M \oplus M'$. Recall also that the *radical* of A, $Rad(A)$, is defined as the largest nil ideal of A. A is called *semisimple* if $Rad(A) = 0$, and *radical* if $Rad(A) = A$.

PROPOSITION 1.3.- *If $Rad(A)$ is a direct summand of A (for instance, if A is either semisimple or radical), then all spectrally dominant norms on A are stable.*

Proof.- Assume that $Rad(A)$ is a direct summand of A. Then, by Wedderburn's theory, we have

$$A = \left(\oplus_{i=1}^{m} M_i \right) \ \oplus \ Rad(A) \ ,$$

where m is a natural number and, for $i = 1, ..., m$, M_i is an ideal of A isomorphic to a full matrix complex algebra. Now we keep in mind Lemma 1.2 and the Friedland-Zenger theorem to realize that the proof is concluded by showing that all norms on $Rad(A)$ are stable. Let $\|.\|$ be a norm on $Rad(A)$. Since the product of $Rad(A)$ is $\|.\|$-continuous, there exists $\sigma \geq 1$ satisfying $\|xy\| \leq \sigma\|x\|\|y\|$ for all x, y in $Rad(A)$. On the other hand, if d denotes the dimension of $Rad(A)$, then for every x in $Rad(A)$ we have $x^{d+1} = 0$. It follows $\|x^n\| \leq \sigma^{d-1}\|x\|^n$ for every x in $Rad(A)$ and all natural numbers n. ■

As the next example shows, the converse of Proposition 1.3 is not true.

EXAMPLE 1.4.- Take A equal to the complex associative algebra with basis $\{u, v\}$ and multiplication table given by $u^2 = u$, $uv = v$, and $vu = v^2 = 0$. Then $Rad(A)$ $(= \mathbb{C}v)$ is the unique nonzero proper ideal of A, and hence it is not a direct summand of A. However, for arbitrary $x = \lambda u + \mu v$ in A the equalities $x^2 = \lambda x$ and $\rho(A, x) = |\lambda|$ hold, and hence for any spectrally dominant norm $\|.\|$ on A and every n in $\mathbb{N}$ we have

$$\|x^n\| = |\lambda|^{n-1}\|x\| = (\rho(A, x))^{n-1}\|x\| \leq \|x\|^n \ .$$

Therefore all spectrally dominant norms on A are (strongly) stable.

A norm $\|.\|$ on A is said to be *weakly stable* if, for each x in A, there exists a positive constant σ_x satisfying $\|x^n\| \leq \sigma_x \|x\|^n$ for every n in $\mathbb{N}$. The argument showing that stable norms are spectrally dominant works without changes for weakly stable norms.

PROPOSITION 1.5.- *Assume that all spectrally dominant norms on A are weakly stable. Then, for every subalgebra B of A such that $A = B \oplus Rad(A)$, we have $BAB \subseteq B$.*

Proof.- Suppose that there exists a subalgebra B of A such that $A = B \oplus Rad(A)$ but BAB is not contained in B. Then $B\ Rad(A)\ B \neq 0$, and hence, denoting by e the unit of B (whose existence is not in doubt because B is semisimple), we have $eRad(A)e \neq 0$. Now, denote by P the unique linear projection on A satisfying $P(A) = B$ and $Ker(P) = Rad(A)$, and choose a norm $\|.\|$ on A satisfying the following three properties:

i) $\|e\| = 1$.

ii) The restriction of $\|.\|$ to B is an algebra norm on B.

iii) $\|x\| = \max\{\|P(x)\|, \|x - P(x)\|\}$ for every x in A.

(Indeed, take an arbitrary norm $|\!|\!| \cdot |\!|\!|$ on A, and for x in A put

$$\|x\| := \max\{\ Sup\{|\!|\!|\, P(x)b \,|\!|\!| : b \in B, |\!|\!|\, b \,|\!|\!| \leq 1\}\ ,\ \ |\!|\!|\, x - P(x) \,|\!|\!|\ \ \}\ .)$$

Then, since elements in $Rad(A)$ do not perturb the spectrum [A, Lemma 2, p. 2], for x in A we have

$$\rho(A, x) = \rho(A, P(x)) = \rho(B, P(x)) \leq \|P(x)\| \leq \|x\| \ ,$$

and therefore the norm $\|.\|$ is spectrally dominant. Since $eRad(A)e$ is a non-zero nil subalgebra of A, there must exist some v in $eRad(A)e$ such that $v^2 = 0$ and $\|v\| = 1$. Putting $u := e + v$, for n in $\mathbb{N}$ we have $u^n = e + nv$, and hence

$$\|u^n\| = \max\{\|e\|, \|nv\|\} = n \ .$$

It follows that the spectrally dominant norm $\|.\|$ on A is not weakly stable.
■

COROLLARY 1.6.- *Assume that A is commutative. Then the following assertions are equivalent:*

i) $Rad(A)$ is a direct summand of A.

ii) All spectrally dominant norms on A are stable.

iii) All spectrally dominant norms on A are weakly stable.

Proof.- By Proposition 1.3, certainly $i)$ implies $ii)$. On the other hand, the implication $ii) \Rightarrow iii)$ is clear. Let us complete the proof by showing that $iii)$ implies $i)$. By Wedderburn's principal theorem, there exists a subalgebra B of A such that $A = B \oplus Rad(A)$. Now, if $iii)$ holds, then Proposition 1.5 and the commutativity of A give $B^2A \subseteq B$. Since $B^2 = B$ (recall that B has a unit), actually B is an ideal of A, and $i)$ follows. ■

COROLLARY 1.7.- *Assume that A has a unit. Then the following assertions are equivalent:*

i) A is semisimple.

ii) $Rad(A)$ is a direct summand of A.

iii) All spectrally dominant norms on A are stable.

iv) All spectrally dominant norms on A are weakly stable.

Proof.- We know that the chain of implications $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$ is true. Assume that $iv)$ holds. Then, choosing a subalgebra B of A such that $A = B \oplus Rad(A)$, Proposition 1.5 provides us with the inclusion $BAB \subseteq B$. Denoting by $\mathbf{1}$ the unit of A, and writing $\mathbf{1} = e + v$ with e in B and v in $Rad(A)$, for every b in B we have $b = be + bv = eb + vb$. Since be and eb

lie in B, and bv and vb lie in $Rad(A)$, we obtain $be = eb = b$, hence e is a unit for B. Then v $(= \mathbf{1} - e)$ is an idempotent in $Rad(A)$, and hence $v = 0$. Therefore $\mathbf{1}$ belongs to B, so that $A = \mathbf{1}A\mathbf{1} \subseteq BAB \subseteq B$, and hence $Rad(A) = 0$. That is, $i)$ is satisfied. ■

2 A general version of the Friedland-Zenger theorem

In this section we prove that, when almost stable norms replace stable norms, and arbitrary finite dimensional associative complex algebras replace full matrix complex algebras, the Friedland-Zenger theorem remains true. The key tool in the proof is the following result on spectrally dominant almost extension of spectrally dominant norms.

PROPOSITION 2.1.- *Let $\|.\|$ be a spectrally dominant norm on A, B a finite-dimensional associative complex algebra containing A as a subalgebra, and ε a positive number. Then there exists a spectrally dominant norm on B whose restriction to A is equal to $(1+\varepsilon)\|.\|$.*

Proof.- Choose an arbitrary norm $|\!|\!| \,.\, |\!|\!|$ on B, and put

$$R := \{x \in A \ : \ \|x\| \leq 1\} \ ,$$

$$S := \{y \in B \ : \ \rho(B, y) < 1\} \ ,$$

$$T := \{y \in B \ : |\!|\!| \, y \, |\!|\!| \leq 1\} \ .$$

Since the mapping $\rho(B, .)$ is upper semicontinuous, S is open in B. On the other hand, since the norm $\|.\|$ on A is spectrally dominant, and for x in A the equality $\rho(B, x) = \rho(A, x)$ holds, we have

$$(1+\varepsilon)^{-1}R \ \subseteq \ S \ .$$

It follows from the compactness of $(1+\varepsilon)^{-1}R$ that there exists a positive number δ satisfying

$$(1+\varepsilon)^{-1}R \ + \ \delta T \ \subseteq \ S \ .$$

(Indeed, if $S = B$, then the assertion is clearly true, and otherwise it is enough to define δ as a half of the minimum value on $(1+\varepsilon)^{-1}R$ of the continuous function $d(., Fr(S))$, where d denotes the distance on B relative to the norm $|||\,.\,|||$, and $Fr(S)$ stands for the boundary of S in B.) Moreover, since the restriction of $|||\,.\,|||$ to A is equivalent to $\|.\|$, such a δ can be chosen small enough to have additionally

$$(\delta T) \cap A \ \subseteq\ (1+\varepsilon)^{-1} R \ .$$

It follows that, if U denotes the convex hull of $[(1+\varepsilon)^{-1}R] \cup (\delta T)$ in B, then we have

$$U \ \subseteq\ (1+\varepsilon)^{-1}R \ +\ \delta T \ \subseteq\ S \quad \text{and} \quad U \cap A \ =\ (1+\varepsilon)^{-1} R \ .$$

Since U is an absorbent, absolutely convex, and radially compact subset of B, there exists a norm $\|.\|_\varepsilon$ on B such that

$$U \ =\ \{y \in B \ :\ \|y\|_\varepsilon \leq 1\} \ .$$

Since $U \cap A \ =\ (1+\varepsilon)^{-1}R$, the restriction of $\|.\|_\varepsilon$ to A coincides with $(1+\varepsilon)\|.\|$. Finally, since $U \subseteq S$, $\|.\|_\varepsilon$ is spectrally dominant. ■

Recall that a norm $\|.\|$ on A is called almost stable if, for every positive number ε, $(1+\varepsilon)\|.\|$ is stable.

THEOREM 2.2.- *Let $\|.\|$ be a norm on A. Then $\|.\|$ is spectrally dominant (if and) only if it is almost stable.*

Proof.- Let A_1 denote the unital hull of A. By identifying each element x of A with the matrix (relative to a given basis of A_1) of the operator of left multiplication by x on A_1, we can see A as a subalgebra of a full complex matrix algebra (say B). Assume that $\|.\|$ is spectrally dominant. Then, by Proposition 2.1, for each $\varepsilon > 0$ there exists a spectrally dominant norm $\|.\|_\varepsilon$ on B whose restriction to A is equal to $(1+\varepsilon)\|.\|$. By the Friedland-Zenger theorem, $\|.\|_\varepsilon$ is stable. Therefore $(1+\varepsilon)\|.\|$ is stable too. ■

REMARK 2.3.- Let $N(A)$ denote the set of all norms on A, endowed with the topology of the distance $d(\|.\|_1, \|.\|_2) := \log(k)$, where k is the smallest positive number satisfying $k^{-1}\|x\|_1 \leq \|x\|_2 \leq k\|x\|_1$ for every x in A. It is easily seen that the set $SDN(A)$ of all spectrally dominant norms on A is closed in $N(A)$. Now, keeping in mind that norms on A majorizing some stable norm are stable, we can reformulate Theorem 2.2 by saying that $SDN(A)$ is the closure in $N(A)$ of the set of all stable norms on A.

References

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