A NON-ASSOCIATIVE RICKART'S DENSE-RANGE-HOMOMORPHISM THEOREM

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The aim of this paper is to discuss the non-associative side of a celebrated theorem of C. E. Rickart asserting the automatic continuity of dense range homomorphisms from complete normed associative algebras to complete normed strongly semisimple associative algebras. As main result, we prove that associativity can be removed in Rickart's theorem whenever the range algebra is Jordan-admissible.

1. Introduction and previously known results

In this paper we deal with the following problem.

Problem 1.1. *Let A be a complete normed algebra over $\mathbb{K}$, B a complete normed strongly semisimple algebra over $\mathbb{K}$, and $\varphi : A \to B$ a dense range homomorphism. Is φ automatically continuous?*

Here $\mathbb{K}$ denotes the field of real or complex numbers, and by an algebra we mean a possibly non-associative algebra. We recall that an algebra B is said to be **strongly semisimple** if its strong radical is zero, that the **strong radical** of B is defined as the intersection of all modular maximal (two-sided) ideals of B, and that an ideal M of B is called **modular** if there exists some element u in B such that $y - yu$ and $y - uy$ belong to M for every y in B.

According to a celebrated theorem of C. E. Rickart (see for instance [17, Theorem 6.18]), *Problem 1.1 has an affirmative answer whenever the algebra A is associative* (note that, if in Problem 1.1 A is associative, then so is B). However, as far as we know, in the general non-associative formulation we are considering, Problem 1.1 remains open to date. This situation contrasts with that of Johnson's continuity-of-epimorphisms theorem [7], since in this last case a satisfactory non-associative generalization is available [10, Theorem 3.3]. In fact, it follows easily from the results in [10] that *Problem 1.1 has an affirmative answer if φ is actually surjective.* Indeed, strongly semisimple algebras have at most one complete algebra norm topology [10, Remark 2.4.(i)], and homomorphisms from complete normed algebras onto strongly semisimple algebras have closed kernels [3, Lemma 1].

2000 *Mathematics Subject Classification.* 46H40, 46H70.

Partially supported by Junta de Andalucía grant FQM 0199.

Recall that an algebra is said to be **simple** if it has nonzero product and has no nonzero proper ideals. It is worth mentioning that both an affirmative and a negative answer to Problem 1.1 reduce to the case that the complete normed algebra B is simple with a unit. Indeed, simple algebras with a unit are strongly semisimple. On the other hand, if M is a modular maximal ideal of an algebra B, then B/M is a simple algebra with a unit, and if additionally B is complete normed, then B/M is a complete normed algebra because M is closed in B [4, Lemma 4]. Now assume that Problem 1.1 has an affirmative answer whenever B is simple with a unit. Then, when B is not subjected to any additional requirement, an affirmative answer can be obtained by the following standard closed graph argument. Let $\{x_n\}$ be a sequence in A with $\{x_n\} \to 0$ and $\{\varphi(x_n)\} \to y \in B$. Let M be a modular maximal ideal of B, and let π denote the natural quotient homomorphism $B \to B/M$. Since the composition $\pi \circ \varphi$ becomes a dense range homomorphism from A to B/M, and B/M is a complete normed simple algebra with a unit, our assumption applies, so that $\pi \circ \varphi$ is continuous, and hence we have $\{(\pi \circ \varphi)(x_n)\} \to 0$. Since $\{(\pi \circ \varphi)(x_n)\} = \{\pi(\varphi(x_n))\} \to \pi(y)$, we deduce $\pi(y) = 0$, or equivalently $y \in M$. Finally, since M is an arbitrary modular maximal ideal of B, and B is strongly semisimple, we obtain $y = 0$.

The above argument actually shows a more general fact, which is collected in the following lemma.

Lemma 1.2. *Let A be a complete normed algebra over $\mathbb{K}$, and let $\mathcal{C}$ be a class of algebras over $\mathbb{K}$ closed under quotients. Then the following assertions are equivalent:*

1. *For every complete normed strongly semisimple algebra B in $\mathcal{C}$, all dense range homomorphisms from A to B are continuous.*
2. *For every complete normed simple algebra B in $\mathcal{C}$ having a unit, all dense range homomorphisms from A to B are continuous.*

The reduction method given by Lemma 1.2 is applied in the proofs of all known partial affirmative answers to Problem 1.1. For instance, this is the case of Rickart's original theorem, as well as that of the result proved in [4] that *Problem 1.1 has an affirmative answer whenever the algebra B is algebraic*. We recall that an algebra B is said to be **algebraic** if all single-generated subalgebras of B are finite-dimensional. We note that, while complete normed (strongly) semisimple algebraic associative algebras are finite-dimensional [8], a similar fact is far from being true if associativity is removed. An illustrative example is the following.

Example 1.3. Let X be a Banach space over $\mathbb{K}$ with $\dim(X) \geq 2$, and f a continuous non-degenerate symmetric bilinear form on X with $\|f\| \leq 1$. Then the complete normed algebra B over $\mathbb{K}$ whose underlying Banach space is $\mathbb{K} \times X$ with the sum norm, and whose product is defined by

$$(\lambda, x)(\mu, y) := (\lambda\mu + f(x, y), \lambda y + \mu x)\,,$$

is algebraic and simple, and has a unit. The algebra B above is even **power-associative**, i.e., all single-generated subalgebras of B are associative.

Another partial affirmative answer to Problem 1.1, which in fact generalizes Rickart's theorem, is provided in [11]. There, some previous ideas in [1] are applied to show that *Problem 1.1 has an affirmative answer whenever the algebras A and B are power-associative.* In fact it is easy to see that, if A, B, and φ are as in Problem 1.1, and if A is power-associative, then so is B (see Remark 1.6 below). The result in [11] just reviewed contains the one in [9] that *Problem 1.1 has an affirmative answer whenever A* (and hence B) *is a Jordan algebra.* A slight refinement of the result in [11] quoted above, as well as a simplification of its proof, is given in what follows. As in the associative case, we define the **spectral radius** $r(x)$ of an element x in a normed power-associative algebra by

$$r(x) := \lim_{n\to\infty}\{\|x^n\|^{\frac{1}{n}}\} = \inf\{\|x^n\|^{\frac{1}{n}} : n \in \mathbb{N}\}\,.$$

Lemma 1.4. *Let A be a complete normed power-associative algebra over $\mathbb{K}$, B a normed algebra over $\mathbb{K}$ with a unit $\mathbf{1}$, and $\varphi : A \to B$ a homomorphism. Then for x in A we have*

$$1 \leq r(x) + \|\mathbf{1} - \varphi(x)\|\,.$$

Proof. Let x be in A. Denote by A' the closure in A of the subalgebra of A generated by x, put $B' := \mathbb{K}\mathbf{1} + \varphi(A')$, and let φ' stand for the mapping $y \to \varphi(y)$ from A' to B'. Then A' is a complete normed associative algebra, B' is an associative normed algebra, and $\varphi' : A' \to B'$ is a homomorphism. Therefore we have $r(\varphi(x)) \leq r(x)$. Finally, since $\varphi(x)$ and $\mathbf{1} - \varphi(x)$ are commuting elements of the normed associative algebra B', we obtain

$$1 = r(\mathbf{1}) \leq r(\varphi(x)) + r(\mathbf{1} - \varphi(x)) \leq r(x) + \|\mathbf{1} - \varphi(x)\|\,,$$

as desired. ∎

Let B be an algebra. We denote by B^+ the algebra whose vector space is the one of B, and whose product is defined by $x \circ y := \frac{1}{2}(xy + yx)$. Clearly, if B has a unit $\mathbf{1}$, then $\mathbf{1}$ is also a unit for B^+, and if B is normed, then B^+ becomes a normed algebra under the norm of B. We say that B **admits power-associativity** if B^+ is power-associative. We note that the class of algebras admitting power-associativity strictly enlarges that of power-associative algebras.

Theorem 1.5. *Problem 1.1 has an affirmative answer whenever the algebra A admits power-associativity.*

Proof. Let A be a complete normed algebra over $\mathbb{K}$ admitting power-associativity, B a complete normed strongly semisimple algebra over $\mathbb{K}$, and $\varphi : A \to B$ a dense range homomorphism. We are going to show that φ is continuous. In view of Lemma 1.2 (with $\mathcal{C}$ equal to the class of all algebras over $\mathbb{K}$), we can assume that the strongly semisimple algebra B is in fact simple with a unit $\mathbf{1}$. By the denseness of the range of φ, the separating subspace $\mathcal{S}(\varphi)$ of φ is an ideal of B [5, p. 140]. Therefore, by the simplicity of B and the closed graph theorem, it is enough to prove that $\mathbf{1}$ does not belong to $\mathcal{S}(\varphi)$. Assume that $\mathbf{1}$ lies in $\mathcal{S}(\varphi)$. Then we have $\{x_n\} \to 0$ and $\{\varphi(x_n)\} \to \mathbf{1}$ for some sequence $\{x_n\}$ in A. Now, regarding φ as a homomorphism from A^+ to B^+, and applying Lemma 1.4, we obtain

$$1 \leq \lim_{n\to\infty} \{\|x_n\| + \| \mathbf{1} - \varphi(x_n) \|\} = 0 \ ,$$

which is a contradiction. ∎

Remark 1.6. Let A be an algebra over $\mathbb{K}$, and let B be a normed algebra over $\mathbb{K}$ such that there exists a dense range homomorphism φ from A to B. It is almost obvious that, if A satisfies some identity (for instance, associativity $(\mathbf{xy})\mathbf{z} - \mathbf{x}(\mathbf{yz}) = 0$, or commutativity $\mathbf{xy} - \mathbf{yx} = 0$), then B also satisfies such an identity. For the precise meaning of the sentence **an algebra satifies a given identity** the reader is referred to [6, p. 25]. Since power-associative algebras are nothing but those algebras satisfying all identities in the set

$$\{\mathbf{x}^{n+m} - \mathbf{x}^n\mathbf{x}^m = 0 : (m,n) \in \mathbb{N} \times \mathbb{N}\}$$

(where the n-th left power $\mathbf{x}^n$ of the indeterminate $\mathbf{x}$ is defined inductively by $\mathbf{x}^1 = \mathbf{x}$ and $\mathbf{x}^{n+1} = \mathbf{x}\mathbf{x}^n$), it follows that B is power-associative whenever A is. As a consequence, regarding φ as a homomorphism from A^+ to B^+, we obtain that B admits power-associativity whenever A does.

According to the above remark, if we were able to answer affirmatively Problem 1.1 whenever B admits power-associativity, then we would be provided with a generalization of Theorem 1.5. Unfortunately, we do not know if such a generalization is true. We note that, if A, B, and φ are as in Problem 1.1, if B admits power-associativity, and if $\ker(\varphi)$ is closed in A, then φ is continuous. This is so because in this case $A' := A/\ker(\varphi)$ is a complete normed algebra admitting power-associativity, φ induces a dense range homomorphism $\varphi' : A' \to B$, and Theorem 1.5 applies with A' and φ' instead of A and φ, respectively. The main result in this paper, which will be proved in Section 2, asserts that Problem 1.1 has an affirmative answer whenever B enjoys a property slightly stronger than that of admitting power-associativity. This property is the so-called Jordan-admissibility, and will be presented also in Section 2.

The concluding section of the paper (Section 3) is devoted to collect those previously known automatic continuity results which give some light about Problem 1.1. In some cases, the adaptation of such previously known results to our framework needs a certain amount of work.

2. The main result

Jordan algebras are defined as those commutative algebras satisfying the identity $\mathbf{x}(\mathbf{y}\mathbf{x}^2)-(\mathbf{x}\mathbf{y})\mathbf{x}^2=0$. To provide the reader with some examples of Jordan algebras, let us say that, if A is an associative algebra, then A^+ is a Jordan algebra. Also, the algebra B built in Example 1.3 is a Jordan algebra. We note that Jordan algebras are power-associative [6, Theorem 8, p. 36]. Let B be a Jordan algebra with a unit $\mathbf{1}$. An element x of B is said to be **invertible** in B if there exists some y in B satisfying $xy=\mathbf{1}$ and $x^2y=x$. In the particular case that $B=A^+$ for some associative algebra A, the unit $\mathbf{1}$ of B is also a unit for A, and invertible elements of B in the Jordan sense are nothing but invertible elements of A in the usual associative meaning [6, p. 51]. Now, let B be a complex Jordan algebra with a unit $\mathbf{1}$. The **spectrum** of an element x of B, denoted by $\mathrm{sp}(x)$, is defined as the set of those complex numbers λ such that $x-\lambda\mathbf{1}$ is not invertible in B. If in addition B is complete normed, then, for x in B, the fundamental Gelfand-Beurling formula $r(x)=\max\{|\lambda|:\lambda\in\mathrm{sp}(x)\}$ holds [18].

For elements x,y in an algebra C, we put

$$L_x^C(y):=xy \quad \text{and} \quad U_x^C(y):=x(xy+yx)-x^2y\,.$$

Lemma 2.1. *Let A be a complete normed algebra over $\mathbb{K}$, B a normed Jordan algebra over $\mathbb{K}$ with a unit $\mathbf{1}$, and $\phi:A\to B$ a homomorphism. Then for every x in A we have $r(\phi(x))\leq 3\|x\|$.*

Proof. Regarding φ as a homomorphism from A to the completion of B, there is no loss of generality in assuming that B is complete.

First suppose that $\mathbb{K}=\mathbb{C}$. Then, in view of the Gelfand-Beurling formula, it is enough to show that $\mathbf{1}-\phi(x)$ is invertible in B whenever x is in A and satisfies $\|x\|<\frac{1}{3}$. Let x be in A with $\|x\|<\frac{1}{3}$. Then we have

$$\|2L_x^A+L_{x^2}^A-2(L_x^A)^2\|\leq 2\|x\|+3\|x\|^2<1\,.$$

Therefore, denoting by I_A the identity mapping on A, the operator $I_A-(2L_x^A+L_{x^2}^A-2(L_x^A)^2)$ is an invertible element of the complete normed associative algebra of all bounded linear operators on A, and hence there exists some y in A satisfying

$$(I_A-(2L_x^A+L_{x^2}^A-2(L_x^A)^2))(y)=x^2-2x\,,$$

that is

$$x^2-2x-y+2xy+x^2y-2x(xy)=0\,.$$

Now we have the equality

$$\phi(x)^2-2\phi(x)-\phi(y)+2\phi(x)\phi(y)+\phi(x)^2\phi(y)-2\phi(x)(\phi(x)\phi(y))=0\,,$$

which can be re-formulated as

$$U^B_{\mathbf{1}-\phi(x)}(\mathbf{1}-\phi(y))=\mathbf{1}\,.$$

Therefore $\mathbf{1}$ lies in the range of $U^B_{\mathbf{1}-\phi(x)}$, and hence, by [6, Theorem 13, p. 52], $\mathbf{1}-\phi(x)$ is invertible in B, as desired.

Now suppose $\mathbb{K}=\mathbb{R}$. Consider the complete normed algebras $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$, complexifications of A and B respectively (argue as in the associative case [2, Proposition 13.3]), and note that $B_{\mathbb{C}}$ is a Jordan algebra [16, p. 91]. Then, by extending ϕ to a complex-linear homomorphism from $A_{\mathbb{C}}$ to $B_{\mathbb{C}}$, the preceding part of the proof applies. ∎

We say that an algebra B is **Jordan-admissible** if B^+ is a Jordan algebra. We remark that, since Jordan algebras are power-associative, Jordan admissible algebras admit power-associativity.

Theorem 2.2. *Problem 1.1 has an affirmative answer whenever B is Jordan-admissible.*

Proof. Let A, B, and φ be as in Problem 1.1, and suppose that B is Jordan-admissible. Since the class of all Jordan admissible algebras over $\mathbb{K}$ is closed under quotients, Lemma 1.2 applies, so that we can assume that B is simple with a unit $\mathbf{1}$. On the other hand, regarding φ as a homomorphism from A^+ to B^+, we can apply Lemma 2.1 to obtain $r^+(\varphi(x))\leq 3\|x\|$ for every x in A, where $r^+(\cdot)$ denotes spectral radius relative to B^+. Since, for x in A, the subalgebra of B^+ generated by $\mathbf{1}$ and $\varphi(x)$ is associative, for such an x we have

$$1\leq r^+(\varphi(x))+r^+(\mathbf{1}-\varphi(x))\leq 3\|x\|+\|\mathbf{1}-\varphi(x)\|\,.$$

Therefore $\mathbf{1}$ cannot lie in the separating subspace of φ, and the continuity of φ follows as in the proof of Theorem 1.5 ∎

Most important classes of non-associative algebras arising in the literature consist of Jordan-admissible algebras. For instance, the so-called **non-commutative Jordan algebras** (see [16] for a definition) are Jordan-admissible. Here the words "non-associative" and "non-commutative" must be understood as "possibly non-associative" and "possibly non commutative", respectively. We note that the class of non-commutative Jordan algebras contain both that of Jordan algebras and that of **alternative algebras** (see again [16]). By the way, associative algebras are alternative. For the sake of convenience, we emphasize in the next corollary the specialization of Theorem 2.2 to the case that B is associative.

Corollary 2.3. *Problem 1.1 has an affirmative answer whenever B is associative.*

For readers only interested in Corollary 2.3, we note that a proof of such a corollary can be given without involving Jordan theory. Indeed, after the usual reduction to the case that B is simple with a unit, it is enough to follow the lines of the proof of Theorem 2.2 replacing Lemma 2.1 above with Lemma 2.4 immediately below.

Lemma 2.4. *Let A be a complete normed algebra over $\mathbb{K}$, B a normed associative algebra over $\mathbb{K}$ with a unit $\mathbf{1}$, and $\phi : A \to B$ a homomorphism. Then for every x in A we have $r(\phi(x)) \leq \|x\|$.*

Proof. We can assume that $\mathbb{K} = \mathbb{C}$. Then it is enough to show that $\mathbf{1} - \phi(x)$ is invertible in B whenever x is in A and satisfies $\|x\| < 1$. Let x be in A with $\|x\| < 1$. By [4, Lemma 1], there exist y, z in A satisfying $xy = y - x$ and $zx = z - x$. Now we have

$$(\mathbf{1} - \varphi(x))(\mathbf{1} + \varphi(y)) = (\mathbf{1} + \varphi(z))(\mathbf{1} - \varphi(x)) = \mathbf{1}\,,$$

and hence $\mathbf{1} - \varphi(x)$ is invertible in B as desired. ■

3. Other dense-range-homomorphism theorems in relation to Problem 1.1

Automatic continuity theorems for dense range homomorphisms between complete normed algebras have been proved in [13] and [14] under certain additional requirements which, in a first instance, are not too close to those in Problem 1.1. In this section we investigate how such automatic continuity results can give some light on that problem.

Let B be an algebra. We denote by B^2 the linear hull of the set $\{xy : x, y \in B\}$. We remark that, if B is simple, then $B^2 = B$ [16, p. 15]. A bilinear form $< \cdot, \cdot >$ on B is said to be **associative** if the equalities $< xy, z >=< x, yz >=< y, zx >$ hold for all x, y, z in B.

Proposition 3.1. *Let A be a complete normed algebra over $\mathbb{K}$, B a complete normed simple algebra over $\mathbb{K}$, and $\varphi : A \to B$ a dense range homomorphism. Assume that there exists a nonzero continuous symmetric associative bilinear form $< \cdot, \cdot >$ on B. Then φ is continuous.*

Proof. First we note that, since B is simple and $< \cdot, \cdot >$ is a nonzero associative bilinear form, $< \cdot, \cdot >$ is in fact nondegenerate.

Assume that $\mathbb{K} = \mathbb{C}$. Then, by [12, Corollary B.14] and [13, Proposition 2], φ is continuous.

Now assume $\mathbb{K} = \mathbb{R}$. Consider the complete normed algebras $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ (complexifications of A and B, respectively), the unique complex-linear mapping $\varphi_{\mathbb{C}} : A_{\mathbb{C}} \to B_{\mathbb{C}}$ which extends φ, and the unique complex-bilinear form $< \cdot, \cdot >_{\mathbb{C}}$ on $B_{\mathbb{C}}$ which extends $< \cdot, \cdot >$. Note that $\varphi_{\mathbb{C}}$ is a dense range homomorphism, and that $< \cdot, \cdot >_{\mathbb{C}}$ is a continuous nondegenerate symmetric associative bilinear form. If $B_{\mathbb{C}}$ is simple, then, by the preceding paragraph, $\varphi_{\mathbb{C}}$ (and hence φ) is continuos. Suppose additionally that $B_{\mathbb{C}}$ is not simple. Let τ denote the unique real-linear operator on $B_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} B$ whose values on elementary tensors are given by $\tau(\lambda \otimes x) = \overline{\lambda} \otimes x$. Then τ is an involutive conjugate-linear automorphism of $B_{\mathbb{C}}$ satisfying

$$B = \{u \in B_{\mathbb{C}} : \tau(u) = u\}\,,$$

and therefore the simplicity of B implies that $B_{\mathbb{C}}$ is τ-**simple**, i.e., $B_{\mathbb{C}}$ has nonzero product and has no nonzero proper τ-invariant ideals. Now, for every nonzero proper ideal M of $B_{\mathbb{C}}$, we have $B_{\mathbb{C}} = M \oplus \tau(M)$. This implies that such an ideal M of $B_{\mathbb{C}}$ is a simple algebra and is closed in $B_{\mathbb{C}}$. Indeed, closedness of M in $B_{\mathbb{C}}$ follows from the equality

$$M = \{u \in B_{\mathbb{C}} : u\tau(M) = \tau(M)u = 0\},$$

which is easily verified using the simplicity of $\tau(M)$ and the fact that $B_{\mathbb{C}} = M \oplus \tau(M)$. Let us fix a nonzero proper ideal M of $B_{\mathbb{C}}$ (the existence of which is not in doubt because $B_{\mathbb{C}}$ is not simple). We claim that the restriction of $< \cdot, \cdot >_{\mathbb{C}}$ to $M \times M$ is nondegenerate. Indeed, if u is in M, and if the equality $< u, M >_{\mathbb{C}} = 0$ holds, then, by the simplicity of $\tau(M)$, roughly writing we have

$$< u, \tau(M) >_{\mathbb{C}} = < u, \tau(M)^2 >_{\mathbb{C}} = < u\tau(M), \tau(M) >_{\mathbb{C}} = < 0, \tau(M) >_{\mathbb{C}} = 0,$$

so $< u, B_{\mathbb{C}} >_{\mathbb{C}} = < u, M \oplus \tau(M) >_{\mathbb{C}} = 0$, and so $u = 0$, which proves the claim. Now, let p and q denote the projections from $B_{\mathbb{C}}$ onto M and $\tau(M)$, respectively, corresponding to the decomposition $B_{\mathbb{C}} = M \oplus \tau(M)$. Then, by the preceding paragraph, $p \circ \varphi_{\mathbb{C}}$ and $q \circ \varphi_{\mathbb{C}}$ are continuous. Since the direct sum $B_{\mathbb{C}} = M \oplus \tau(M)$ is topological, we obtain that $\varphi_{\mathbb{C}}$ (and hence φ) is continuous. ∎

Let B be an algebra. For x, y, z in B, we write $[x, y] := xy - yx$ and $[x, y, z] := (xy)z - x(yz)$. The symbols $[B, B]$ and $[B, B, B]$ will stand for the linear hull of the sets $\{[x, y] : x, y \in B\}$ and $\{[x, y, z] : x, y, z \in B\}$, respectively.

Corollary 3.2. *Let A be a complete normed algebra over $\mathbb{K}$, let B be a complete normed simple algebra over $\mathbb{K}$ such that $[B, B] + [B, B, B]$ is not dense in B, and let $\varphi : A \to B$ be a dense range homomorphism. Then φ is continuous.*

Proof. Choose a nonzero continuous linear functional f on B such that $f([B, B] + [B, B, B]) = 0$, and for x, y in B put $< x, y > := f(xy)$. Then $< \cdot, \cdot >$ becomes a continuous symmetric associative bilinear form on B. Moreover $< \cdot, \cdot >$ is nonzero because, by the simplicity of B, roughly writing we have $0 \neq f(B) = f(B^2) = < B, B >$. Now Proposition 3.1 applies. ∎

In general Corollary 3.2 could collect less information than Proposition 3.1. However, this is not the case if the algebra B in that corollary has a unit. Indeed, if B is a normed algebra with a unit $\mathbf{1}$, and if there exists a nonzero continuous symmetric associative bilinear form $< \cdot, \cdot >$ on B, then the continuous linear functional f on B defined by $f(x) := < x, \mathbf{1} >$ is nonzero (because $< x, y > = f(xy)$ for x, y in B) and satisfies $f([B, B] + [B, B, B]) = 0$, so that $[B, B] + [B, B, B]$ is not dense in B.

For a vector space E over $\mathbb{K}$, we denote by I_E the identity operator on E, and by $L(E)$ the associative algebra over $\mathbb{K}$ of all linear operators on E. Now let C be an algebra over $\mathbb{K}$. For c in C we denote by L_c^C (respectively, R_c^C) the operator of left (respectively, right) multiplication by c on C. The **unital multiplication algebra** of C is defined as the subalgebra of $L(C)$ generated by $\{I_C\} \cup \{L_x^C : x \in C\} \cup \{R_y^C : y \in C\}$, and is denoted by $\mathcal{M}(C)$.

Proposition 3.3. *Let A be a complete normed algebra over $\mathbb{K}$, B a complete normed simple algebra over $\mathbb{K}$, and $\varphi : A \to B$ a dense range homomorphism. Assume that there exists a nonzero element in $\mathcal{M}(\varphi(A))$ with finite-dimensional range. Then φ is continuous.*

Proof. Suppose that $\mathbb{K} = \mathbb{C}$. Then, since B is simple, $\mathcal{M}(B)$ is an irreducible algebra of linear operators on B, and hence a prime algebra. Moreover, clearly, simple algebras have zero annihilator. It follows from our assumption and [13, Proposition 1] that φ is continuous.

Now suppose $\mathbb{K} = \mathbb{R}$. We note that, since φ has dense range, and $\mathcal{M}(\varphi(A))$ consists of bounded linear operators on $\varphi(A)$, we can identify each element F in $\mathcal{M}(\varphi(A))$ with the unique bounded linear operator on B which extends F, and consequently we can see $\mathcal{M}(\varphi(A))$ as the subalgebra of $\mathcal{M}(B)$ generated by $\{I_B\} \cup \{L_x^B : x \in \varphi(A)\} \cup \{R_y^B : y \in \varphi(A)\}$. We remark that an element of $\mathcal{M}(\varphi(A))$ has finite-dimensional range if and only if it has finite-dimensional range regarded as an element of $\mathcal{M}(B)$ in the above identification. Let $A_\mathbb{C}$, $B_\mathbb{C}$, $\varphi_\mathbb{C}$, and τ be as in the concluding paragraph of the proof of Proposition 3.1. For F in $L(B)$, let $F_\mathbb{C}$ denote the unique complex-linear operator on $B_\mathbb{C}$ which extends F. Note that, for such an F, we have $\tau \circ F_\mathbb{C} = F_\mathbb{C} \circ \tau$. By assumption, there exists a nonzero element G in $\mathcal{M}(\varphi(A))$ with finite-dimensional range, so that, easily, $G_\mathbb{C}$ becomes a nonzero element in $\mathcal{M}(\varphi_\mathbb{C}(A_\mathbb{C}))$ having also finite-dimensional range. Now, if $B_\mathbb{C}$ is simple, then, by the first paragraph of this proof, $\varphi_\mathbb{C}$ (and hence φ) is continuous. Suppose additionally that $B_\mathbb{C}$ is not simple. Then, fixing a nonzero proper ideal M of $B_\mathbb{C}$, we know (by the concluding paragraph of the proof of Proposition 3.1) that M is a simple algebra, that M is closed in $B_\mathbb{C}$, and that $B_\mathbb{C} = M \oplus \tau(M)$. Let us consider the set J of those elements F in $L(B)$ such that M is invariant under $F_\mathbb{C}$, and for F in J let $\psi(F)$ stand for the mapping $m \to F_\mathbb{C}(m)$ from M to M. Then J is a subalgebra of $L(B)$, and ψ becomes a homomorphism from J into the real algebra underlying $L(M)$. If F is in J, and if $\psi(F) = 0$, then we have $F_\mathbb{C}(M) = 0$ and, since $\tau \circ F_\mathbb{C} = F_\mathbb{C} \circ \tau$, also $F_\mathbb{C}(\tau(M)) = 0$, and hence $F_\mathbb{C} = 0$. Thus ψ is injective. On the other hand, for x in B, L_x^B and R_x^B lie in J, and the equalities $\psi(L_x^B) = L_{p(x)}^M$ and $\psi(R_x^B) = R_{p(x)}^M$ hold, where p denotes the projection from $B_\mathbb{C}$ onto M corresponding to the decomposition $B_\mathbb{C} = M \oplus \tau(M)$. In particular, for a in A, we have $\psi(L_{\varphi(a)}^B) = L_{p(\varphi(a))}^M$ and $\psi(R_{\varphi(a)}^B) = R_{p(\varphi(a))}^M$, and hence $\psi(L_{\varphi(a)}^B)$ and $\psi(R_{\varphi(a)}^B)$ belong to $\mathcal{M}(p \circ \varphi_\mathbb{C}(A_\mathbb{C}))$ (when this last algebra is "naturally" regarded as a subalgebra of $\mathcal{M}(M)$). Since $\psi(I_B) = I_M$, it follows that $\mathcal{M}(\varphi(A))$ is contained in J, and that $\psi(\mathcal{M}(\varphi(A)))$ is contained in $\mathcal{M}(p \circ \varphi_{\mathbb{C}}(A_{\mathbb{C}}))$. Now, since we had a nonzero element G in $\mathcal{M}(\varphi(A))$ with finite-dimensional range, we are also provided with a nonzero element of $\mathcal{M}(p \circ \varphi_{\mathbb{C}}(A_{\mathbb{C}}))$, namely $\psi(G)$, which has a finite-dimensional range. By the first paragraph in the proof, $p \circ \varphi_{\mathbb{C}}$ is continuous. Similarly $q \circ \varphi_{\mathbb{C}}$ is also continuous, where q denotes the projection from $B_{\mathbb{C}}$ onto $\tau(M)$ corresponding to the decomposition $B_{\mathbb{C}} = M \oplus \tau(M)$. Since the direct sum $B_{\mathbb{C}} = M \oplus \tau(M)$ is topological, the continuity of $\varphi_{\mathbb{C}}$ follows. ∎

Thanks to Lemma 2.1, we know that, if Problem 1.1 has a negative answer, then in fact there exists a complete normed algebra A, a complete normed simple algebra B with a unit, and a dense range discontinuous homomorphism $\varphi : A \to B$. In the next theorem we collect all the information we have concerning the above situation. A linear operator F on a normed space X is said to be **bounded below** if there is a positive constant k satisfying $k\|x\| \leq \|F(x)\|$ for every x in X. Now, let B be a normed algebra. An element x of B is said to be a **left** (respectively, **right**) **topological divisor of zero** in B if L_x^B (respectively, R_x^B) is not bounded below. Elements of B which are both left and right (respectively, either left or right) topological divisors of zero are called **two-sided** (respectively, **one-sided**) **topological divisors of zero** in B.

Theorem 3.4. *Let A be a complete normed algebra over $\mathbb{K}$, B a complete normed simple algebra over $\mathbb{K}$, and $\varphi : A \to B$ a discontinuous dense range homomorphism. Then we have:*

1. *φ is not surjective.*
2. *$[B, B] + [B, B, B]$ is dense in B.*
3. *Every nonzero element of $\mathcal{M}(\varphi(A))$ has infinite-dimensional range.*
4. *If $\mathbb{K} = \mathbb{C}$, then B has a nonzero two-sided topological divisor of zero.*

Moreover, if in addition B has a unit, then:

5. *Every element in $\mathcal{M}(B)$ is either bijective or non bounded below.*
6. *A does not admit power-associativity, and B is neither algebraic nor Jordan-admissible.*

Proof. Assertions 1, 2, 3, and 4 follow from [10, Remark 3.4.(ii)], Corollary 3.2, Proposition 3.3, and [14, Theorem 3.5], respectively. Now assume that B has a unit. Then B has some element which is not a two-sided divisor of zero in B (namely, the unit of B). Since φ is a discontinuous dense range homomorphism, and B is simple, the separating subspace $\mathcal{S}(\varphi)$ of φ coincides with B, and hence $\mathcal{S}(\varphi)$ does not consist only of two-sided topological divisors of zero in B. Therefore Assertion 5 is a consequence of [14, Corollary 2.2]. Finally, Assertion 6 follows from results which have been either reviewed or proved in Sections 1 and 2. ∎

According to the actual statement of [14, Theorem 3.5], neither the simplicity of B nor the density of the range of φ are needed for the validity of

Assertion 4 in Theorem 3.4. However, even under the restrictive environmental requirements of Theorem 3.4, we do not know if the extra condition $\mathbb{K} = \mathbb{C}$ in Assertion 4 can be removed. Even more, we are unable to prove or disprove the automatic continuity of dense range homomorphisms from complete normed real algebras to complete normed real algebras WITH A UNIT and having no nonzero ONE-SIDED topological divisors of zero. We note that, by [14, Lemma 4.2], these last algebras are in fact **two-sided division algebras** (i.e., all operators of left and right multiplication by their nonzero elements are bijective), and hence they are simple. It follows from either Assertion 1 or Assertion 3 in Theorem 3.4 that this "bleeding" particular case of Problem 1.1 would have an affirmative answer if the old conjecture [19], that complete normed two-sided division real algebras are finite-dimensional, were verified. For a more detailed information about the scope of Assertion 4 in Theorem 3.4, as well as of its variants for $\mathbb{K} = \mathbb{R}$, the reader is referred to [14] and [15].

Again in relation to Assertion 4 of Theorem 3.4, let us note that the real and complex versions of Problem 1.1 are equivalent. Indeed, simple complex algebras with a unit remain simple when they are regarded as real algebras. Therefore, with the help of Lemma 1.2, we deduce that Problem 1.1 has an affirmative answer for $\mathbb{K} = \mathbb{C}$ whenever it has an affirmative answer affirmatively for $\mathbb{K} = \mathbb{R}$. To see the converse, assume that Problem 1.1 has an affirmative answer for $\mathbb{K} = \mathbb{C}$. Let A, B, and φ be as in Problem 1.1 with $\mathbb{K} = \mathbb{R}$. We are going to show that φ is continuous, so that, by a new application of Lemma 1.2, we can suppose that B is simple and has a unit. Let $A_{\mathbb{C}}$, $B_{\mathbb{C}}$, and $\varphi_{\mathbb{C}}$ as in the proof of Proposition 3.1. By such a proof we know that $B_{\mathbb{C}}$ is either simple or a direct sum of two simple ideals. Since $B_{\mathbb{C}}$ has a unit, this implies that $B_{\mathbb{C}}$ is strongly semisimple. Therefore, by our assumption, $\varphi_{\mathbb{C}}$ (and hence φ) is continuous.

In the above paragraph we noted that simple complex algebras with a unit are also simple when regarded as real algebras. It is worth mentioning that this obvious fact remains true, now not so obviously, if the assumption of existence of a unit is removed. Actually the simplicity of a possibly non unital algebra does not depend on the field over which the algebra is defined [16, pp. 15-16]. For the sake of convenience, we give here a new proof of this result. **Ring ideals** of an algebra B are defined as those additive subgroups M of B such that MB and BM are contained in M. The algebra B is said to be **simple as a ring** if it has nonzero product and has no nonzero proper ring ideals.

Proposition 3.5. *Let B be an algebra over a field $\mathbb{F}$. Then B is simple as an algebra over $\mathbb{F}$ (if and) only if B is simple as a ring.*

Proof. Assume that B is simple as an algebra over $\mathbb{F}$. Let M be a proper ring ideal of B. Then $\bigcap_{\lambda\in\mathbb{F}\setminus\{0\}} \lambda M$ is a proper ideal of B, and hence $\bigcap_{\lambda\in\mathbb{F}\setminus\{0\}} \lambda M = 0$. But, for λ in $\mathbb{F}\setminus\{0\}$, we have $BM = \lambda BM \subseteq \lambda M$, and hence $BM \subseteq \bigcap_{\lambda\in\mathbb{F}\setminus\{0\}} \lambda M = 0$. Analogously, we obtain $MB = 0$. It follows that M is contained in the annihilator of the simple algebra B, and therefore $M = 0$. ■

References

[1] V. K. BALACHANDRAN and P. S. REMA, Uniqueness of the norm topology in certain Banach Jordan algebras, *Publ. Ramanujan Inst.* **1** (1969), 283-289.

[2] F. F. BONSALL and J. DUNCAN, *Complete normed algebras*, Springer Verlag, Berlin, 1973.

[3] A. CEDILNIK and A. RODRÍGUEZ, Automatic continuity of homomorphism into normed quadratic algebras, *Publ. Math. Debrecen* (to appear).

[4] A. CEDILNIK and A. RODRÍGUEZ, Continuity of homomorphisms into complete normed algebraic algebras, to appear.

[5] H. G. DALES, Automatic continuity: a survey, *Bull. London Math. Soc.* **10** (1978), 129-183.

[6] N. JACOBSON, *Structure and representations of Jordan algebras*, Amer. Math. Soc., Providence, Rhode island, 1968.

[7] B. E. JOHNSON, The uniqueness of the (complete) norm topology, *Bull. Amer. Math. Soc.* **73** (1967), 537-539.

[8] I. KAPLANSKY, Topological methods in valuation theory, *Duke Math. J.* **14** (1947), 527-541.

[9] P. S. PUTTER and B. YOOD, Banach Jordan $*$-algebras, *Proc. London Math. Soc.* **41** (1980), 21-44.

[10] A. RODRÍGUEZ, The uniqueness of the complete norm topology in complete normed non algebras, *J. Functional Analysis* **60** (1985), 1-15.

[11] A. RODRÍGUEZ, An aproach to Jordan-Banach algebras from the theory of non complete normed algebras, *Ann. Sci. Univ. Blaise Pascal, Clermont II, Sér. Math.* **27** (1991), 1-57.

[12] A. RODRÍGUEZ, Jordan structures in Analysis, in *Proceedings of the Conference held in Oberwolfach, Germany, August 9-15, 1992* (Edited by W. Kaup, K. McCrimmon, and H. P. Petersson), Walter de Gruyter, Berlin, 1994, 97-186.

[13] A. RODRÍGUEZ, Continuity of densely valued homomorphisms into H^*-algebras, *Quart. J. Math. Oxford* **2** (1995), 107-118.

[14] A. RODRÍGUEZ, Continuity of homomorphisms into normed algebras without topological divisors of zero, *Rev. Real Acad. Cienc. Exact. Fis. Natur. Madrid* (to appear).

[15] A. RODRÍGUEZ and M. V. VELASCO, A note on topological divisors of zero (paper in progress).

[16] R. D. SCHAFER, *An introduction to non algebras*, Academic Press, New York, 1966.

[17] A. M. SINCLAIR, *Automatic continuity of linear operators*. London Math. Soc. Lecture Note Series **21**, Cambridge University Press, 1976.

[18] C. VIOLA DEVAPAKKIAM, Jordan algebras with continuous inverse, *Math. Japon.* **16** (1971), 115-125.

[19] F. B. WRIGHT, Absolute valued algebras, *Proc. Math. Acad. Sci. U.S.A.* **39** (1953), 330-332.

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