2&3.An Overview of General Equilibrium Theory

Ana-Isabel Guerra
Department of International Economics
Faculty of Economics and Business Administration
Universidad de Granada
Spain
International Input-Output School. Lisbon_2014

2&3. An Overview of General Equilibrium Theory

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-2. Competitive Equilibrium: Introduction

Why economists like so much competitive market?
Typology of Competitive Markets:
Partial Equilibrium Analysis vs General Equilibrium Analysis.

-3. Positive Analysis of Walrasian Equilibrium:

3.1. Walras' Law and the Walrasian Equilibrium: Definitions.
How a competitive economy works?
Which are its mechanisms?
We will address these two questions using a **Pure Exchange Economy**.

3.2. Existence and Uniqueness of Walrasian Equilibrium:
Existence: Proof that that **"point of agreement" exists**…
Uniqueness: If it exists, is it **unique**?. **Relevance for Comparative Statics.**

2. Competitive Equilibrium:Introduction

- **It is an agreement reached between a vast number of individual that take seemingly separate decisions.**

(…) Hence, the need for coordination, for some means of seeing that plans of diverse agents have balanced totals, remains. How this coordination takes place has been a **central preoccupation of economic theory** since Adam Smith and received a reasonably clear answer in the 1870's with the work of Jevons, Menger, and above all, Leon Walras: **it was the fact that all agents in the economy faced the same set of prices that provided the common flow of information needed to coordinate the system**. There was, so it was argued, a set of prices, one for each commodity which would equate supply and demand for all commodities; and if supply and demand were unequal anywhere, at least some prices would change, while none would change in the opposite case.

(*Kenneth Arrow, Nobel Memorial Lecture, December, 1972*)

2. Competitive Equilibrium:Introduction

- **This agreement realized through a price**: the equilibrium price that balances supply and demand of an specific market.
- **Equilibrium Price is conditioned by agents' preferences, income levels and technology (cost structures).**
- Depending on **Market laws that affect equilibrium prices** and consequently, the amount of quantities supplied and demanded:

-Perfectly Competitive Markets.

-Imperfectly Competitive Markets.

2. Competitive Equilibrium: Introduction

- **Perfectly Competitive Markets:**
  Characteristics:

-**Agents are price-takers**; the price is known by all participants.

-**Big Markets**(with many participants).

-**Implicit characteristic** (from the **Supply side**):
Constant Returns to Scale (CRS)

$$P=\frac{\partial C(q,(.))}{\partial q}=\frac{C(q,(.))}{q}$$

2. Competitive Equilibrium: Introduction

- Perfectly Competitive Markets:

-Why Increasing Returns to Scale (IRS) technologies (non-convex technologies) are incompatible with Competitive Markets?

-With an **IRS** technology: $\frac{C(q,(.))}{q}>\frac{\partial C(q,(.))}{\partial q}$

-However, a producer who behaves in 'competitive' manner would incur in losses since in competitive markets:

$$P=\frac{\partial C(q,(.))}{\partial q}$$

The profit maximization condition would make the supplier to produce more than it is demanded in that market.

2. Competitive Equilibrium: Introduction

- Perfectly Competitive Markets:

-Why do economists fancy competitive markets so much?

-These markets are very attractive to economists since they present good welfare properties.
Actually, most economic policies are specifically designed for that purpose:

"altering economic agents' behavior to approximate market rules to those present in perfectly competitive markets"

-The mechanisms (laws) that govern competitive markets lead to **efficient and optimal results (The Invisible Hand Theorem).**

-These efficient mechanisms behind competitive markets wouldn't be possible if Non-Convex production sets were present. e.g. IRS.

Competitive Equilibrium ⟶ **Walras' Equilibrium**

2. Competitive Equilibrium: Introduction

- To Know More:

If any of you are interested in working with non-convexities in the production set in the context of General equilibrium, a good starting point is the work of:

Villar, A. 1994. ***"Existence and Efficiency of Equilibrium with Increasing Returns to Scale: An Exposition".*** Investigaciones Económicas, Volume XVIII (2), pp. 205-243

2. Competitive Equilibrium: Introduction

- Typology of Competitive Markets:

  -Partial Equilibrium:

  -One-side Analysis: **Only Direct Effects are considered.**

  -Demand and supply of a single market only depend on **'an equilibrium price'**: **only one price determines the equilibrium in that market**.

  -General Equilibrium:

  -Complex and Comprehensive Analysis: Direct Effects, Indirect Effects and Induced Effects.

  -Demand and supply of a single market depend on 'a set of prices': **the equilibrium of a group of markets is simultaneously ascertained**.

2. Competitive Equilibrium: Introduction

- Typology of Competitive Markets:

  -General Equilibrium:

  -It considers all effects (in terms of prices and quantities) of a potential change (event) in an economy: economy-wide impacts.

  Example: **An increase in the activity levels of the Touristic Sector in an specific region.**

  -Direct Effects: **An increase in the final demand of Touristic activities in that region.**

  -Indirect Effects: **An increase in intermediate demand needed to produce services of the Touristic Sector (it favors intermediate suppliers of that sector)**

  -Induced Effects: **expenditure levels in the economy also raise due to the derived positive effects on employment levels that are connected either directly or indirectly to the Touristic Sector.**

3. Positive Analysis of Walrasian Equilibrium.

- Objective:

  -Analyze the Laws and Conditions that govern Competitive Markets.

  -Existence and Uniqueness: Relevance for Comparative Static Analysis.

- Context:

  -Pure Exchange Economy ("Edgeworth Box Economy")

  - **Each Agent** is endowed with a bundle of commodities, **trading his excess demand** (**Net Demand**).
  - **No Production possibilities** are present in this economy.
  - All agents are price-takers (competitive markets); there are many consumers behind each Demand function (**Gross Demand**).
  - There is a **"Walrasian auctioneer"**.

3.1. Walras' Law Analysis and the Walrasian Equilibrium: Definitions.

-There exists a finite number of:

Agents: $h = 1, \ldots, H$

Commodities: $i = 1, \ldots, N$

-The Stock of Commodities (and thus, endowments) is fixed:

$$x = (x_1, \ldots, x_i, \ldots, x_N) \in \mathbb{R}^N$$

-Each commodity is valued with a non-negative unit of market value:

$$p = (p_1, \ldots, p_i, \ldots, p_N) \in \mathbb{R}_+^N$$

-Each Agent initially owns a part of those endowments:

$$e_h \in \mathbb{R}_+^N$$

3.1.Walras' Law Analysis and Walrasian Equilibrium: Definitions.

- Necessary Equilibrium Condition: Walras' Law

-**All information related to preferences and other issues of each agent** i.e. utility maximizing, prices, initial wealth (endowment) is reflected through a **non-negative demand function for each agent and for each commodity**:

$$\chi_{ih}(p, e_h) \quad \textbf{(1)}$$

-Since the Analysis of this economy is done from a General Equilibrium perspective, some laws must be considered (the economy is a **closed and interdependent**):

$$\sum_{i=1}^{N} p_i \cdot \chi_{ih}(p, e_h) = \sum_{i=1}^{N} p_i \cdot e_{ih} \qquad \forall h = 1, \ldots, H \quad \textbf{(2)}$$

-The Restriction in (2) implies that: "for each agent the value of total consumption should be equal to the value of the owned wealth"

3.1. Walras' Law Analysis and Walrasian Equilibrium: Definitions.

- **Necessary Equilibrium Condition: Walras' Law**

-**Restriction (2)** will be verified at the level of the whole economy: **Walras' Law**

$$\sum_{h=1}^{H}\sum_{i=1}^{N} p_i \cdot \chi_{ih}(p, e_h) = \sum_{h=1}^{H}\sum_{i=1}^{N} p_i \cdot e_{ih} \quad \textbf{(3)}$$

Now, we define the Excess Demand function (**Net Demand**):

$$\zeta_i = \chi_i(p) - e_i \quad \textbf{(4)}$$

-With some algebraic rearrangement in **(3)** and the definition in **(4)**, the **Walras' Law** will read as:

$$\sum_{i=1}^{N} p_i \cdot \zeta_i(p) = 0 \quad \textbf{(5)}$$

This equality in **(5)** is telling us something quite interesting**: "the value of market excess demands equals zero at all prices (whether or not they are Equilibrium prices).**

3.1. Walras' Law Analysis and Walrasian Equilibrium: Definitions.

- **Sufficient Equilibrium Condition**

-For a set of prices to become an equilibrium in this closed and interdependent economic system the **sufficient condition** is:

$$\zeta_i(p^*) = \chi_i(p^*) - e_i = 0 \qquad \forall i = 1, \ldots, N \quad \textbf{(6)}$$

-Condition in (6) must be fulfilled for all markets *i* =1,…,N for a set of positive prices:

$$p^* > 0$$

Note: The condition of exact equality in (6) is a bit stronger than needed. All that is required is that demand should not be greater than supply at equilibrium prices.

3.1. Walras' Law Analysis and Walrasian Equilibrium: Definitions.

- **Definition of Walrasian Equilibrium:**

-A **Walrasian equilibrium** is defined as an **allocation** of **the available level of commodities** (consumption bundle for each agent) and **a set of prices** such that condition **(6)** is fulfilled (**market clearing condition**).

Alternatively, in considering agents' behavioral issues (explicitly):

↓

-A Walrasian equilibrium for an exchange economy is a vector of prices, and a consumption bundle for each agent, such that (*i*) every agent's consumption maximizes utility at a given set of prices and (*ii*) markets clear.

3.1. Walras' Law Analysis and Walrasian Equilibrium: Definitions.

- **Key Corollary from Walras' Law**

-"If all markets but one are cleared, then the remaining market must also be cleared".

-Therefore, **to determine equilibrium prices in this economy, we just need to pick up *N-1* of the system equations in (6) and find a solution.**

-Consequently, since we have *N* variables and *N-1* independent equations, only relative equilibrium prices can be determined.

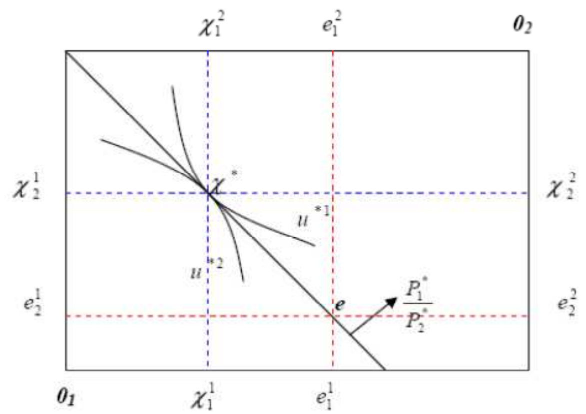
-How can we resolve the system?

We need some 'reference' price fixed from outside (the **numéraire**).

With an adequate selection of numéraire, all prices are expressed as relative distances to it, both when determining the benchmark equilibrium and when exploring possible changes in this economy.

3.1. Walras' Law Analysis and Walrasian Equilibrium: Definitions.

Figura 3.1. Walrasian Equilibrium in a pure exchange economy



3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Existence and Uniqueness: Relevance.

-Walras (1846): did not offer any formal proof on the existence and uniqueness of equilibrium. He just took all this for granted.

-The works of Wald during the 30s and that of Arrow and Debreu (1954) significantly contributed to prove the conditions under which equilibrium was unique and stable.

-Why existence and uniqueness are so relevant?

- In using as tools AGEM, we researchers want to carried out an Comparative Static Analysis.
- Then, if several equilibria exist…Ups…we are in trouble: **"we do not know where we are and where we go"**

↓

we unknown both the benchmark equilibrium and the new equilibrium (after evaluating an specific change in the economy. e.g. changes in the initial endowments).

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- **Existence and Uniqueness**

Existence

In the context of a simple exchange economy, we should look for at least one specific 'point' determined by a set of prices and quantities for which everybody agrees.

Uniqueness

and that this 'point' or 'agreement' is unique…

But how?...We economists fancy taking advantage of mathematics. 'Maths' make our lives easier…or may be not?

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Existence:

-How can we prove it ? ⟶ **Topology Theorems**

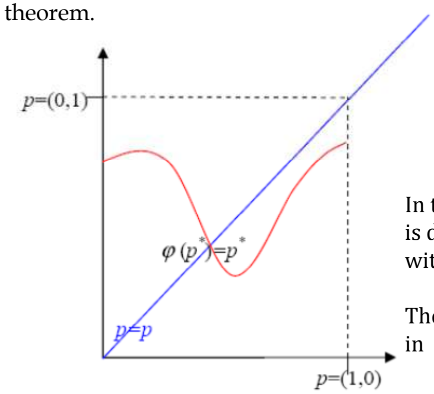
Brouwer's Fixed point theorem (1911-1912) and the extension (in a more integrated way) of **Kakutani (1941)** were very useful for this purpose.

Define S as the subset of points $p \in \mathbb{R}_{+}^{N}$ such that $p_1 + ... + p_i + ... + p_N = 1$
(S is a closed and convex set)

Brouwer's fixed Point Theorem: Let $\varphi(p)$ be a **continuous function** $\varphi: S \rightarrow S$ with $p \in S$ then there is a $p^* \in S$ such that $\varphi(p^*) = p^*$.

3.2. Existence and Uniqueness of Walrasian Equilibrium:

Figura. 3.2. Illustration of Brouwer's fixed point theorem.



$\varphi: S \rightarrow S$

In this case the simplex (S) is defined in $0 \leq p \leq 1$ with dimension N-1=1.

Then we can represent φ in $\mathbb{R}^2$

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Existence:

-Implications of the Fixed Point Theorem:

-If **the function** $\varphi(p)$ **is continuous,** its graph cannot go from the left edge to the right edge without intersecting the diagonal at least once (the fixed point)**.**

-But, should the graph of that function intersect the diagonal a second time, then it must necessarily do it a third time:

- The Theorem says that, under the properties of $\varphi(p)$ **there must be a finite and odd number of fixed points (equilibria or agreements).**
- **It is a good property hold for the so-called 'Regular Economies' (Debreu, 1970).**

Don't worry! Good News! (almost) All economies are regular.

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Existence:

-Implications of the Fixed Point Theorem:

-A Key Aspect: to ensure the existence of equilibrium, the aggregate excess demand function in each market must be continuous.

This occurs when:

-i) Each agent demand function is continuous (**sufficient condition**). That is to say that **preferences are convex**: "small variations in prices lead to small variations in quantities'

-ii) Even though, **each agent demand functions were non-continuous, the aggregate demand function is continuous (necessary and sufficient condition**).

Regular Economies fulfilled property i) or ii) (among others); unless they are known as Non-Regular economies.

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Existence

-How can we link the Fixed Point Theorem to the Existence of Equilibrium?

-How can we define the continuous mapping $\varphi(p)$ **?**

What about this one? "The Price Adjustment Function" (Gale-Nikaido function)

$$\varphi_i(p) = \frac{p_i + \max[0, \zeta_i(p)]}{1 + \sum_{j=1}^{N} \max[0, \zeta_j(p)]} \quad (7)$$

In addition, since $\zeta_i(p)$ **is homogenous of degree zero in** p we can normalize prices $\sum_{i=1}^{N} p_i = 1$ such that $p \in S$

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Existence:

-"The Price Adjustment Function":

$$\varphi_i(p) = \frac{p_i + \max[0, \zeta_i(p)]}{1 + \sum_{j=1}^{N} \max[0, \zeta_j(p)]} \quad (7)$$

We depart from a situation whereby the Walrasian auctioneer offers a initial set of prices that is not a fixed point:

-If $\zeta_i(p) > 0$ then p_i should be increased precisely by the value of the positive excess demand.

-If $\zeta_i(p) < 0$ then by the Corollary of Walras' Law, there has to be another market where $\zeta_k(p) > 0$. The adjustment rule now acts modifying the price of commodity *k*.

This process will continue until:

$$\zeta_i(p^*) = 0 \quad \forall i = 1, \ldots, N$$

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Existence:

-We know that the Price Adjustment Function fulfils the requirements for a fixed point to exists. i.e. $\varphi(p^*) = p^*$.

- (i) It is a continuous function.
- (ii) Price adjustments take always place in the simplex S

-We have to show now that the fixed point of that function is indeed an equilibrium, i.e. $\zeta_i(p^*) = 0 \quad \forall i = 1, \ldots, N$

-Let's prove it…

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- **Existence: Proof**

-The definition of a fixed point says that $\varphi(p^*) = p^*$**. If we substitute this definition in condition (7), we find:**

$$p_i^* = \frac{p_i^* + \max[0, \zeta_i(p^*)]}{1 + \sum_{j=1}^{N} \max[0, \zeta_j(p^*)]} \quad (8)$$

-Reorganizing and simplifying the terms in (8), we get:

$$p_i^* \cdot \sum_{j=1}^{N} \max[0, \zeta_j(p^*)] = \max[0, \zeta_i(p^*)] \quad (9)$$

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- **Existence: Proof**

Necessary Condition

-Multiplying now by the excess demand function for good i and add all of them up to obtain:

$$\left(\sum_{i=1}^{N} p_i^* \cdot \zeta_i(p^*)\right) \cdot \left(\sum_{j=1}^{N} \max\left[0, \zeta_j(p^*)\right]\right) = \sum_{i=1}^{N} \max\left[0, \zeta_i^2(p^*)\right] \qquad \textbf{(10)}$$

-Recall that , by Walras' Law, the left-hand side of (10) is zero and so must therefore be the right-hand side. Then, it follows that:

$$\zeta_i(p^*) = 0$$

-…but! We have not finished yet!

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- **Existence: Proof:**

Sufficient Condition

-We know that if there is an equilibrium, this equilibrium is a Walrasian equilibrium, now we move to prove that it is effectively an Equilibrium.

-Is this condition $\zeta_i(p^*) = 0$fulfilled?

If there exists any market k with $\zeta_k(p^*) \neq 0$, then $\zeta_k^2(p^*) > 0$ and $\sum_{i=1}^{N} \max\left[0, \zeta_i(p^*)\right] \neq 0$ in (11) could not be zero, A CONTRADICTION!

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- **Uniqueness:**

-To Demonstrate the Uniqueness was always more 'complex' than Existence (but not less relevant).

-To Proof 'global' uniqueness is very ambitious (perhaps) as stated by Debreu (1970). **Local uniqueness is enough**.

From the point of view of the Comparative Static Analysis, it is only necessary:

-that the new equilibrium is unique within the vicinity of the initial or benchmark equilibrium, i.e. the new equilibrium would stem from 'modest' changes in the initial parameters.

Continuity was an essential for Existence

Which is the key assumption for Uniqueness?

For a pure exchange economy, uniqueness of equilibrium can be assured if aggregate excess demand functions satisfy the so-called gross substitutability property.

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Uniqueness:

-If in a pure exchange economy, the "Marshallian" demand function of each agent satisfies the condition of gross substitutability property, then the aggregated demand function too.

-The property of gross substitutability is sufficient though no necessary for uniqueness.

-If the demand function fulfils this property, the Walrasian equilibrium is Unique in this economy.

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Uniqueness:

- **Gross substitutability** property implies that if we have two price vectors, i.e. p and p', such that for some good j we have:

$$p'_j > p_j \text{ but } p'_i = p_i \quad \forall i \neq j$$

-Then:

$$\zeta_i(p') > \zeta_i(p) \forall i \neq j$$

-This would imply that if the price of one commodity *j* increases then the excess demand of the other commodities *i* would increase too, since:

$$\partial \zeta_i(p)/\partial p_j > 0$$

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- **Uniqueness: Proof**

-Consider two set of different equilibrium prices p and p' (not collinear)

-If they are equilibrium prices, then:

$$\zeta_i(p') = \zeta_i(p) = 0$$

- By homogeneity of degree zero, we can normalize in such a way that:

$$p'_j \geq p_j \; \forall j = 1, \ldots N \quad \text{with} \quad p'_k = p_k$$

-Then, if we move from p to p' in a series of *N-1* steps and because Gross Substitutability, the aggregate demand for commodity k must increase:

$$\zeta_k(p') > \zeta_k(p) = 0$$ **A CONTRADICTION!**

3.2. Existence and Uniqueness of Walrasian Equilibrium:

- Uniqueness:

- Which are the consequences if demand functions do not satisfy the condition of gross substitutability?

If this is the case, it is necessary:

-Weak Axiom of Reveal Preferences (Kehoe, 1992).

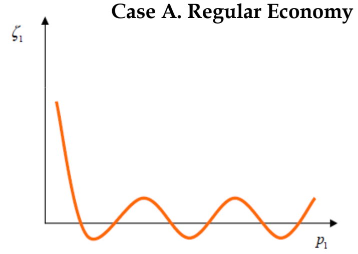
This means that consumer decisions are stable in the sense that when they face the same price, they behave in the same manner, i.e. demand is exactly the same.

It is not necessary then transitivity o coherence (**Strong Axiom of Reveal Preferences**).

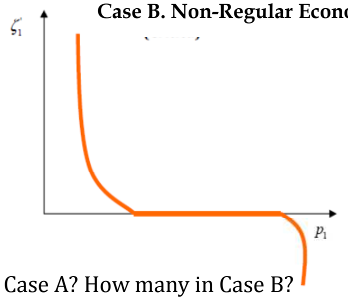
3.2. Existence and Uniqueness of Walrasian Equilibrium:

- **Regular Economies vs Non-Regular Economies.**

Case A. Regular Economy



Case B. Non-Regular Economy



-How many equilibria are there in Case A? How many in Case B?

-Respect to $\frac{\partial \zeta_1}{\partial p_1}$ which is the difference between Case A and B?

2&3. An Overview of General Equilibrium Theory

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Related to Fixed Point Theorems and Comparative Statics:

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4.A 'Simple' CGE: Part I

Ana-Isabel Guerra
Universidad de Granada

4.1.Description of the Economy.

-The economy is 1×2×2

-1 household (type)
-2 producers/firms (type): two heterogeneous commodities.
-2 primary factors (type): labor and capital.

-Households:

Behavior: They maximize utility subject to a budget constraint that depends on income levels obtained from their labor and capital endowments.

-Firms:

Behavior: With the existing market prices, they maximize profits from selling what they produce with a RCS technology.

4.1.Description of the Economy:

-Households:

Behavior: C-D Utility:

$$\begin{aligned} &Max U(c,\beta) = c_1^{0,4} c_2^{0,6} \\ &s.t. \quad p_1 \cdot c_1 + p_2 \cdot c_2 = m = r \cdot K + w \cdot L \end{aligned} \qquad \textbf{(1)}$$

After solving (1) we get the Marshallian Demand Functions:

$$\begin{aligned} c_1^* &= \frac{m \cdot 0,4}{P_1} \\ c_2^* &= \frac{m \cdot 0,6}{P_2} \end{aligned} \qquad \textbf{(2)}$$

4.1.Description of the Economy.

-Firms:

Behavior:

$$MimC(\bullet) = \underbrace{r \cdot K_j + w \cdot L_j}_{\text{Value Added Consumption}} + \underbrace{p_1 \cdot z_{1j} + p_2 \cdot z_{2j}}_{\text{Intermediate Consumption Costs}} \qquad (3)$$

$$s.t. \quad y_j = f(VA_j, z_{1j}, z_{2j})$$

We have to solve the problem in (3) in two parts:

a) **Primary factors combine to obtain a ' composite factor': Value Added.**

b) **Then, this 'composite factor' combines with intermediate inputs to get final output.**

4.1.Description of the Economy

-Firms: Exemple for producer/firm *j=1*

Technology Description (Cost Structures):

$$y_1 = \min\left(\underbrace{\frac{VA_1}{v_j}}_{\Uparrow VA_1}, \underbrace{\frac{z_{11}}{a_{11}}, \frac{z_{21}}{a_{21}}}_{\Uparrow [A]_{i1}} \right) \qquad Leontief$$

$$VA_1 = \mu_1 \cdot L_1^{0,8} \cdot K_1^{0,2} \qquad Cobb-Douglas$$

Value Added Coefficient $v_1 = 0,5$

Intermediate Consumption Coefficient $A = \begin{bmatrix} 0,2 & a_{12} \\ \underbrace{0,3}_{\Uparrow [A]_{i1}} & a_{22} \end{bmatrix}$

4.1.Description of the Economy.

-Firms:

Demand for Value-Added: Labor and Capital

From the solution to the problem:

$$Mim(r \cdot K_1 + w \cdot L_1)$$
$$s.t. \quad VA_1 = \mu_1 \cdot L_1^{0,8} \cdot K_1^{0,2}$$

We obtain the cost function for Value Added:

$$C_1(r, w, VA_1) = \underbrace{\frac{1}{\mu_1} \cdot 0,8^{-0,8} \cdot 0,2^{-0,2}}_{=1 \Leftrightarrow \mu_1 = 1,649} \cdot w^{0,8} \cdot r^{0,2} \cdot VA_1$$

$$C_1(r, w, VA_1) = w^{0,8} \cdot r^{0,2} \cdot VA_1$$

4.1.Description of the Economy.

-Producers:

Demand for Value-Added: Labor and Capital.

From Shephard Lemma, we obtain the demand function for each factor:

$$L_1(r, w, VA_1) = \frac{\partial C_1(r, w, VA_1)}{\partial w} = 0,8 \cdot \left(\frac{r}{w}\right)^{0,2} \cdot VA_1$$

$$K_1(r, w, VA_1) = \frac{\partial C_1(r, w, VA_1)}{\partial r} = 0,2 \cdot \left(\frac{w}{r}\right)^{0,8} \cdot VA_1$$

Technical coefficients for labor and capital (variables):

$$l_1 = \frac{L_1(r, w, VA_1)}{VA_1} = 0,8 \cdot \left(\frac{r}{w}\right)^{0,2}$$

$$k_1 = \frac{K_1(r, w, VA_1)}{VA_1} = 0,2 \cdot \left(\frac{w}{r}\right)^{0,8}$$

4.2. Equilibrium.

-Description of the System:

We are looking for…

A vector of Equilibrium prices:

$$(p_1^*, p_2^*, r^*, w^*)$$

that simultaneously determines Equilibrium quantities:

$$(y_1^*, y_2^*)$$

taking into account the primary factors endowments (fixed):

$$(L, K)$$

We have to define a system of de <u>6 equations and 6 unknowns…</u>

4.2.Equilibrium.

-Equations of the GEM:

Equilibrium in commodity markets:

In the markets of commodity 1 y 2: "demand must equal supply"

$$\underbrace{c_1^*(p,m)}_{\text{Final Demand}} + \underbrace{a_{11}\cdot y_1 + a_{12}\cdot y_2}_{\text{Intermediate Demand}} = \underbrace{y_1}_{\text{Supply}} \qquad (1)$$

$$\underbrace{c_2^*(p,m)}_{\text{Final Demand}} + \underbrace{a_{21}\cdot y_1 + a_{22}\cdot y_2}_{\text{Intermediate Demand}} = \underbrace{y_2}_{\text{Supply}} \qquad (2)$$

Where final demand takes the form:

$$c_1^*(p,m) = \frac{m\cdot 0,4}{P_1} = \frac{(L\cdot w^* + K\cdot r^*)\cdot 0,4}{P_1}$$

$$c_2^*(p,m) = \frac{m\cdot 0,6}{P_2} = \frac{(L\cdot w^* + K\cdot r^*)\cdot 0,6}{P_2}$$

4.2.Equilibrium

-Equations of the GEM:

Equilibrium in factor markets:

$$\underbrace{\underbrace{l_1\cdot v_1\cdot y_1}_{L_1} + \underbrace{l_2\cdot v_2\cdot y_2}_{L_2}}_{\text{Demand}} = \underbrace{L}_{\text{Supply}} \qquad (3)$$

$$\underbrace{\underbrace{k_1\cdot v_1\cdot y_1}_{k_1} + \underbrace{k_2\cdot v_2\cdot y_2}_{k_2}}_{\text{Demand}} = \underbrace{K}_{\text{Supply}} \qquad (4)$$

Factor Demands are determine by:

Input needs for Commodity 1:

$$l_1 = \frac{L_1(r,w,VA_1)}{VA_1} = 0,8\cdot\left(\frac{r}{w}\right)^{0,2}$$

$$k_1 = \frac{K_1(r,w,VA_1)}{VA_1} = 0,2\cdot\left(\frac{w}{r}\right)^{0,8}$$

Input needs for Commodity 2:

$$l_2 = \frac{L_2(r,w,VA_2)}{VA_2} = 0,6\cdot\left(\frac{r}{w}\right)^{0,4}$$

$$k_2 = \frac{K_2(r,w,VA_2)}{VA_2} = 0,4\cdot\left(\frac{w}{r}\right)^{0,6}$$

4.2.Equilibrium.

-Equations of the GEM:

Zero Profit Condition:

$$\underbrace{p_1}_{\text{Price}} = \underbrace{p_{va_1}\cdot v_1 + p_1\cdot a_{11} + p_2\cdot a_{21}}_{\text{Unitary Cost}} \qquad (5)$$

$$\underbrace{p_2}_{\text{Price}} = \underbrace{p_{va_2}\cdot v_2 + p_1\cdot a_{12} + p_2\cdot a_{22}}_{\text{Unitary Cost}} \qquad (6)$$

Where:

$$\underbrace{p_{va_1}}_{\text{Price}} = \underbrace{w^{0,8}\cdot r^{0,2}}_{\text{Unitary Cost}}$$

$$\underbrace{p_{va_2}}_{\text{Price}} = \underbrace{w^{0,6}\cdot r^{0,4}}_{\text{Unitary Cost}}$$

But we just need 5 out of the 6 equations ⟶ Walras' Law

4.3.Calibration of the Model.

-Now, we can start with "GAMS"…(other mathematical programs are possible)

Why is so relevant to calibrate the model?

-a) It allows obtaining the initial necessary parameters to find the benchmark equilibrium (where we are)

Example: In the case of a C-D Consumer, for instance, the parameters to calibrate demand functions are captured from the SAM.

-b) It makes possible to obtain the initial values of the variables of the model that will define the vicinity where we will be looking for the equilibrium.

-c) Through calibration, we can check if the initial or benchmark equilibrium is correct, i.e. it can reproduce the initial SAM.

4.3.Calibration of the Model.

-The initial information about the economy: SAM

	Social Accounting Matrix (SAM)					
	Firm 1	Firm 2	Labor	Capital	Household	Total
Firm 1	20	50	0	0	30	100
Firm 2	30	25	0	0	45	100
Labor	40	10	0	0	0	50
Capital	10	15	0	0	0	25
Household	0	0	50	25	0	75
Total	100	100	50	25	75	

-In addition to the decision parameters:
Elasticities (Not in this case) and technological structure.

-We have to calibrate the model, find the initial/benchmark equilibrium coherent with the quantitative information reflected in the SAM.

Recall that in Comparative Statics is important to know 'the point of departure'!

4.3.Calibration of the Model

-STEP 1: Define the subsets/accounts of the SAM:

***One Household (H)**
***Two Primary Factors(F)**
***Two firms (Two production sectors) (I)**

```
set ibig /i1,i2,L,K,H/; SAM ACCOUNTS
set i(ibig) /i1,i2/; TWO FIRMS
set f(ibig) /L,K/; TWO PRIMARY FACTORS
```

***Two carry out algebraic operations, we need to name the previous defined subsets using alternative 'names':**

```
alias  (ibig,jbig), (i,j,k);
```


4.3.Calibración de Modelo.

-STEP 2: Upload the SAM:

We can do that defining a Table:

```
Table SAM (ibig,jbig)
          i1   i2   L   K   H
     i1   20   50           30
     i2   30   25           45
      L   40   10
      K   10   15
      H             50  25    ;
```

***Check if the SAM fulfils the accounting principle (Resources=Uses)**

```
parameter    sumcolumn(jbig) sum columns of theSAM,
             sumrow(ibig)  sum rows of  the SAM;
             sumcolumn(jbig)=sum(ibig, SAM(ibig,jbig));
             sumrow(ibig)=sum(jbig, SAM(ibig,jbig));
            display sumcolum, sumfila;
```

4.3.Calibration of the Model.

-STEP 3: Calibration of the Model:

Ejemplo: Calibration of the Marshallian Demand Functions:

```
parameter m0 household income,
          alphac(i) coeffient of the  C-D utility function
          m0=r0*KT+w0*LT;
```

Accordint to our numerical example, alphac(i) is equal to 0.4 and 0.6:

```
alphac(i)=SAM(i,'H')/m0;
```

Calibration of demand functions:

```
parameter cf0(i) final demand function for each commodity;
        cf0(i)=alphac(i)*m0/p0(i);
```


4.4.Solving the Benchmark Equilibrium.

-STEP 4: Define the variables and the initial values that are close to the Walrasian equilibrium (equilibrium allocation).

Variables

Z variable dummy (that we want to maximize or minimize)

Our 5 Unknowns:

w initial price of labor, labor unitary cost.

p commodity prices (2 prices)

Y(i) total production of each commodity (2 commodities)

Remaining set of variables that help to determine the equilibrium:

pva(i) price of the input composite (value added)

cL(i) technical coefficient of labor

cK(i) technical coefficient of capital

cf(i) final consumption for each commodity

m household income

Ld(i) Demand for labor

Kd(i) Demand for capital;

4.4.Solving the Benchmark Equilibrium.

-STEP 4: Define the variables and the initial values that are close to the Walrasian equilibrium (equilibrium allocation).

Example: Variables prices, define equilibrium prices within the interval (0.001,10).

```
parameter lb nivel inferior,
          ub nivel superior;
          lb=0.001;
          ub=10;
          p.l(i)=p0(i);
          p.lo(i)=p0(i)*lb;
          p.up(i)=p0(i)*ub;
```


4.4.Solving the Benchmark Eqilibrium

-STEP 5: Define the System of Equations:

```
Equations
Eqz          "Auxiliary" Equation
Eqp(i)       Final prices (Zero profit condition)
Eqpva(i)     Price for Value Added
EqcL(i)      Technical Coefficient for Labor
EqcK(i)      Technical Coefficient for Capital
EqLd(i)      Demand for Labor
EqKd(i)      Demand for Capital
Eqm          Household Income
Eqcf(i)      Final Demand
Eqy(i)       Equilibrium quantities
EqL          Equilibrium in labor market
EqK          Equilibrium in capital market;
```

4.4.Solving the Benchmark Equilibrium.

-STEP 6: Introduce the system and ask GAMS to solve it.

```
Eqz..        Z=e=0;
Eqp(i)..     p(i)=e=pva(i)*cva(i)+sum(j,p(j)*A(j,i));
Eqpva(i)..   pva(i)=e=(w**(alphaL(i)))*(r0**(1-alphaL(i)));
EqcL(i)..    cl(i)=e=alphaL(i)*((r0/w)**(1-alphaL(i)));
Eqck(i)..    ck(i)=e=(1-alphaL(i))*((w/r0)**(alphaL(i)));
EqKd(i)..    Kd(i)=e=Y(i)*ck(i)*cva(i);
EqLd(i)..    Ld(i)=e=Y(i)*cl(i)*cva(i);
EqL..        LT=N=sum(i,Ld(i));        ---->  Walras' Law (Corollary)
EqK..        KT=E=sum(i,Kd(i));
Eqm..        m=e=KT*r0+w*LT;
Eqcf(i)..    cf(i)=e=alphac(i)*m/p(i);
EqY(i)..     Y(i)=e=sum(j,A(i,j)*Y(j))+cf(i);
model equilibrio /all/;
solve equilibrio using  nlp maximazing z;
```


4.4. Solving the Benchmark Equilibrium.

-STEP 7: Check if the initial equilibrium is 'well-defined"

display p.l,p0, y.l,y0;

If the initial equilibrium is correct, then:

p.l → $p^* = (1,1)$

w.l → $w^* = 1$

y.L → $y^* = (100,100)$

But…We have not finish yet since we have to check if our initial equilibrium fulfils Walras' Law:

Parameter LT0 oferta total de capital de acuerdo con el equilibrio
LT0=sum(i,Ld.l(i));

In our numerical example, LT0 must be equal to 50 units.

4.A 'Simple' CGE: Part II: Introducing a Government Sector

Ana-Isabel Guerra
Universidad de Granada

4.5.An Indirect Tax on Output with Lump-sum Transfers.

The Government (another economic agent) can obtain income from the private sector by taxing, for instance, transactions among firms.

-A tax on output can take several formats:

-Unit or Excise Tax: A tax on the number of units being trade.

-Ad Valorem Tax: A tax on the value of transactions.

↓

-We will focus our attention on Ad Valorem Taxes.

-We will assume that the amount of output taxes is given back in full to household (Lump-sum transfers).

4.6.Description of the Economy (extended with government).

-Let τ_j be the ad valorem output tax rate on the output of firm *j* in our (previously defined) simple model and let now p denote the gross-of tax vector of commodity prices.

-Because of the presence of the advalorem indirect tax, average cost (*ac*) of commodity *j=1,2* becomes:

$$\underbrace{p_j}_{\text{Price}} = ac_j = (1+\tau_j)\left(p_{va\,j} + p_j \cdot a_{1j} + p_2 \cdot a_{2j}\right)$$

4.6.Description of the Economy (extended with government).

-The ammount collected by the goverment thanks to the output tax will be denoted by T .

-T is return to consumers (households) in lump sum form according to δ_h weights that add up to unity. The budget constraint of consumer *h* reads as:

$$p_1 \cdot c_1^h + p_2 \cdot c_2^h = m_h = r \cdot K_h + w \cdot L_h + \delta_h \cdot T$$

-In our numerical example, we have only one household then the utility maximization problem becomes:

$$\begin{aligned} &MaxU(c,\beta) = c_1^{0,4} c_2^{0,6} \\ &s.t. \quad p_1 \cdot c_1 + p_2 \cdot c_2 = m = r \cdot K + w \cdot L + T \end{aligned} \qquad \textbf{(1a)}$$

4.6.Description of the Economy (extended with government).

-Because in this case we have a new unknown, we need to introduce a new independent equation to keep the solvability of the equilibrium system.

- In this case we introduce a government revenue function that makes explicit its source of income:

$$R = \tau_1 \cdot p_1 \cdot y_1 + \tau_2 \cdot p_2 \cdot y_2 = T \qquad \textbf{(7)}$$

4.7. Equilibrium Equations (extended with Government)

-An Equilibrium with output taxes will now be characterized by a vector $Y^* = (y_1^*, y_2^*)$ **of output levels, a set of prices** $p^* = (p_1^*, p_2^*)$ $\omega^* = (r^*, w^*)$ **and a level of tax collections** T^* **such that:**

(i) Market of commodities Clear:

$$Y^* = FD(p^*, \omega^*, T^*) + A \cdot Y^*$$

(ii) Market of Primary Factors Clear:

$$\overline{L} = Ld(p^*, \omega^*, Y^*)$$

$$\overline{K} = Kd(p^*, \omega^*, Y^*)$$

(iii) Zero Profit Condition:

$$p^* = \left[p_{va}^*(\omega^*) \cdot V^* + p^* \cdot A \right] \cdot \Gamma$$

→ **Diagonal matrix with** $(\Gamma)_{ii} = (1+\tau_i)$

(iv) Government Revenue function:

$$R(p^*, \omega^*, Y^*; (\Gamma)_{ii}) = T^*$$

→ **The additional Equation**

4.8. Calibration of the 'Extended' Model:

-Our new data Set (Extended SAM): Two additional rows/columns (Value Added Tax and Government)

	Social Accounting Matrix (SAM).Extended							
	Firm 1	Firm 2	Labor	Capital	Household	Government	Value Added Tax	Total
Firm 1	20	50	0	0	40	0	0	110
Firm 2	30	25	0	0	55	0	0	110
Labor	40	10	0	0	0	0	0	50
Capital	10	15	0	0	0	0	0	25
Household	0	0	50	25	0	20	0	95
Government	0	0	0	0	0	0	20	20
Value Added Tax	10	10	0	0	0	0	0	20
Total	110	110	50	25	95	20	20	

4.6.Calibration of the Extended Model.

-The most difficult thing in this case is to **calibrate Taxes**: **Two possibilities**.

-**(i)** Calibrate Taxes such a way that initial prices are unitary: Gross Prices are unitary. (the prefered one).

-**(ii)** Calibrate Taxes in such a way that initial prices are not unitary: Net Prices are unitary.

4.6.Calibration of the Extended Model

-STEP 1: Define the subsets/accounts of the SAM:

***One Household (H)**
***Two Primary Factors(F)**
***Two firms (Two production sectors) (I)**
***Government (G)**

```
set ibig /i1,i2,L,K,H,G,TVA/; SAM ACCOUNTS
set i(ibig) /i1,i2/; TWO FIRMS
set f(ibig) /L,K/; TWO PRIMARY FACTORS
alias  (ibig,jbig), (i,j,k);
```

- STEP 2A: Unload the SAM (as in the previous case):

4.6.Calibration of the Extended Model

STEP 2B: Calibrating Taxes: Option (i)

```
parameter  Y0(j) output production (gross of taxes)
           Y1(j) output production (net of taxes)
           tao(j) Value added tax;
Y0(j)=sumcolumn(j);
Y1(j)=Y0(j)-SAM('TVA',j);
tao(j)=SAM('TVA',j)/(Y1(j));
```

We re-scale the system in such a way that gross production in volume terms correspond to production in physical terms, with unitary gross prices.

Then, to calibrate technological coefficients we must use Y0(j).

4.6.Calibration of the Extended Model.

-STEP 3: Calibration of the Model:

Differently to the previous example: Introduce the Revenue function of the Government.

```
parameter RT0 government revenue,
          T0 lump-sum transfers;

RT0=sum(i,y0(i)*tao(i)*(pva0(i)*cva(i)+sum(j,p0(j)*A(j,i))));
T0=RT0;
```


4.6.Calibration of the Extended Model.

-STEP 3: Calibration of the Model:

Ejemplo: Calibration of the Marshallian Demand Functions (we have to Introduce government transfers within households income):

```
parameter m0 household income,
          alphac(i) coeffient of the  C-D utility function
          m0=r0*KT+w0*LT+T0;
```

Accordint to our numerical example, alphac(i) is equal to 0.4 and 0.6:

```
alphac(i)=SAM(i,'H')/m0;
```

Calibration of demand functions:

```
parameter cf0(i) final demand function for each commodity;
     cf0(i)=alphac(i)*m0/p0(i);
```


4.7.Solving the Benchmark Equilibrium.

-STEP 4: Define the variables and the initial values that are close to the Walrasian equilibrium (equilibrium allocation).

Variables

Z variable dummy (that we want to maximize or minimize)

Our 6 Unknowns:

w initial price of labor, labor unitary cost.

p commodity prices (2 prices)

Y(i) total production of each commodity (2 commodities)

T government transfers/ RT revenues from output taxes

Remaining set of variables (as in the previous case):

pva(i) price of the input composite (value added)

cL(i) technical coefficient of labor

cK(i) technical coefficient of capital

cf(i) final consumption for each commodity

m household income

Ld(i) Demand for labor

Kd(i) Demand for capital;

4.7.Solving the Benchmark Equilibrium.

-STEP 4: Define the variables and the initial values that are close to the Walrasian equilibrium (equilibrium allocation).

STEP 5: Define the System of Equations (2 additional equations(in fact 1)):

Equations

Eqz	"Auxiliary" Equation
Eqp(i)	Final prices (Zero profit condition)
Eqpva(i)	Price for Value Added
EqcL(i)	Technical Coefficient for Labor
EqcK(i)	Technical Coefficient for Capital
EqLd(i)	Demand for Labor
EqKd(i)	Demand for Capital
Eqm	Household Income
Eqcf(i)	Final Demand
Eqy(i)	Equilibrium quantities
EqL	Equilibrium in labor market
EqK	Equilibrium in capital market;
EqR	Government Revenue Function
EqT	Government Transfers

4.7.Solving the Benchmark Equilibrium.

-STEP 6: Introduce the system and ask GAMS to solve it.

```
Eqz..         Z=e=0;
Eqp(i)..      p(i)=e=pva(i)*cva(i)+sum(j,p(j)*A(j,i));
Eqpva(i)..    pva(i)=e=(w**(alphaL(i)))*(r0**(1-alphaL(i)));
EqcL(i)..     cl(i)=e=alphaL(i)*((r0/w)**(1-alphaL(i)));
Eqck(i)..     ck(i)=e=(1-alphaL(i))*((w/r0)**(alphaL(i)));
EqKd(i)..     Kd(i)=e=Y(i)*ck(i)*cva(i);
EqLd(i)..     Ld(i)=e=Y(i)*cl(i)*cva(i);
EqL..         LT=N=sum(i,Ld(i));
EqK..         KT=E=sum(i,Kd(i));
Eqm..         m=e=KT*r0+w*LT;
Eqcf(i)..     cf(i)=e=alphac(i)*m/p(i);
EqY(i)..      Y(i)=e=sum(j,A(i,j)*Y(j))+cf(i);
EqR..         RT=e=sum(i,y(i)*tao(i)*(pva(i)*cva(i)+sum(j,p(j)*A(j,i))))
EqT..         T=e=RT
model equilibrio /all/;solve equilibrio using  nlp maximazing z;
```


4.8.Small Exercise with the Extended Model

-A 5% increase in Output Taxes:

```
tao(i)=tao(i)*1.05;

solve equilibrio using  nlp maximazing z;
```

-Check (again) if Walras' Law is fulfilled

-Check (again) if the circular flow of income is fulfilled.

```
GDPincome=sum(i,w.l*Ld.l(i))+sum(i,r0*Kd.l(i))+RT.l;
GDPexpenditure=sum(i,p.l(i)*cf.l(i));

display GDPincome, GDPexpenditure;
```


4.8.Small Exercise with the Extended Model

-We will compare the initial equilibrium to the 'evaluated' change.

-In doing so, we have to take into account that prices are always expressed in terms of the numeraire (in our case the rental price of capital)

-For instance, in our case, the impact on prices

Prices New Equilibrium	$\Delta\tau = 5\%$
p_1	1.0109
p_2	1.0134
r	1.0000
w	1.0004

All we can say is that in the counterfactual equilibrium one unit of commodity one would by 1.109 units of capital.

4. A Simple CGE.

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