The Hausdorff entropic moment problem

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Our aim in this paper is twofold. First, to find the necessary and sufficient conditions to be satisfied by a given sequence of real numbers $\{\omega_n\}_{n=0}^{\infty}$ to represent the ''entropic moments'' $\int_{[0,a]}[\rho(x)]^n dx$ of an unknown non-negative, decreasing and differentiable (*a.e.*) density function $\rho(x)$ with a finite interval support. These moments are called entropic moments because they are closely connected with various information entropies (Renyi, Tsallis, ...). Second, we outline an efficient method for the reconstruction of the density function from the knowledge of its first N entropic moments.
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I. INTRODUCTION

The problem of moments[1,2] asks when a given sequence of complex numbers may be represented as the moments around the origin of a non-negative measure, defined on the line (Hamburger), on a half-line (Stieltjes), on a finite interval (Hausdorff) or on the unit circumference (the trigonometric moment problem).

This is a classical topic in analysis which has illuminated an extraordinary number of scientific subjects from both standpoints, theoretical and applied. Indeed, it has facilitated many developments[3] in function theory, in functional analysis, in spectral representations of operators, in Fourier analysis as well as in probability and statistics. Also, it has numerous applications not only in approximation theory, in numerical mathematics and for the prediction of stochastic processes, but also in linear prediction, in inverse scattering, in digital filtering and for the determination of rigorous relationships among physical quantities of many-particle systems within the framework of the density functional theory as well as in the design of algorithms for simulating physical systems. The latter should not surprise anybody since the own terminology ''problem of moments'' was taken by Stieltjes from Mechanics. Moreover, he used very often physical concepts (mass, stability, electrostatic properties, ...) in solving analytical problems.[1,4]

In this paper, we shall focus our attention on the problem of entropic moments, which differs from the ordinary moment problem above mentioned in that it does not consider the moments-around-the-origin of a density function $\rho(x)$ defined by

$$\mu_n = \int_K x^n \rho(x) dx, \tag{1}$$

but the quantities

$$\omega_n = \int_K [\rho(x)]^n dx, \tag{2}$$

which are called frequency moments of $\rho(x)$, $x \in K$, in probability and statistics.[5–9] The study of these quantities was initiated by Yule following a suggestion of Pearson. Then Sichel[6,7] usefully employed them for the fitting of certain frequency curves. It happens that estimators based on

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frequency moments are, at times, much better than the ordinary moment estimates. Moreover, the frequency moments are fairly efficient in the range where the ordinary moments are very inefficient.[10] This is so in some cases where the range K is unlimited and the density is poorly known.[8]

It is interesting to remark that the frequency moments ω_n are location independent when $K=\mathbb{R}$ (Hamburger case); that is, two densities differing only in location have identical frequency moments. In these cases, the location parameter can be provided by the mode, the median or any other appropriate quantity.[9]

We shall call the quantities ω_n the ''entropic'' moments of the density function $\rho(x)$, because they are closely connected to the so-called Renyi and Tsallis entropies of $\rho(x)$ defined[11,12] by

$$S_q^R := \frac{1}{1-q} \ln \int_K [\rho(x)]^q \, dx; \quad q>0, \quad q\neq 1, \quad \int_K \rho(x) dx = 1, \tag{3}$$

and

$$S_q^T := \frac{1}{q-1} \left[1 - \int_K [\rho(x)]^q \, dx \right]; \quad q>0, \quad q\neq 1, \quad \int_K \rho(x) dx = 1, \tag{4}$$

respectively. The entropic adjective allows us to identify more appropriately the moments ω_n from the other type of moments[8] (moments around the origin, central moments, factorial moments, absolute moments, ...) of a frequency distribution.

In addition, the entropic moments ω_α have various physical meanings depending on the nature of the associated density function ρ (charge density, momentum density, ...). Indeed, they characterize some density functionals which describe certain physical quantities of fundamental and/or experimentally accessible character such as, up to a constant factor, the Thomas–Fermi kinetic energy ($\omega_{5/3}$), the Dirac exchange energy ($\omega_{4/3}$) and the electron average density (ω_2) of the many-electron systems; see, e.g., Ref. 24.

This paper has a twofold aim. First we solve the Hausdorff entropic moment problem in Sec. II, which allows us to characterize a density function by means of its entropic moments. Then, in Sec. III, we describe a practical procedure to reconstruct the density from its entropic moments.

II. THE HAUSDORFF ENTROPIC MOMENT PROBLEM

Let $K=[0,a]$ with $a>0$, and $\mathcal{M}(K)$ the set of real density functions $f(x)$ bounded on K and such that $f(0)=1$ and $f(a)=0$. We have obtained the following result for this set of functions.

Theorem 1: *The necessary and sufficient conditions which the given sequence of positive numbers* $\omega_0, \omega_1, \ldots, \omega_n, \ldots$, *must satisfy in order that a positive, decreasing and differentiable (a.e.) density function* $f(x), x\in K$, *having these entropic moments (2) may exist, are given by*

$$\Sigma^k \frac{\omega_{m+1}}{m+1} \geqslant 0 \quad \text{and} \quad \Sigma^k \omega_m \geqslant 0, \tag{5}$$

for $k,m=0,1,2,\ldots$, *and being*

$$\Sigma^k \omega_m = \omega_m - \binom{k}{1} \omega_{m+1} + \ldots + (-1)^k \omega_{m+k}.$$

Proof: Let us first prove the sufficiency condition. For convenience we adopt the notations

$$\mu_m \equiv \frac{\omega_{m+1}}{m+1}, \quad \nu_m \equiv \omega_m; \quad m=0,1,2,\ldots,$$

so that $\nu_m = m\mu_{m-1}$ for $m=1,2,\dots$, and $\mu_{-1}\equiv a$. If conditions (5) are fulfilled, the Hausdorff theorem for the ordinary moment problem on the interval $[0,1]$ allows us to state[1] that

$$\exists ! z(t) \geqslant 0 \quad \text{on } [0,1], \quad \text{such that } \int_0^1 t^m z(t)\,dt = \nu_m$$

and

$$\exists ! g(t) \geqslant 0 \quad \text{on } [0,1], \quad \text{such that } \int_0^1 t^m g(t)\,dt = \mu_m .$$

On the other hand, let us define

$$h(t) = \int_t^1 z(s)\,ds, \quad t \in [0,1].$$

So, $h'(t) = -z(t)$. Moreover, $h(t)$ has the same ordinary moments as $g(t)$; then, they are equal. Thus $g(t)$ is a decreasing function since $g'(t) = -z(t)$. We can define $f(x)$ as its inverse with $x \in [g(1)=0, g(0)=a]$, which will be positive, decreasing and differentiable (*a.e.*). One should realize that in case that $g(t)$ is a constant $c>0$ on some subintervals, this would provoke a jump discontinuity for $f(x)$ in $x=c$ and *vice versa*. Then, it is straightforward to obtain that

$$\int_0^a [f(x)]^m\,dx = \nu_m = \omega_m ; \quad m = 0,1,2,\dots \quad .$$

To prove necessity, we define the inverse of $f(x)$ as $h(t)$, $t \in [0,1]$, which is decreasing and differentiable (*a.e.*). A simple change of variable $t = f(x)$ allows us to find the following relationship between the entropic moments of $f(x)$ and the ordinary moments of $h(t)$:

$$m\int_0^a t^m h(t)\,dt = \frac{\omega_{m+1}}{m+1}, \quad m = 0,1,2,\dots \quad .$$

Now we consider the function $z(t) = -h'(t)$, $t \in [0,1]$, and we realize that its ordinary moments are given by ω_m. Then, the direct application of the classical Hausdorff moment above mentioned leads us to the relations (5). □

III. DENSITY RECONSTRUCTION

Associated to any moment problem there exists an inverse problem, namely that of the reconstruction of the corresponding density function. Moreover, in practical purposes we have at our disposal only a finite number of moments. The inverse Hausdorff (ordinary) moment problem (1), that is the determination of the density $\rho(x)$ from the moments around the origin $\{\mu_n\}_{n=0}^{\infty}$, was first proposed by Pafnuty Chebyshev.[13] It is a severely ill-conditioned problem because of the lack of *a priori* information and the large involved numerical instabilities.[14–19] To avoid these instabilities, various regularization methods (Tikhonov, maximum-entropy methods, orthogonal-polynomials based methods,...) have been proposed; see Ref. 16 for a brief survey. The maximum-entropy method has been widely and efficiently used for scientific applications.[16,20–22] It consists in maximizing an entropic functional, and it allows us to find a density estimate which converges to the solution of the problem when the number of the involved moments increases.

Here we shall use a maximum entropy method to solve the inverse Hausdorff entropic moment problem discussed in the previous section when the number of known entropic moments is finite. Based on the proof of Theorem 1, this method first computes the maximum-entropy estimate to the solution $z(t)$ of the inverse Hausdorff problem related to the sequence $\{\mu_n\}_{n=0}^{\infty}$ with $\mu_n \equiv \omega_{n+1}/n+1$. Then, the inverse of the estimated $z(t)$ is the desired approximated solution of

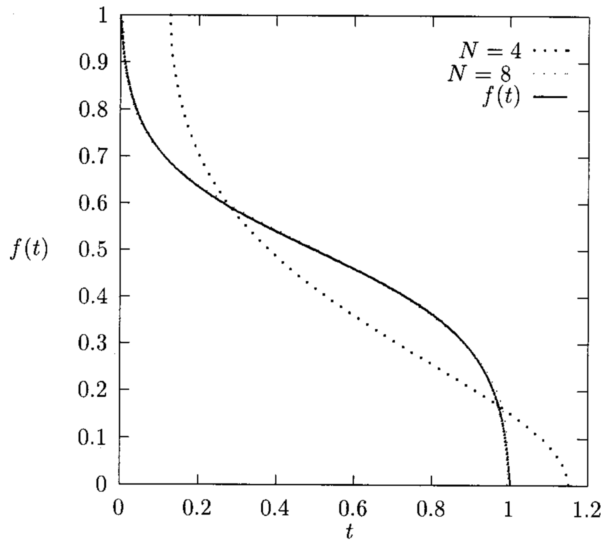

FIG. 1. Graphical representation of the function $f(t)=\frac{1}{2}+\frac{1}{10}\ln[1/(At+B)-1]$ and its estimates from the entropic moments ω_n, $n=0,1,\ldots,N$ with $N=4$ and $N=8$.

our problem. Let us notice that, although we know that the asymptotic ($N\to\infty$) approach to $z(t)$ is inversible, the different Nth estimates to $z(t)$ may not have this property. In the case that there is not any inversible approach, our method is not applicable.

Although we may use any entropic functional to be maximized, we have chosen the Fisher information measure defined by

$$E_f \equiv \int_{[0,a]} \frac{[f'(x)]^2}{f(x)} dx,$$

if $f(x)>0$, and $E_f=0$ if $f(x)\equiv 0$. Contrary to other entropic functionals (e.g., Boltzmann–Shannon information entropy, Burg entropy, positive L^2 entropy), this choice has the advantage of taking into account information from the derivative of the function, what is expected to have a strong smoothing effect on the estimate. In doing so we follow the operation lines of Borwein, Limber, and Noll[23] to which we refer for further details.

To illustrate the method and the rate of convergence of the Fisher-information estimates for a function $f(t)$ from its first $N+1$ entropic moments $\omega_n=\int_0^1[f(t)]^n dt$, $n=0,1,\ldots,N$, we have represented in Fig. 1 the exact values and the Fisher estimates for the cases $N=4$ and $N=8$ of a specific function, namely,

$$f(t)=\frac{1}{2}+\frac{1}{10}\ln\left(\frac{1}{At+B}-1\right), \quad \text{with } A=\frac{1}{1+e^5} \quad \text{and } B=\frac{1}{1+e^{-5}}-\frac{1}{1+e^5}. \tag{6}$$

We visually notice in the figure the fast convergence of the method for this function as well as the good precision reached with nine entropic moments.

Finally we show in Fig. 2 the reconstruction of the function $f(t)$ given by (6) from the first $N+1$ moments around the origin $\mu_n=\int_0^1 f(t)t^n\, dt$ in the cases $N=4$ and 8. The comparison of the two figures for the two corresponding Nth cases illustrates that there are functions that may be better estimated or reconstructed from the entropic moments (2) that from the ordinary moments (1). Needless to say that there exist other functions where the reciprocal situation occurs; consider, for example, the inverse of the function $f(t)$ given by Eq. (6).

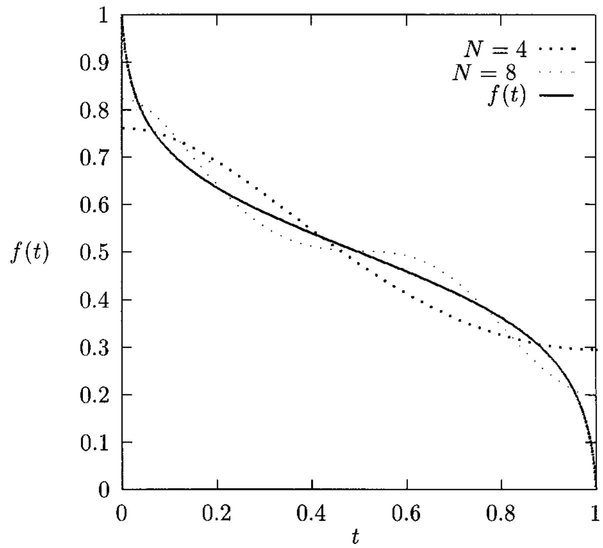

FIG. 2. Graphical representation of the function $f(t)=\frac{1}{2}+\frac{1}{10}\ln[1/(At+B)-1]$ and its estimates from the moments μ_n, $n=0,1,\ldots,N$ with $N=4$ and $N=8$.

IV. CONCLUSIONS

In this paper we have posed the entropic moment problem, whose elements are information measures of an unknown density function. Physically, the entropic moments may also describe some fundamental and/or experimentally accessible quantities of quantum-mechanical systems as already pointed out. Then, we have solved the Hausdorff entropic moment problem by use of some specific properties of the inverse function of the density according to the lines of a recent work of the authors.[24] Moreover, our strategy has let us outline a maximum-entropy method based on an algorithm of minimization of the Fisher information measure[23] which allows one to solve the inverse finite Hausdorff entropic moment problem; that is, to determine the density function from its first few entropic moments. We realize that other density reconstruction procedures which do not include the previous determination of the inverse density function (which would avoid the requirement of decreasing behavior for the density) would be desirable.

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