Contact Holomorphic Curves and Flat Surfaces

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Abstract

In this paper we study flat surfaces in the hyperbolic 3-space and the de Sitter 3-space with the conformal structure induced by its second fundamental form and give a conformal representation of such surfaces in terms of holomorphic data.

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1 Introduction

Partial differential equations on surfaces whose solutions could be represented in terms of holomorphic functions on Riemann surfaces have been extensively investigated. Famous examples are Laplace’s equation $\Delta u = 0$ and Liouville’s equation $\Delta u = e^u$.

An example from geometry is the minimal surface equation in the Euclidean space $\mathbb{R}^3$ whose holomorphic representation gives the global version of the Enneper-Riemann-Weierstrass representation, which is essentially due to Osserman [O]. This representation has been crucial in both reaching a rather exhaustive understanding and finding examples of complete minimal surfaces. In spaces of other constant sectional curvature such as the hyperbolic 3-space $\mathbb{H}^3$ or the de Sitter 3-space $\mathbb{S}^3_1$ the equation of a surface of constant mean curvature admits a holomorphic resolution that provides a global complex representation which has been used in the study of global properties of these surfaces, (see [AA], [B], [UY]).

The fully non-linear Monge-Ampère equation $\det \nabla^2 u = 1$ which arises in affine differential geometry (see [FMM], [J]) and in the study of the second fundamental form of flat surfaces in $\mathbb{H}^3$ and $\mathbb{S}^3_1$, can be solved using holomorphic data. In this paper we consider flat surfaces in $\mathbb{H}^3$ and $\mathbb{S}^3_1$ with the conformal structure induced by its second fundamental form. We will prove that these surfaces share a fundamental property with minimal surfaces in $\mathbb{R}^3$.

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and surfaces of constant mean curvature in $\mathbb{H}^3$ and $S^3$, they possess a “conformal representation” in terms of holomorphic data which involve its “hyperbolic” Gauss map (Theorem 1).

2 Some Preliminaries

Let $\mathbb{L}^4$ be the Minkowski 4-space endowed with linear coordinates $(x_0, x_1, x_2, x_3)$ and the scalar product, $\langle \cdot , \cdot \rangle$ given by the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2$. We set the two hyperquadrics

$$\mathbb{H}^3 = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{L}^4 / -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1, \ x_0 > 0 \right\},$$

$$S^3_1 = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{L}^4 / -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \right\},$$

with the induced metric from $\mathbb{L}^4$. Then, $\mathbb{H}^3$ is a Riemannian 3-manifold of constant sectional curvature $-1$ which is called the hyperbolic 3-space. $S^3_1$ is a 3-dimensional Lorentzian manifold of constant sectional curvature 1 and it is called the de Sitter 3-space.

Let $N^3$ denote the positive null cone, that is

$$N^3 = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{L}^4 / -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \ x_0 > 0 \right\}.$$  

If one considers for all $v \in N^3$ the halfline $[v]$ spanned by $v$, then this gives a partition of $N^3$ and the ideal boundary $\mathbb{S}^\infty_3$ of $\mathbb{H}^3$ can be regarded as the quotient of $N^3$ under the associated equivalence relation. Thus, the induced metric is well-defined up to a factor and $\mathbb{S}^\infty_3$ inherits a natural conformal structure as the quotient $\mathbb{N}^3/\mathbb{R}^+$. 

We consider $\mathbb{L}^4$ identified with the space of $2 \times 2$ Hermitian matrices, $\text{Herm}(2)$, by identifying $(x_0, x_1, x_2, x_3) \in \mathbb{L}^4$ with the matrix

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} = \sum_{j=0}^{3} x_j e_j,$$

where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Under this identification, one has $\langle m, m \rangle = -\det(m)$, for all $m \in \text{Herm}(2)$, and the complex Lie group $\text{SL}(2, \mathbb{C})$ of $2 \times 2$ complex matrices with determinant 1 acts naturally on $\mathbb{L}^4$ by the representation

$$g \cdot m = gmg^*,$$

where $g \in \text{SL}(2, \mathbb{C})$, $g^* = ^t\overline{g}$ and $m \in \text{Herm}(2)$. Consequently, $\text{SL}(2, \mathbb{C})$ preserves the scalar product and orientations. The kernel of this action is $\{ \pm I_2 \} \subseteq \text{SL}(2, \mathbb{C})$ and $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C}) / \{ \pm I_2 \}$ can be regarded as the identity component of the special Lorentzian group $SO(1, 3)$. This action can be restricted to $\mathbb{H}^3$ and $S^3_1$ as an isometric and transitive one. Thus, $\mathbb{H}^3$ and $S^3_1$ can also be represented as

$$\mathbb{H}^3 = \{ g \cdot e_0 / g \in \text{SL}(2, \mathbb{C}) \}$$
and

\[ S_1^3 = \{ g \cdot e_j / g \in \text{SL}(2, \mathbb{C}) \}, \quad j \in \{1, 2, 3\}. \]

The space \( \mathbb{N}^3 \) is seen as the space of positive semi-definite \( 2 \times 2 \) Hermitian matrices of determinant 0 and its elements can be written as \( a^t \alpha \), where \( ^t a = (a_1, a_2) \) is a non-zero vector in \( \mathbb{C}^2 \) uniquely defined up to multiplication by an unimodular complex number. The map \( a^t \alpha \rightarrow [(a_1, a_2)] \in \mathbb{CP}^1 \) becomes the quotient map of \( \mathbb{N}^3 \) on \( S_2^2 \) and identifies \( S_2^2 \) with \( \mathbb{CP}^1 \). So the natural action of \( \text{SL}(2, \mathbb{C}) \) on \( S_2^2 \) is the action of \( \text{SL}(2, \mathbb{C}) \) on \( \mathbb{CP}^1 \) by Möbius transformations.

## 3 Contact Holomorphic Curves

On \( \text{SL}(2, \mathbb{C}) = \left\{ \zeta = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \mid \det(\zeta) = 1 \right\} \)

we shall consider the canonical contact structure induced by the contact 1-form

\[ \Omega \equiv z_{22}dz_{11} - z_{12}dz_{21}. \]

Let \( \Sigma \) be a Riemann surface and \( g : \Sigma \rightarrow \text{SL}(2, \mathbb{C}) \),

\[ g = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \quad G_{11}G_{22} - G_{12}G_{21} = 1, \]

be a holomorphic map such that \( g^* \Omega \) vanishes on \( \Sigma \), then

\[ g^{-1}dg = \begin{pmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{pmatrix} \begin{pmatrix} dG_{11} \\ dG_{21} \end{pmatrix} \begin{pmatrix} dG_{12} \\ dG_{22} \end{pmatrix} = \begin{pmatrix} 0 & \alpha_{12} \\ \alpha_{21} & 0 \end{pmatrix}. \]

**Definition 1** A holomorphic map \( g : \Sigma \rightarrow \text{SL}(2, \mathbb{C}) \) is called a contact holomorphic map if \( g^* \Omega \equiv 0 \) and \( \alpha_{21} \) never vanishes on \( \Sigma \).

Thus, if we set

\[ f = \frac{\alpha_{12}}{\alpha_{21}}, \quad \omega = \alpha_{21}, \]

the pair \((f, \omega)\) satisfies the following equality

\[ g^{-1}dg = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} \omega. \]

Conversely, let \( f \) be a holomorphic function and \( \omega \) a holomorphic 1-form on \( \Sigma \) such that \( \omega \neq 0 \) everywhere. Then the ordinary differential equation (2) is integrable and a solution \( g \) is a contact holomorphic map into \( \text{SL}(2, \mathbb{C}) \) but \( g \) may not be well-defined on \( \Sigma \). In fact, when we consider \((f, \omega)\) written in an arbitrary complex parameter \( \zeta \) as \((f(\zeta), h(\zeta)d\zeta)\), every solution \( g \) of (2) is given as

\[ g = \begin{pmatrix} V \\ W \end{pmatrix} \begin{pmatrix} \frac{1}{h}V_{\zeta} \\ \frac{1}{h}W_{\zeta} \end{pmatrix} \]

3
where $V$ and $W$ are linearly independent solutions of the ordinary linear differential equation

\[(4) \quad X_{\zeta\zeta} - \frac{h\zeta}{h} X_{\zeta} - fh^2 X = 0,\]

and

\[h = VW_{\zeta} - WV_{\zeta}.\]

**Definition 2** The pair $(f, \omega)$ will be called Weierstrass data (complex representation) of $g$.

### 4 A Conformal Representation

**Theorem** A) Let $\Sigma$ be a Riemann surface and $g : \Sigma \rightarrow \mathrm{SL}(2, \mathbb{C})$ a contact holomorphic map with Weierstrass data $(f, \omega)$.

- I) If $|f| < 1$, then $\psi_0 = g \cdot e_0 : \Sigma \rightarrow \mathbb{H}^3$ and $\psi_3 = g \cdot e_3 : \Sigma \rightarrow \mathbb{S}_1^3$ are well-defined flat Riemannian immersions.

- II) If the imaginary part of $f$, $\Im(f)$, never vanishes on $\Sigma$, then $\psi_1 = g \cdot e_1 : \Sigma \rightarrow \mathbb{S}_1^3$ is a well-defined flat Lorentzian immersion.

B) Conversely, let $M$ be a simply connected surface and $\psi : M \rightarrow N$ a flat immersion, where $N$ is either $\mathbb{H}^3$ or $\mathbb{S}_1^3$. If on $M$ we consider the conformal structure determined by the second fundamental form of $\psi$, then there exists a contact holomorphic map $g : M \rightarrow \mathrm{SL}(2, \mathbb{C})$ with Weierstrass data $(f, \omega)$ such that $\psi = \psi_j$ for some $j \in \{0, 1, 3\}$. Moreover $g$ is unique up to right multiplication by a constant matrix $g_0$ with $g_0 \cdot e_j = e_j$.

**Proof:** We consider $\psi = g \cdot e$ with

\[e = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}, \quad \epsilon = \pm 1.\]

Then from (2),

\[d\psi = g \begin{pmatrix} 0 & \epsilon f \omega + \overline{\omega} \\ \omega + \epsilon \overline{f} \omega & 0 \end{pmatrix} \, g^*, \]

and the induced metric is given by

\[ds^2 = \langle d\psi, d\psi \rangle = -\det(d\psi) = (1 + f \overline{f}) \omega \overline{\omega} + \epsilon f \omega^2 + \epsilon \overline{f} \omega^2.\]

Since $f$ is a holomorphic function

\[d \left( \frac{1}{2} (1 + \epsilon f) \omega + \frac{1}{2} (1 + \epsilon \overline{f} \overline{\omega}) \right) = 0, \quad d \left( \frac{i}{2} (1 - \epsilon f) \omega - \frac{i}{2} (1 - \epsilon \overline{f} \overline{\omega}) \right) = 0\]

and there exist local functions $x, y$ such that

\[dx = \frac{1}{2} (1 + \epsilon f) \omega + \frac{1}{2} (1 + \epsilon \overline{f} \overline{\omega}), \quad dy = \frac{i}{2} (1 - \epsilon f) \omega - \frac{i}{2} (1 - \epsilon \overline{f} \overline{\omega}).\]

with

\[dx \wedge dy = \frac{-i}{2} (1 - |f|^2) \omega \wedge \overline{\omega}.\]
Because of $|f| < 1$, $(x, y)$ are new coordinates with $ds^2 = dx^2 + dy^2$. Therefore, the immersion is flat.

In the same way, when $\psi = \psi_1$ one gets $ds^2 = 2dx\,dy$, where

$$dx = -f\omega - \overline{f}\omega, \quad dy = \frac{1}{2}\omega + \frac{1}{2}\overline{\omega}.$$ 

Conversely, if $\psi : M \to \mathbb{S}^1_1$ is a Lorentzian immersion with flat induced metric, then there exists an asymptotic coordinate immersion $x + iy : M \to \mathbb{C}$ such that

$$(5) \quad ds^2 = 2dx\,dy,$$

and if $\eta$ is an unit normal vector field to the immersion, a straight calculation gives the following structure equations

$$(6) \quad \psi_{xx} = E\eta, \quad \psi_{xy} = F\eta - \psi, \quad \psi_{yy} = G\eta, \quad \eta_x = -F\psi_x - E\psi_y, \quad \eta_y = -G\psi_x - F\psi_y,$$

where $E$, $F$ and $G$ are smooth functions on $M$ and by $(\cdot)_x$ and $(\cdot)_y$ we shall denote the usual partial derivatives respect to $x$ and $y$, respectively.

Using the Gauss’ and Codazzi-Mainardi’s equations we have $EG - F^2 = 1$, $E_x = F_x$ and $F_y = G_x$. Hence, as $M$ is simply connected, there exists a well-defined function $\phi$ on $M$ such that $E = \phi_{xx}$, $F = \phi_{xy}$, $G = \phi_{yy}$ and the second fundamental form of the immersion is given by

$$(7) \quad d\sigma^2 = \phi_{xx}dx^2 + \phi_{yy}dy^2 + 2\phi_{xy}dx\,dy,$$

with

$$(8) \quad \phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1.$$

We shall regard $M$ as a Riemann surface with the conformal structure determined by the second fundamental form $d\sigma^2$.

From (8), we can choose $\eta$ such that $\phi_{xx} > 0$ and consider the new coordinate immersion

$$(9) \quad z = u + iv = y - i\phi_x.$$

Then, a straight computation gives,

$$(10) \quad \psi_u = -\frac{\phi_{xy}}{\phi_{xx}}\psi_x + \psi_y, \quad \psi_v = -\frac{1}{\phi_{xx}}\psi_x.$$

Now, from (6), (7), (8), (9) and (10) we have

$$(11) \quad d\sigma^2 = \frac{1}{\phi_{xx}}|dz|^2.$$
and

\[(\psi_x)_u = -\psi, \quad (\psi_x)_v = -\eta.\]

Thus, from (5), (9), (11) and (12), \(z : M \to \mathbb{C}\) is a conformal coordinate immersion and \([\psi_x] : M \to S^2_{\infty}\) is a conformal map, which induces on \(M\) the flat Riemannian metric \(|dz|^2\).

Moreover, from the above expressions, we obtain

\[(\psi_x)_{uu} = \frac{\phi_{xy}}{\phi_{xx}} \psi_x - \psi_y, \quad (\psi_x)_{uv} = -\frac{\phi_{xy}}{\phi_{xx}} \psi_x - \psi_y,\]

and by using standard notations of complex analysis, one has

\[(13) \quad 4(\psi_x)_{z\overline{z}} = -2\psi_y.\]

Now, let \(A, B : M \to \mathbb{C}\) be global holomorphic functions on \(M\) such that \([\psi_x]\) is represented as \([(A, B)] \in \mathbb{CP}^1 \equiv S^2_{\infty}\), then

\[\psi_x = \lambda \begin{pmatrix} A \\ B \end{pmatrix} (A, B) = \lambda \begin{pmatrix} A\overline{A} & AB \\ AB & B\overline{B} \end{pmatrix},\]

for some positive function \(\lambda \in C^\infty(M)\). Thus, from (5), (9) and (12), one gets

\[\frac{1}{2} = ((\psi_x)_z, (\psi_x)_{\overline{z}}) = \frac{1}{2} \lambda^2 |AB_x - BA_x|^2\]

and as \(AB_x - BA_x\) does not vanish on the simply connected surface \(M\), there exists a holomorphic function \(R : M \to \mathbb{C}\) with \(R^2 = AB_x - BA_x\). Hence, we can write

\[(14) \quad \psi_x = \begin{pmatrix} C \overline{C} \\ C \overline{D} \end{pmatrix} + \begin{pmatrix} C \overline{D} \\ D \overline{D} \end{pmatrix} = \begin{pmatrix} C \overline{C} \\ C \overline{D} \end{pmatrix} + \begin{pmatrix} C \overline{D} \\ D \overline{D} \end{pmatrix},\]

where \(C = A/R\) and \(D = B/R\). Consequently, from (12) and (14) we have the following expression for the immersion

\[(15) \quad \psi = -\begin{pmatrix} C \overline{C} + C_z \overline{C} & C \overline{D} + C_z \overline{D} \\ C \overline{D}_z + C_z \overline{D} & D \overline{D}_z + D_z \overline{D} \end{pmatrix},\]

and for its unit normal

\[\eta = -i \begin{pmatrix} -C \overline{C} + C_z \overline{C} & -C \overline{D} + C_z \overline{D} \\ C \overline{D}_z - C_z \overline{D} & D \overline{D}_z - D_z \overline{D} \end{pmatrix}.\]

If we consider the function \(f : M \to \mathbb{C}\) defined by

\[(16) \quad f = \frac{\phi_{xy} - i}{2\phi_{xx}},\]

then, from (6), (10), (12), (13) and (14) we obtain \((\psi_y)_z = -2f(\psi_x)_{\overline{z}}\) and

\[\begin{pmatrix} C_z \overline{C}_z & C \overline{D}_z \\ C \overline{D}_z & D \overline{D}_z \end{pmatrix} = f \begin{pmatrix} C \overline{C}_z & C \overline{D}_z \\ C \overline{D}_z & D \overline{D}_z \end{pmatrix}.\]
As $C_z$ and $D_z$ cannot vanish simultaneously, we have

$$
(C_{zz} = fC, \quad D_{zz} = fD).
$$

Thus, from (8), (16) and (17), $f$ is a holomorphic function which satisfies

$$
\Im(f) \neq 0.
$$

Finally, from (15) and (17), the immersion $\psi$ can be recovered as $\psi = -g \cdot e_1$, where $g : M \to \text{SL}(2, \mathbb{C})$ is the contact holomorphic map given by

$$
g = \begin{pmatrix} C & C_z \\ D & D_z \end{pmatrix}
$$

such that

$$
g^{-1}dg = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} \omega
$$

and $\omega = dz$. On the other hand, if $\tilde{g} : M \to \text{SL}(2, \mathbb{C})$ is a holomorphic immersion with $\psi = -\tilde{g} \cdot e_1$, then there exists a holomorphic map $g_0 : M \to \text{SL}(2, \mathbb{C})$ such that $g = \tilde{g}g_0$, with $g_0 \cdot e_1 = e_1$. Thus $(g_0)_z = 0$ and $g_0$ must be constant.

Now, let $\psi : M \to N$ be a Riemannian immersion with flat induced metric, where $N = \mathbb{H}^3$ or $N = \mathbb{S}^3_1$, then there exists a coordinate immersion $x + iy : M \to \mathbb{C}$ such that

$$
ds^2 = dx^2 + dy^2.
$$

The structure equations are given by

$$
\begin{align*}
\psi_{xx} &= E\eta + \epsilon\psi, \\
\psi_{xy} &= F\eta, \\
\psi_{yy} &= G\eta + \epsilon\psi, \\
\eta_x &= -\epsilon E\psi_x - \epsilon F\psi_y, \\
\eta_y &= -\epsilon F\psi_x - \epsilon G\psi_y,
\end{align*}
$$

where $\epsilon = 1$ if $N = \mathbb{H}^3$, $\epsilon = -1$ if $N = \mathbb{S}^3_1$, $E$, $F$ and $G$ are smooth functions on $M$ and $\eta$ is the well-oriented unit normal vector field. From the integrability conditions, there exists a well-defined function $\phi$ on $M$ such that $E = \phi_{xx}$, $F = \phi_{xy}$, $G = \phi_{yy}$ satisfying (8) and the second fundamental form of the immersion is given by (7).

We consider the new coordinate immersion

$$
z = u + iv = x + \phi_x + i(y + \phi_y).
$$

Then

$$
\frac{1}{2 + \phi_{xx} + \phi_{yy}}|dz|^2
$$

and

$$
(\psi - \eta)_u = \psi_x, \quad (\psi - \eta)_v = \psi_y.
$$
Thus, $z : M \rightarrow \mathbb{C}$ is a conformal coordinate immersion and $[\psi - \eta] : M \rightarrow S^2_\infty$ is a conformal map.

Calculating $(\psi - \eta)_{uu}$, $(\psi - \eta)_{vv}$, we obtain

$$\psi = \frac{1}{2}(\psi - \eta) + 2\epsilon(\psi - \eta)z\tau.$$

As in the above case, (see [GMM]), the immersion can be calculated as

$$\psi = \begin{pmatrix} C\overline{C} + 4\epsilon Cz\overline{C}z & C\overline{D} + 4\epsilon Cz\overline{D}z \\ \overline{C}D + 4\epsilon \overline{C}zDz & D\overline{D} + 4\epsilon Dz\overline{D}z \end{pmatrix}$$

where $C$ and $D$ are linearly independent solutions of the ordinary linear differential equation

$$X_{zz} = \frac{1}{4}fX,$$

and $f : M \rightarrow \mathbb{C}$ is the holomorphic function defined by

$$f = \frac{\phi_{yy} - \phi_{xx} + 2i\phi_{xy}}{2 + \phi_{xx} + \phi_{yy}}.$$

Moreover, from (8), $|f| < 1$.

Finally, the immersion $\psi$ can be recovered as $\psi = g \cdot e_0$ or $\psi = g \cdot e_3$ if $N = \mathbb{H}^3$ or $N = S^3_1$, respectively, where $g : M \rightarrow SL(2, \mathbb{C})$ is the contact holomorphic map given as in (18) satisfying (19) and $\omega = \frac{1}{2}dz$. Q.E.D.

**Remark** The above conformal representation can be used in the study of global properties of flat surfaces in $\mathbb{H}^3$ and $S^3_1$. For the particular case of flat surfaces in $\mathbb{H}^3$ the reader can see [GMM].

**References**


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