A bilinear version of Holsztynski's theorem on isometries of $C(X)$-spaces

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Abstract. We prove that, for a compact metric space X not reduced to a point, the existence of a bilinear mapping $\diamond : C(X) \times C(X) \to C(X)$ satisfying $\|f \diamond g\| = \|f\|\|g\|$ for all $f, g \in C(X)$ is equivalent to the uncountability of X. This is derived from a bilinear version of Holsztynski's theorem [**3**] on isometries of $C(X)$-spaces, which is also proved in the paper.

1. Introduction

A celebrated theorem of W. Holsztynski [**3**] asserts that, *if X and Z are compact Hausdorff topological spaces, and if $T : C(X) \to C(Z)$ is a linear isometry, then there exist a closed subset Z_0 of Z, a continuous surjective mapping $\varphi : Z_0 \to X$, and a norm-one element $\alpha \in C(Z)$ satisfying $|\alpha(z)| = 1$ and $T(f)(z) = \alpha(z)f(\varphi(z))$ for every $(z, f) \in Z_0 \times C(X)$.* As main result, we prove that, *if X, Y, Z are compact Hausdorff topological spaces, and if $\diamond : C(X) \times C(Y) \to C(Z)$ is a bilinear mapping satisfying $\|f \diamond g\| = \|f\|\|g\|$ for every $(f, g) \in C(X) \times C(Y)$, then there exist a closed subset Z_0 of Z, a continuous surjective mapping $\varphi : Z_0 \to X \times Y$, and a norm-one element $\alpha \in C(Z)$ satisfying $|\alpha(z)| = 1$ and*

$$(f \diamond g)(z) = \alpha(z)f(\pi_X(\varphi(z))g(\pi_Y(\varphi(z))$$

for every $(z, f, g) \in Z_0 \times C(X) \times C(Y)$, where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ stand for the natural coordinate projections. We note that Holsztynski's original theorem follows from the new bilinear version by taking the space Y reduced to a point.

We looked for the main result just reviewed in the attempt to determine those compact Hausdorff topological spaces X such that the Banach space $C(X)$ is "absolute-valuable". That a Banach space E is absolute-valuable means that there exists a bilinear mapping $\diamond : E \times E \to E$ satisfying

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$\|\xi \diamond \chi\| = \|\xi\|\|\chi\|$ for all $\xi, \chi \in E$. The reader is referred to [**1**] for a view of the present status of the theory of such spaces. We derive from the main result that, if X is a compact Hausdorff topological space such that $C(X)$ is absolute-valuable, then X must be either reduced to a point or not scattered. Thus we rediscover the fact, first proved in [**1**], that $C(X)$ is not absolute-valuable when we take X equal to the one-point compactification of any infinite discrete space. We also deduce that, in the case that the compact space X is metrizable and not reduced to a point, the Banach space $C(X)$ is absolute-valuable if and only if X is uncountable.

2. The main result

Throughout this paper $\mathbb{K}$ will denote the field of real or complex numbers. The field $\mathbb{K}$ will remain fixed, and, for a compact Hausdorff topological space X, $C(X)$ will stand for for the Banach space over $\mathbb{K}$ of all $\mathbb{K}$-valued continuous functions on X. That

Theorem 2.1. *Let X, Y, Z be compact Hausdorff topological spaces, and let $\diamond : C(X) \times C(Y) \to C(Z)$ be a bilinear mapping satisfying*

$$\|f \diamond g\| = \|f\|\|g\|$$

for every $(f, g) \in C(X) \times C(Y)$. Then there exist a closed subset Z_0 of Z, a continuous surjective mapping $\varphi : Z_0 \to X \times Y$, and a norm-one element $\alpha \in C(Z)$ satisfying $|\alpha(z)| = 1$ and

$$(f \diamond g)(z) = \alpha(z) f(\pi_X(\varphi(z)) g(\pi_Y(\varphi(z))$$

for every $(z, f, g) \in Z_0 \times C(X) \times C(Y)$. Here, $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the natural coordinate projections.

Proof. Given a compact Hausdorff topological space K, we denote by $\mathbf{1}_K$ the constant function equal to 1 on K, and, for k in K, we put

$$S_k := \{f \in C(K) : \|f\| = 1 = |f(k)|\}.$$

Given compact Hausdorff topological spaces K and L, an element k of K, and a linear isometry $T : C(K) \to C(L)$, we put

$$Q_k^T := \{l \in L : T(S_k) \subseteq S_l\}.$$

We will apply several times the following result, proved by W. Holsztynski [**3**]:

(*) *If l is in Q_k^T, then we have $T(f)(l) = T(\mathbf{1}_K)(l) f(k)$ for every $f \in C(K)$.*

Now, for $(x, y) \in X \times Y$ we define

$$Q_{x,y} := \{z \in Z : S_x \diamond S_y \subseteq S_z\},$$

and organize the proof in several steeps.

Steep (i).- If $(x, y) \in X \times Y$ and $(f, g) \in C(X) \times C(Y)$ are such that $f(x) = 0$ or $g(y) = 0$, then we have $(f \diamond g)(z) = 0$ for every $z \in Q_{x,y}$. Let us fix (x, y, g) in $X \times Y \times C(Y)$ with $g \in S_y$, consider the linear isometry

$T : C(X) \to C(Z)$ defined by $T(h) := h \diamond g$, and note that $Q_{x,y} \subseteq Q_x^T$. Assume that $f \in C(X)$ satisfies $f(x) = 0$. Then, by $(*)$, we have

$$(f \diamond g)(z) = (T(f))(z) = 0$$

for every $z \in Q_x^T$, and in particular $(f \diamond g)(z) = 0$ for every $z \in Q_{x,y}$. Now, the restriction that g lies in S_Y can be removed by keeping in mind that, since $S_y S_y \subseteq S_y$, the linear hull of S_y in $C(Y)$ is a subalgebra of $C(Y)$, which is self-adjoint, contains the constants, and separates the points of Y, so that the Stone-Weierstrass theorem applies.

Steep (ii).- If (x, y) and (x', y') are in $X \times Y$ with $(x, y) \neq (x', y')$, then $Q_{x,y} \cap Q_{x',y'} = \emptyset$. Let $x, x' \in X$ and $y, y' \in Y$ be such that $x \neq x'$, and assume that there exists $z \in Q_{x,y} \cap Q_{x',y'}$. Then, taking $(f, g) \in S_x \times S_y$ with $f(x') = 0$, we have $|(f \diamond g)(z)| = 1$ (by the definition of $Q_{x,y}$) and $(f \diamond g)(z) = 0$ (by Steep (i)), a contradiction.

Steep (iii).- For every $(x, y) \in X \times Y$ we have $Q_{x,y} \neq \emptyset$. Let (x, y) be in $X \times Y$, and let $f_1, ..., f_n$ and $g_1, ..., g_n$ be in S_x and S_y, respectively. Putting $F := \sum_{i=1}^n \overline{f_i(x)} f_i$ and $G := \sum_{i=1}^n \overline{g_i(y)} g_i$, we have $|F(x)| = n = \|F\|$, $|G(y)| = n = \|G\|$, and hence $\|F \diamond G\| = n^2$. Therefore there exists $z \in Z$ satisfying

$$n^2 = |(F \diamond G)(z)| = \Big| \sum_{i,j=1}^n \overline{f_i(x) g_j(y)} (f_i \diamond g_j)(z) \Big|.$$

This implies $|(f_i \diamond g_j)(z)| = 1$ for all $i, j = 1, ..., n$. In this way we have shown that, denoting by $\mathbb{T}$ the unit sphere of $\mathbb{K}$, the family

$$\{(f \diamond g)^{-1}(\mathbb{T}) : (f, g) \in S_x \times S_y\}$$

has the finite intersection property. By the compactness of Z, we have in fact $Q_{x,y} = \bigcap_{(f,g) \in S_x \times S_y} (f \diamond g)^{-1}(\mathbb{T}) \neq \emptyset$.

Now, we consider the norm-one element α of $C(Z)$ defined by

$$\alpha(z) := (\mathbf{1}_X \diamond \mathbf{1}_Y)(z).$$

Steep (iv).- For (x, y, z) in $X \times Y \times Z$ with $z \in Q_{x,y}$, we have

$$(f \diamond g)(z) = \alpha(z) f(x) g(y)$$

for every $(f, g) \in C(X) \times C(Y)$, and $|\alpha(z)| = 1$. Since $(\mathbf{1}_X, \mathbf{1}_Y)$ belongs to $S_x \times S_y$ whenever (x, y) is in $X \times Y$, it follows from the definitions of α and $Q_{x,y}$ that $|\alpha(z)| = 1$ whenever z is in $Q_{x,y}$. Now, let us fix (x, y, f, g) in $X \times Y \times C(X) \times C(Y)$ with $g \in S_y$, and consider the linear isometries $T : C(X) \to C(Z)$ and $R : C(Y) \to C(Z)$ defined by $T(h) := h \diamond g$ and $R(h) := \mathbf{1}_X \diamond h$, respectively. Keeping in mind the inclusion $Q_{x,y} \subseteq Q_x^T \cap Q_y^R$, and applying $(*)$, for $z \in Q_{x,y}$ we derive

$$\begin{aligned}(f \diamond g)(z) &= T(f)(z) = T(\mathbf{1}_X)(z) f(x) = (\mathbf{1}_X \diamond g)(z) f(x) = R(g)(z) f(x) \\ &= R(\mathbf{1}_Y)(z) g(y) f(x) = (\mathbf{1}_X \diamond \mathbf{1}_Y)(z) f(x) g(y) = \alpha(z) f(x) g(y).\end{aligned}$$

The restriction that g lies in S_y can be removed by arguing as in the conclusion of the proof of Steep (i).

Now, we define $Z_0 := \bigcup_{(x,y)\in X\times Y} Q_{x,y}$. In view of Steep (ii), *for every* $z \in Z_0$ *there exists a unique* $\varphi(z) \in X \times Y$ *such that* z *belongs to* $Q_{\pi_X(\varphi(z)),\pi_Y(\varphi(z))}$. Moreover, by Steep (iii), *the mapping* $\varphi : Z_0 \to X \times Y$ *defined in this way is surjective.* On the other hand, by Steep (iv), *the norm-one element* $\alpha \in C(Z)$ *satisfies* $|\alpha(z)| = 1$ *whenever* z *lies in* Z_0*, and the equality*

$$(f \diamond g)(z) = \alpha(z) f(\pi_X(\varphi(z)) g(\pi_Y(\varphi(z)) \tag{2.1}$$

holds for every $(z, f, g) \in Z_0 \times C(X) \times C(Y)$. Thus, to conclude the proof of the theorem it is enough to establish the following.

Steep (v).- Z_0 is closed in Z, and the mapping $\varphi : Z_0 \to X \times Y$ is continuous. Let A be a closed subset of $X \times Y$, let z_0 be in $Z \setminus \varphi^{-1}(A)$, and let $a = (x, y)$ be in A. Since $\varphi^{-1}(A) = \bigcup_{(x,y)\in A} Q_{x,y}$, there exists $(f_a, g_a) \in S_x \times S_y$ such that $\varepsilon_a := \frac{1-|(f_a\diamond g_a)(z_0)|}{2} > 0$. Now, we consider the open subset U_a of $X \times Y$ given by

$$U_a := \{(x', y') \in X \times Y : |f_a(x')g_a(y')| > 1 - \varepsilon_a\},$$

and the disjoint open subsets V_a and G_a of Z defined by

$$V_a := \{z \in Z : |(f_a \diamond g_a)(z)| > 1 - \varepsilon_a\}$$

and

$$G_a := \{z \in Z : |(f_a \diamond g_a)(z)| < 1 - \varepsilon_a\}.$$

We claim that $\varphi^{-1}(U_a) \subseteq V_a$. Indeed, if z is in Z_0 with $\varphi(z) = (x', y') \in U_a$, then, by the sentence containing equality (2.1), and the definition of U_a, we have

$$|(f_a \diamond g_a)(z)| = |f_a(x')g_a(y')| > 1 - \varepsilon_a,$$

which means that z lies in V_a, as claimed. On the other hand, since clearly a lies in U_a, we can move a in A, and apply the compactness of A to find $a_1, ..., a_n \in A$ such that $A \subseteq \bigcup_{i=1}^n U_{a_i}$. Then, invoking the claim, we derive $\varphi^{-1}(A) \subseteq \bigcup_{i=1}^n V_{a_i}$, and hence

$$(\cap_{i=1}^n G_{a_i}) \cap \varphi^{-1}(A) \subseteq (\cap_{i=1}^n G_{a_i}) \cap (\cup_{i=1}^n V_{a_i}) = \emptyset.$$

In this way $\bigcap_{i=1}^n G_{a_i}$ becomes a neighbourhood of z_0 in Z contained in $Z \setminus \varphi^{-1}(A)$. Since z_0 is an arbitrary element of $Z \setminus \varphi^{-1}(A)$, we realize that $\varphi^{-1}(A)$ is closed in Z. Finally, Since A is an arbitrary closed subset of $X \times Y$ we obtain that Z_0 is closed in Z (by noticing that $Z_0 = \varphi^{-1}(X \times Y)$) and that φ is continuous. ∎

Taking in Theorem 2.1 the space Y reduced to a point, we immediately get the following.

Corollary 2.2. [**3**] *Let X, Z be compact Hausdorff topological spaces, and let $T : C(X) \to C(Z)$ be a linear isometry. Then there exist a closed* *subset Z_0 of Z, a continuous surjective mapping $\varphi : Z_0 \to X$, and a norm-one element $\alpha \in C(Z)$ satisfying $|\alpha(z)| = 1$ and*

$$T(f)(z) = \alpha(z)f(\varphi(z))$$

for every $(z, f) \in Z_0 \times C(X)$.

Corollary 2.3. *For compact Hausdorff topological spaces X, Z, consider the following conditions:*

(1) *There exists a continuous surjective mapping from Z to X.*
(2) *$C(X)$ is linearly isometric to a subspace of $C(Z)$.*
(3) *There exists a continuous surjective mapping from some closed subset of Z to X.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Moreover, if Z is metrizable, then (2) $\iff$ (3).

Proof. (1) $\Rightarrow$ (2).- If there exists a continuous surjective mapping $\theta : Z \to X$, then the mapping $h \to h \circ \theta$ from $C(X)$ to $C(Z)$ is a linear isometry.

(2) $\Rightarrow$ (3).- By Corollary 2.2.

If Z_0 is any metrizable closed subset of Z, then $C(Z_0)$ is linearly isometric to a subspace of $C(Z)$ (indeed, by the Borsuk-Kakutani theorem [**2**, Theorem 1.21], there is in fact a norm-one linear operator $S : C(Z_0) \to C(Z)$ satisfying $S(f)_{|Z_0} = f$ for every $f \in C(Z_0)$). Now, assume that Z is metrizable, and that there exists a continuous surjective mapping from a closed subset Z_0 of Z to X. Then, since $C(X)$ is linearly isometric to a subspace of $C(Z_0)$ (by (1) $\Rightarrow$ (2)), it follows that $C(X)$ is linearly isometric to a subspace of $C(Z)$. ■

Remark 2.4. (a) Even if Z is metrizable, the implication (1) $\Rightarrow$ (2) in Corollary 2.3 above is not reversible. Many counterexamples can be exhibited by keeping in mind the Banach-Mazur theorem that $C(X)$ is linearly isometric to $C(Z)$ whenever the compact spaces X and Z are metrizable and Z is uncountable. Thus, putting $X := \{0, 1\}$ and $Z := [0, 1]$, Condition (2) in Corollary 2.3 is fulfilled, whereas clearly Condition (1) does not hold. In this case, an elementary embedding $C(X) \hookrightarrow C(Z)$ is the one assigning to each function from $\{0, 1\}$ to $\mathbb{K}$ its unique affine extension to $[0, 1]$.

(b) Without the assumption of metrizability of Z, the implication (2) $\Rightarrow$ (3) in Corollary 2.3 is also not reversible. Indeed, taking $Z := \beta(\mathbb{N})$ (the Stone-Čech compactification of the integers) and $X := \beta(\mathbb{N}) \setminus \mathbb{N}$, Condition (3) is fulfilled in an obvious way, but Condition (2) does not hold. Indeed, the norm of $C(Z)$ is determined by the family of all point evaluations on the set $\mathbb{N}$, whereas the norm of $C(X)$ cannot be determined by any countable subset of the closed unit ball of its dual (see the second paragraph after Proposition II.4.16 of [**4**]).

Corollary 2.5. *For compact Hausdorff topological spaces X, Y, Z, consider the following conditions:*

(1) *There exists a continuous surjective mapping from Z to $X \times Y$.*
(2) $C(X \times Y)$ *is linearly isometric to a subspace of* $C(Z)$.
(3) *There exists a bilinear mapping* $\diamond : C(X) \times C(Y) \to C(Z)$ *satisfying* $\|f \diamond g\| = \|f\|\|g\|$ *for every* $(f, g) \in C(X) \times C(Y)$.
(4) *There exists a continuous surjective mapping from some closed subset of Z to $X \times Y$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *Moreover, if Z is metrizable, then in fact* (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4).

Proof. (1) $\Rightarrow$ (2).- By Corollary 2.3.

(2) $\Rightarrow$ (3).- For $(f, g) \in C(X) \times C(Y)$, define $f \otimes g \in C(X \times Y)$ by $(f \otimes g)(x, y) := f(x)g(y)$. If there exists a linear isometry

$$\phi : C(X \times Y) \to C(Z),$$

then the mapping $\diamond : C(X) \times C(Y) \to C(Z)$ defined by $f \diamond g := \phi(f \otimes g)$ is bilinear and satisfies $\|f \diamond g\| = \|f\|\|g\|$ for every $(f, g) \in C(X) \times C(Y)$.

(3) $\Rightarrow$ (4).- By Theorem 2.1.

In the case that Z is metrizable, the implication (4) $\Rightarrow$ (2) follows from Corollary 2.3. ∎

Remark 2.6. We note that, when in Corollary 2.5 above we take Y reduced to a point, then Conditions (2) and (3) assert the same, and Corollary 2.5 becomes Corollary 2.3. Therefore, by Remark 2.4, none of the implications (1) $\Rightarrow$ (2) (even if Z is metrizable) and (3) $\Rightarrow$ (4) in Corollary 2.5 is reversible.

A more illuminating example that (4) does not imply (3) is the following. Take $Z := \beta(\mathbb{N}) \times \beta(\mathbb{N})$ and $X = Y := \beta(\mathbb{N}) \setminus \mathbb{N}$. Then, since $X \times Y$ is a closed subset of Z, Condition (4) is fulfilled in an obvious way. However, if Condition (3) were satisfied, then, fixing a norm-one element g of $C(Y)$, the mapping $f \to f \diamond g$ from $C(X)$ to $C(Z)$ would be a linear isometry. But, since the norm of $C(Z)$ is determined by the countable family of all point evaluations on the set $\mathbb{N} \times \mathbb{N}$ (because the inclusion $\mathbb{N} \times \mathbb{N} \hookrightarrow \beta(\mathbb{N}) \times \beta(\mathbb{N})$ extends to a continuous surjective mapping from $\beta(\mathbb{N} \times \mathbb{N})$ to $\beta(\mathbb{N}) \times \beta(\mathbb{N})$), the existence of such an isometry is impossible (see Remark 2.4.(b)).

Without the assumption of metrizability of Z, we do not know if the implication (2) $\Rightarrow$ (3) in Corollary 2.5 is reversible.

3. Absolute-valuable $C(X)$-spaces

A Banach space E is said to be **absolute-valuable** if there exists a bilinear mapping $\diamond : E \times E \to E$ satisfying $\|\xi \diamond \chi\| = \|\xi\|\|\chi\|$ for all $\xi, \chi \in E$. Let X be a metrizable compact space. It follows from Corollary 2.5 that $C(X)$ is absolute-valuable if and only if there exists a continuous surjective mapping from some closed subset of X to $X \times X$. In this section we will prove that in fact the absolute valuableness of $C(X)$ can be settled in terms of the cardinality of X. To this end, we need some elementary lemmas of pure topology. We feel that such lemmas are well-known, but we give their proofs for the sake of completeness. As usual, for every topological space X, we define the derived set X' of X as the set of all accumulation points of X.

LEMMA 3.1. *Let X and Y be topological spaces, and let $\varphi : X \to Y$ be a continuous surjective mapping. Assume that X is compact and that Y is Hausdorff. Then $Y' \subseteq \varphi(X')$.*

PROOF. For every point z in a topological space, we denote by $\mathcal{V}(z)$ the set of all neighbourhoods of z. Let y be in Y'. Then, since φ is surjective, for every $V \in \mathcal{V}(y)$ there exists $x_V \in X$ such that $\varphi(x_V) \in V \setminus \{y\}$. Considering in $\mathcal{V}(y)$ the order given by the inverse inclusion, the compactness of X provides us with a cluster point $x \in X$ of the net $\{x_V\}_{V\in\mathcal{V}(y)}$. Since φ is continuous, $\varphi(x)$ is a cluster point of the net $\{\varphi(x_V)\}_{V\in\mathcal{V}(y)}$. Since clearly $\{\varphi(x_V)\}_{V\in\mathcal{V}(y)}$ converges to y, and Y is Hausdorff, it follows that $\varphi(x) = y$. Since x is different from x_V for every $V \in \mathcal{V}(y)$, and is a cluster point of the net $\{x_V\}_{V\in\mathcal{V}(y)}$, it lies in X'. ∎

LEMMA 3.2. *Let X and Y be topological spaces, let $\varphi : X \to Y$ be a continuous mapping, and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a decreasing net of closed subsets of X. Assume that X is compact and that Y is Hausdorff. Then we have $\bigcap_{\lambda\in\Lambda} \varphi(X_\lambda) = \varphi(\bigcap_{\lambda\in\Lambda} X_\lambda)$.*

PROOF. Let y be in $\bigcap_{\lambda\in\Lambda} \varphi(X_\lambda)$. Then, for $\lambda \in \Lambda$ there exists $x_\lambda \in X_\lambda$ with $\varphi(x_\lambda) = y$. Taking a cluster point x of the net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X, and keeping in mind that $\{X_\lambda\}_{\lambda\in\Lambda}$ is a decreasing net of closed subsets of X, we obtain that x belongs to $\bigcap_{\lambda\in\Lambda} X_\lambda$. Since $\varphi(x) = y$, it follows that y lies in $\varphi(\bigcap_{\lambda\in\Lambda} X_\lambda)$. ∎

Given a topological space X and an ordinal α, we apply transfinite induction to define the α-derived set $X^{(\alpha)}$ of X. Indeed, we put $X^{(0)} := X$, $X^{(\alpha+1)} := (X^{(\alpha)})'$, and $X^{(\alpha)} := \bigcap_{\beta<\alpha} X^{(\beta)}$ when α is a limit ordinal.

LEMMA 3.3. *Let X and Y be topological spaces, let $\varphi : X \to Y$ be a continuous surjective mapping, and let α be an ordinal. Assume that X is compact and that Y is Hausdorff. Then $Y^{(\alpha)} \subseteq \varphi(X^{(\alpha)})$.*

PROOF. We argue by transfinite induction on α. The case $\alpha = 0$ is clear.

Assume that the inclusion $Y^{(\alpha)} \subseteq \varphi(X^{(\alpha)})$ is true for some ordinal α. Then, putting $Z := \varphi^{-1}(Y^{(\alpha)}) \bigcap X^{(\alpha)}$ and $\psi := \varphi_{|Z} : Z \to Y^{(\alpha)}$, we can apply Lemma 3.1, with $(Z, Y^{(\alpha)}, \psi)$ instead of (X, Y, φ), to derive that $Y^{(\alpha+1)} \subseteq \varphi(Z')$. Since $Z \subseteq X^{(\alpha)}$, we obtain $Y^{(\alpha+1)} \subseteq \varphi(X^{(\alpha+1)})$.

Now assume that α is a limit ordinal, and that the inclusion $Y^{(\beta)} \subseteq \varphi(X^{(\beta)})$ holds for every ordinal $\beta < \alpha$. Applying Lemma 3.2 we have $Y^{(\alpha)} = \bigcap_{\beta<\alpha} Y^{(\beta)} \subseteq \bigcap_{\beta<\alpha} \varphi(X^{(\beta)}) = \varphi(\bigcap_{\beta<\alpha} X^{(\beta)}) = \varphi(X^{(\alpha)})$. ∎

Lemma 3.4. *Let X and Y be topological spaces, and let α be an ordinal. Then $X^{(\alpha)} \times Y \subseteq (X \times Y)^{(\alpha)}$.*

Proof. Straightforward by transfinite induction on α. ■

We recall that a topological space X is said to be scattered if for every nonempty closed subset Y of X we have $Y \setminus Y' \neq \emptyset$.

Theorem 3.5. *For a compact Hausdorff topological space X, consider the following conditions:*

(1) *There exists a continuous surjective mapping from X to $X \times X$.*
(2) *$C(X \times X)$ is linearly isometric to a subspace of $C(X)$.*
(3) *$C(X)$ is absolute-valuable.*
(4) *There exists a continuous surjective mapping from some closed subset of X to $X \times X$.*
(5) *X is either reduced to a point or non scattered.*
(6) *X is either reduced to a point or uncountable.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$. *Moreover, if X is metrizable, then* $(2) \iff (3) \iff (4) \iff (5) \iff (6)$.

Proof. $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.- By Corollary 2.5.

$(4) \Rightarrow (5)$.- Assume that X is scattered, and that there exist a closed subset X_0 of X and a continuous surjective mapping $\varphi : X_0 \to X \times X$. Since X is scattered and compact, there is an ordinal α such that $X^{(\alpha)}$ is finite and nonempty (see for example [**5**, 8.6.8]). Denote by n and m the cardinal numbers of $X^{(\alpha)}$ and X, respectively. Since $X^{(\alpha)} \times X \subseteq \varphi(X_0^{(\alpha)})$ (by Lemmas 3.3 and 3.4), we have $nm \leq n$. This implies $m = 1$.

$(5) \Rightarrow (6)$.- Since countable compact Hausdorff spaces are scattered.

If X is uncountable and metrizable, then, by the Banach-Mazur theorem (see Remark 2.4.(a)), $C(X \times X)$ is linearly isometric to a subspace of $C(X)$. ■

Remark 3.6. Even if X is metrizable, the implication $(1) \Rightarrow (2)$ in Theorem 3.5 is not reversible. Indeed, taking $X = [0, 1] \cup \{2\}$, Condition (2) is satisfied (by the Banach-Mazur theorem), whereas a connectedness argument shows that Condition (1) is not fulfilled.

Without the assumption of metrizability of X, the implication $(5) \Rightarrow (6)$ is also not reversible. Indeed, if X denotes the one-point compactification of an uncountable discrete space, then X is scattered.

Without the assumption of metrizability of X, we do not know about the reversibility of any of the implications $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$.

Given an infinite set Γ, we denote by $c(\Gamma)$ the vector space over $\mathbb{K}$ of all functions from Γ to $\mathbb{K}$ having a limit along the filter of all co-finite subsets of Γ, endowed with the sup norm. Since $c(\Gamma)$ is linearly isometric to the space of all $\mathbb{K}$-valued continuous functions on the scattered compact

Hausdorff topological space consisting of the one-point compactification of the discrete space Γ, we derive from Theorem 3.5 the following.

COROLLARY 3.7. [**1**] *Let* Γ *be an infinite set. Then* $c(\Gamma)$ *is not absolute-valuable.*

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