# Relatives of Flat Surfaces in $\mathbb{H}^{3}$ 

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In recent years there has been major progress in the classical theory of surfaces in Affine Differential Geometry. Here we will review some of the geometric properties and technical tools that have been useful in studying the family of locally convex surfaces (possibly with singularities ) in Euclidean 3 -space $\mathbb{R}^{3}$ with parallel affine normal lines. This kind of surface keeps an intimate relation to flat surfaces in hyperbolic 3 -space $\mathbb{H}^{3}$ and they are connected with special lagrangian immersions in complex Euclidean plane $\mathbb{C}^{2}$ and minimal surfaces in $\mathbb{R}^{3}$. Although these surfaces are not so well known as minimal surfaces in $\mathbb{R}^{3}$ their local connection to them has oriented their study.

## 1 The regular case

In this Section we discuss the regular case: Improper Affine spheres (IA-spheres in short). Some comments, questions and remarks will motivate this study and show how they are connected with other interesting branches in differential geometry. Their corresponding holomorphic representations let us understand why IA-spheres can be considered as relatives of flat surfaces in $\mathbb{H}^{3}$. We also describe some fundamental facts about their geometry and behaviour at infinity.

### 1.1 Comments and questions

Let us consider $f: \Sigma \longrightarrow \mathbb{H}^{3}$ a flat immersion with first fundamental form $d s^{2}$ and second fundamental form $\sigma$.

In 2000, Gálvez, Milán and the author showed in [13] how to parametrize $f$ by meromorphic data. In fact, when the conformal structure on the surface is induced by the second fundamental form, flat surfaces in $\mathbb{H}^{3}$ have a meromorphic hyperbolic Gauss map and a similar representation to the Weierstrass representation for constant mean curvature one surfaces in $\mathbb{H}^{3}$ (see [3],[26],[27]) also works for flat surfaces. Indeed, we proved that any flat immersion

[^0]$f: \Sigma \longrightarrow \mathbb{H}^{3}$ is the projection of a holomorphic legendrian immersion $E_{f}: \widetilde{\Sigma} \longrightarrow P S L(2, \mathbb{C})$ defined on the universal cover $\widetilde{\Sigma}$ of $\Sigma$, that is,
\[

f=E_{f} E_{f}^{\star}, \quad E_{f}^{-1} d E_{f}=\left($$
\begin{array}{cc}
0 & d G  \tag{1}\\
d F & 0
\end{array}
$$\right),
\]

where by * we denote conjugate and transpose. The induced metric and the second fundamental form of the immersion are given, respectively, by:

$$
\begin{aligned}
I & =|d F|^{2}+|d G|^{2}+2 \Re(d F d G) \\
I I & =|d G|^{2}-|d F|^{2}
\end{aligned}
$$

where $\Re$ represents the real part. The pair $(d F, d G)$ will be called Weierstrass data of $f$.
Some questions came right now:

- Why we can represent flat surfaces by meromorphic data when we consider the underlying conformal structure induced by $\sigma$ ?.
- Is there any special reason for that?

In order to understand the answers we can choose local parameters $(x, y)$ so that,

$$
\begin{align*}
I= & d x^{2}+d y^{2}  \tag{2}\\
I I=I I_{\phi}= & \phi_{x x} d x^{2}+\phi_{y y} d y^{2}+2 \phi_{x y} d x d y  \tag{3}\\
& \operatorname{Det}\left(\nabla^{2} \phi\right)=1 \tag{4}
\end{align*}
$$

where down indices indicate partial derivatives with respect to the corresponding variables.
We will refer to the unimodular Hessian equation (3) as our fundamental equation and the underlying conformal structure induced by $I I_{\phi}$ will be the canonical one associated with the solution $\phi$.

In 1970, Calabi observed (see [6]) that our fundamental equation is, locally, related to the equation of minimal surfaces in $\mathbb{R}^{3}$. In fact, it is well known that a graph of a function $\gamma: \Omega \longrightarrow \mathbb{R}$, on a planar 1 -connected domain $\Omega$, is a minimal surface if and only if $\gamma$ is a solution of the following quasilinear PDE:

$$
\begin{equation*}
\left(1+\gamma_{x}^{2}\right) \gamma_{y y}+\left(1+\gamma_{y}^{2}\right) \gamma_{x x}-2 \gamma_{x} \gamma_{y} \gamma_{x y}=0, \quad \text { on } \Omega \tag{5}
\end{equation*}
$$

If we take $W=\sqrt{1+\gamma_{x}^{2}+\gamma_{y}^{2}}$, then (5) tells us that

$$
\frac{1+\gamma_{x}^{2}}{W} d x+\frac{\gamma_{x} \gamma_{y}}{W} d y, \quad \frac{\gamma_{x} \gamma_{y}}{W} d x+\frac{1+\gamma_{y}^{2}}{W} d y
$$

are closed 1-forms. When $\Omega$ is 1-connected, $\exists \alpha, \beta: \Omega \longrightarrow \mathbb{R}$ s.t.

$$
\alpha_{x}=\frac{1+\gamma_{x}^{2}}{W}, \quad \alpha_{y}=\frac{\gamma_{x} \gamma_{y}}{W}, \quad \beta_{x}=\frac{\gamma_{x} \gamma_{y}}{W}, \quad \beta_{y}=\frac{1+\gamma_{y}^{2}}{W} .
$$

As $\alpha_{y}=\beta_{x}, \exists \phi: \Omega \longrightarrow \mathbb{R}, \phi_{x}=\alpha$ and $\phi_{y}=\beta$. Consequently, $\phi$ verifies $\operatorname{Det} \nabla^{2} \phi=1$. Conversely, given $\phi$ solution of (4) on $\Omega$, exists $\gamma$ solution of (5), satisfying the above relations.

Moreover, from this local connection,

$$
\begin{equation*}
W I I_{\phi}=\left(1+\gamma_{x}^{2}\right) d x^{2}+\left(1+\gamma_{y}\right)^{2} d y^{2}+2 \gamma_{x} \gamma_{y} d x d y \tag{6}
\end{equation*}
$$

that is, the induced metric on the minimal graph and the metric $\sigma_{\phi}$ are conformal. Thus, there is a (local) correspondence between minimal graphs with their usual conformal structure and graphs of our fundamental equation with their canonical complex structure.

Looking for a global situation other questions arise:

- What kind of surfaces in $\mathbb{R}^{3}$ are, locally, graphs $S_{\phi}$ of solutions $\phi$ of our fundamental equation?.
- What about the metric $I I_{\phi}$ ?

Because $I I_{\phi}$ and our fundamental equation are invariant under unimodular affine transformations whose differentials fix the vertical direction, we find some answers in Affine Differential Geometry.

It is easy to chek that $S_{\phi}$ is a locally convex surface in $\mathbb{R}^{3}$ with vertical affine normal lines, which means the tangent lines of the locus of the center of gravity of the slices parallel to the tangent plane at every point are vertical lines (see [2], [19] and Figure 1)


Figure 1: Affine normal lines.

Historically, Tzitzeica was the first to study surfaces in $\mathbb{R}^{3}$ with parallel affine normal lines, [25], while Blaschke, who called these surfaces improper affine spheres (IA-spheres), was the first to study this family in the affine context,[2]. Although important global results were obtained by Jörgens and Calabi ( see [5], [16] ), it has been over the last 20 years that there has been a much wider developement.

Changing, $\phi$ by $-\phi$ if it was necessary, we conclude that $S_{\phi}$ is a locally convex $I A$-sphere with affine normal $\xi=(0,0,1)$ (see [21] [19] for more details). In this case, $\sigma_{\phi}$ is the usual affine metric (also called Berwald-Blaschke metric) on $S_{\phi}$.

From now on we will assume every IA-sphere is locally convex.

### 1.2 An interesting remark

The local connection between IA-spheres and minimal surfaces does not implies same properties or similar behaviour. We keep in mind the following property of minimal surfaces which was proved by Li, Shoen and Yau, [18], on the way they studied an isoperimetric inequality for minimal surfaces. They proved that:

If we move away the boundaries of a connected minimal surface bounded by two Jordan curves, then there is a moment were the surface breaks in two minimal surfaces bounded each one for a Jordan curve.

This property could be checked easily (see Figure 2), because by the immersion of two closed wires-rings into a bucket of soapy water we should get a minimal surface bounded by two circles. But if we move the two wires-ring apart, the soap film breaks in two minimal disks.


Figure 2: A non-existence result of minimal surfaces.
On the contrary, IA-spheres have an opposite behaviour (see [10], Theorems 2 and 3):
If we consider two circles of different radii in parallel planes, always is possible to move them away until we get a connected IA-sphere bounded by the two circles. But if we approximate the two circles, there is a moment where the surface breaks in two pieces each one bounded by a circle.

Now, by using the maximum principle for second order elliptic PDE's in a standard way, an IA-sphere with a compact connected boundary inherits the symmetries of its boundary, and then we can conclude that the broken pieces lie in elliptic paraboloids (see Figure 3).


Figure 3: A non-existence result of IA-spheres.
Other differences appear in the study of IA-spheres that can be foliated by ellipses (see [10]).

### 1.3 The associated special lagrangian immersion

Let $\Sigma$ be an oriented surface and $\psi: \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}, \psi:=(x, u)$ an IA-sphere with conformal structure induced by the Berwald-Blaschke metric $\sigma$. After an unimodular transformation we can assume the affine-Blaschke normal vector field is the constant vertical vector field $\xi=(\overrightarrow{0}, 1)$.

The equiaffine normalization of $\psi$ is given by the affine-Blaschke normal $\xi$ and the affine co-normal vector field $N, N: \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}, N:=(n, 1)$ (see [19], [21]). With this normalization,

$$
\sigma:=-\langle d x, d n\rangle,
$$

where $\langle.,$.$\rangle is the usual inner product in \mathbb{R}^{2}$.
For a complex parameter $z$ on $\Sigma$ we shall use the Cauchy-Riemann operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$. The following expressions can be obtained:

$$
\begin{align*}
N_{z} & =\sqrt{-1} \xi \times \psi_{z}  \tag{7}\\
\sigma & =2 u_{z \bar{z}}|d z|^{2}=-2\left(\left\langle x_{z}, n_{\bar{z}}\right\rangle_{\mathbb{C}}\right)|d z|^{2} \tag{8}
\end{align*}
$$

where $N_{z}=\frac{\partial N}{\partial z}, x_{z}=\frac{\partial x}{\partial z}, u_{z \bar{z}}=\frac{\partial^{2} u}{\partial z \partial \bar{z}}$, etc. By bar we denote the usual conjugation, $\times$ is the standard complex cross product on the complexification $\left(\mathbb{R}^{2} \times \mathbb{R}\right) \otimes \mathbb{C}$ of $\mathbb{R}^{2} \times \mathbb{R}$ and $\langle., .\rangle_{\mathbb{C}}$ is the complex inner product on the complexification $\mathbb{R}^{2} \otimes \mathbb{C}$ of $\mathbb{R}^{2}$.

From (7), (8) and taking into account that $\xi=(\overrightarrow{0}, 1)$ is a transversal vector field, we can obtain that

$$
\begin{equation*}
\left[x_{z}, x_{\bar{z}}\right]=\left[n_{z}, n_{\bar{z}}\right], \quad x_{z \bar{z}}=n_{z \bar{z}}=0, \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
d s^{2}=\langle d x, d x\rangle \text { is non degenerate. } \tag{10}
\end{equation*}
$$

Here $[A, B]$ denotes the determinant functional of any ordered pair of vectors $A, B \in \mathbb{R}^{2} \otimes \mathbb{C}$.
On $\mathbb{C}^{2}=\mathbb{R}^{2} \otimes \mathbb{C}$ with coordinates $\zeta=\left(\zeta_{1}, \zeta_{2}\right), \zeta=x+\sqrt{-1} y, x, y \in \mathbb{R}^{2}$ we are going to consider

$$
\begin{align*}
g^{\prime} & =\left|d \zeta_{1}\right|^{2}+\left|d \zeta_{2}\right|^{2}, \\
\theta^{\prime} & =\frac{\sqrt{-1}}{2}\left(d \zeta_{1} \wedge d \overline{\zeta_{1}}+d \zeta_{2} \wedge d \overline{\zeta_{2}}\right),  \tag{11}\\
\Omega^{\prime} & =d \zeta_{1} \wedge d \zeta_{2}
\end{align*}
$$

Let $L: \Sigma \longrightarrow \mathbb{C}^{2}$ be an special Lagrangian immersion (SL-immersion for short) with respect to the calibration $\Re\left(\sqrt{-1} \Omega^{\prime}\right)$. As in [14], $L$ can be characterized as an immersion in $\mathbb{C}^{2}$ satisfying

$$
\begin{equation*}
\left.\omega^{\prime}\right|_{\Sigma} \equiv 0, \quad \Im\left(\left.\sqrt{-1} \Omega^{\prime}\right|_{\Sigma}\right) \equiv 0 \tag{12}
\end{equation*}
$$

where $\Re$ and $\Im$ represent the real and the imaginary part, respectively.
There is a correspondence between IA-spheres and some "non-degenerate" SL-immersions in $\mathbb{C}^{2}$ in the following way (see [20]):

Theorem 1 The map $L^{\psi}: \Sigma \longrightarrow \mathbb{C}^{2}$ given by

$$
\begin{equation*}
L^{\psi}:=x+\sqrt{-1} n \tag{13}
\end{equation*}
$$

is an SL-immersion such that

1. The induced metric $d \tau^{2}:=\langle d x, d x\rangle+\langle d n, d n\rangle$ is conformal to the Berwald-Blaschke metric $\sigma$ of $\psi$.
2. $d s^{2}=\langle d x, d x\rangle$ is a non-degenerate flat metric,
where $\langle$,$\rangle denotes the usual scalar product in \mathbb{R}^{2}$
Theorem 2 Let $L=x+\sqrt{-1} n: \Sigma \longrightarrow \mathbb{C}^{2}$ be an SL-immersion such that $d s^{2}:=\langle d x, d x\rangle$ is non-degenerate. Then

$$
\psi:=\left(x,-\int\langle n, d x\rangle\right)
$$

is a (perhaps multivaluated) improper affine sphere at its regular points (e.g., where $d s^{2}:=$ $\langle d x, d x\rangle$ is non-degenerate)

### 1.4 Conformal representation

By identifying vectors of $\mathbb{R}^{2}$ with complex numbers in the standard way

$$
(r, s) \equiv r+\sqrt{-1} s, \quad r, s \in \mathbb{R}
$$

one can prove, see [11], that the complex function $x-n$ (respectively, $x+n$ ) is a holomorphic (respectively, antiholomorphic) function on $\Sigma$.

Thus, from Theorem 1 and Theorem 2, IA-spheres can be represented "in a global way" by holomorphic data. Indeed, we have

## Theorem 3 [11] (Conformal representation)

I. Let $\psi:=(x, u): \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}$, be a IA-sphere with affine normal $\xi=(\overrightarrow{0}, 1)$ and conformal structure induced by the affine metric $\sigma$. Then, there exist two holomorphic functions $F, G$ : $\Sigma \longrightarrow \mathbb{C}$, with $|d G|^{2}>|d F|^{2}$ on $\Sigma$ s.t. $\psi$ can be recover, up a vertical translation, as

$$
\begin{equation*}
\psi=\left(G+\bar{F}, \frac{1}{2}\left(|G|^{2}-|F|^{2}\right)+\Re(G F)-2 \Re \int F d G\right) \tag{14}
\end{equation*}
$$

Moreover, the affine metric and the affine conormal map are given by

$$
\begin{equation*}
\sigma=|d G|^{2}-|d F|^{2}, \quad N=(\bar{F}-G, 1) \tag{15}
\end{equation*}
$$

II. Conversely, Let $\Sigma$ be a Riemann surface, $F, G: \Sigma \longrightarrow \mathbb{C}$ two holomorphic functions s.t. $|d G|^{2}>|d F|^{2}$ on $\Sigma$. Then (14) defines an IA-sphere with affine normal $\xi=(\overrightarrow{0}, 1)$ and with affine metric and affine conormal map given as in (15). Moreover, $\psi$ is well defined if and only if $\int F d G$ does not have real periods.

The pair $(F, G)$ will be called Weierstrass data of $\psi$.
Remark 1 IA-spheres can be considered as relatives of flat surfaces in $\mathbb{H}^{3}$. In fact, from (1), every flat immersion $f: \Sigma \longrightarrow \mathbb{H}^{3}, f \equiv(d F, d G)$ is intimately related to the IA-sphere $\psi: \widetilde{\Sigma} \longrightarrow \mathbb{R}^{2} \times \mathbb{R}$ with W-data $(F, G)$. The first and second fundamental form of $f$ coincide, respectively, with the flat fundamental form and the affine metric of $\psi$.

### 1.5 Examples

- A revolution IA-sphere is are conformally equivalent to either a punctured disk or $\mathbb{C}$ (see Figure 4).


Figure 4: Revolution IA-spheres

- Taking as W-data: $\left(1 / z, i c^{2} z\right), c \in \mathbb{R}, c \neq 0$, and $z$ on a punctured disk, we obtain multivaluated IA-spheres with a vertical period (see Figure 5.


Figure 5: A multivaluate IA-sphere

### 1.6 Some results

The above conformal representation for IA-spheres is particularly usuful for the description of their asymptotic behaviour. In that sense, and by a previous study of the Weierstrass data of complete IA-spheres conformally equivalent to pucture disks, we have extended (see [11]) the classical Jörgen's theorem, [16], as follows:

Theorem 4 Consider $\mathcal{A}:=\{A: A$ symmetric positive definite 2 x2-matrix, $\operatorname{Det}(A)=1\}$ and let $u$ be a convex solution of $\operatorname{Det}\left(\nabla^{2} u\right)=1$ in $\mathbb{R}^{2} \backslash \bar{O}$, where $O$ is a bounded domain. Then, there exist $a_{1} \in \mathbb{R}, \vec{b} \in \mathbb{R}^{2}$ and $A \in \mathcal{A}$ s.t.,

$$
u(x)=\frac{1}{2} x^{\prime} A x+\vec{b} \cdot x+a_{1} \log \left(x^{\prime} A x\right)+\mathrm{O}(1)
$$

outside of some bounded domain containing $O$.
For $k$ large, the ellipse

$$
\mathcal{E}_{k} \equiv \frac{1}{2} x^{\prime} A x+\vec{b} \cdot x=k,
$$

gives the shape of the graph of $u$ at infinity. The ellipse $\mathcal{E}_{k}$ and the number $a_{1}$ will be called ellipse at infinity and logarithmic growth rate of $u$, respectively.
By using a different approach, L. Caffarelli and Y. Li, [4] have extended the above result to high dimension.

For its applications the following result (proved in [11])is especially interesting

## Theorem 5 Maximum principle at infinity

Let $u$ and $v$, solutions of (4), on a exterior domain $\Omega$ s.t. $u \geq v$ and $\left|u\left(x_{n}\right)-v\left(x_{n}\right)\right| \rightarrow 0$, $\left\{x_{n}\right\} \in \Omega,\left|x_{n}\right| \rightarrow \infty . u \equiv v$.

Consequences of these two results are that,

- We have uniqueness of the Dirichlet problem on exterior domains for solutions of

$$
\operatorname{Det}\left(\nabla^{2} u\right)=1
$$

with the same ellipse at infinity and the same logarithmic growth rate (see [11]).

- The moduli space of solutions for the Dirichlet problem of

$$
\operatorname{Det}\left(\nabla^{2} u\right)=1, \text { in } \Omega, u_{\mid \partial \Omega}=\gamma
$$

on a exterior domain $\Omega$ is either empty or a 5 -dimensional differentiable manifold (see [12])

## 2 The general case

This Section deals with the general case: Improper Affine spheres with singularities which will be called $I A$-maps. Very interesting results obtained by Kokubu, Umehara and Yamada in [17] for flat surfaces with singularities in $\mathbb{H}^{3}$, let us connect IA-maps with flat fronts in $\mathbb{H}^{3}$ and study global properties of their geometry. This connection could be analogous to the well-known correspondence between minimal surfaces in $\mathbb{R}^{3}$ and surfaces of constant mean curvature one in $\mathbb{H}^{3}$.

### 2.1 New directions: IA-spheres with "natural" singularities

By studying the Cauchy problem for IA-spheres, one learns that singularities can determine the immersion, see [1]. Thus, it is natural to consider locally convex immersions with parallel affine normal lines which have some "natural" singularities. In that sense, the correspondence between some SL-immersion and IA-spheres, given in Subsection §1.3, motivates the following definition:

Definition 1 A map $\psi=(x, u): \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}$ is called an Improper affine map, (IA-map for short), if there is a SL-immersion $L:=x+\sqrt{1} n: \Sigma \longrightarrow \mathbb{C}^{2}$ s. t.

$$
\psi^{L}:=\left(x,-\int<n, d x>\right)
$$

coincides with $\psi$ up to a vertical translation.
Non-regular points of $\psi$ correspond with degenerate points of $\left.d s^{2}=<d x, d x\right\rangle$. We shall refer to $d s^{2}$ as the flat fundamental form of $\psi$.

Remark 2 It is clear from Theorems 1 and 2, that at the non-degenerate points of $d s^{2}$, the induced metric $d \tau^{2}$ is conformal to the affine metric $\sigma$.

From now on, for any IA-map $\psi: \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}, \psi:=(x, u)$ we are going to consider on $\Sigma$ the conformal structure given by the induced metric $d \tau^{2}$ of its associated SL-immersion $L_{\psi}$.
B.Y. Chen and J.M. Morvan proved in [9] that up to a change of the complex structure in $\mathbb{C}^{2}$, every minimal Lagrangian immersion in $\mathbb{C}^{2}$ is a complex curve. To be precise, with the above notation and as every SL-immersion is a minimal Lagrangian immersion (see [14]), we can see, after some straightforward computations, that there exists a complex regular curve $\alpha: \Sigma \longrightarrow \mathbb{C}^{2}, \alpha:=(F, G)$, such that if we identify vectors of $\mathbb{R}^{2}$ with complex numbers in the standard way, then we can write

$$
\begin{equation*}
x=G+\bar{F}, \quad n=\bar{F}-G \tag{16}
\end{equation*}
$$

and since the inner product of two vectors $\zeta_{i}=r_{i}+\sqrt{-1} s_{i}, i=1,2$, is given by $\left\langle\zeta_{1}, \zeta_{2}\right\rangle=$ $\Re\left(\zeta_{1} \overline{\zeta_{2}}\right)$, then the flat fundamental form, the induced metric and $\sigma:=-\langle d x, d n\rangle$ are given, respectively, by

$$
\begin{align*}
d s^{2} & =|d F|^{2}+|d G|^{2}+d G d F+\overline{d G d F} \\
d \tau^{2} & =2\left(|d G|^{2}+|d F|^{2}\right)  \tag{17}\\
\sigma & =|d G|^{2}-|d F|^{2} .
\end{align*}
$$

Moreover, the non trivial part of the Gauss map of $L_{\psi}$ (see [9]) is the holomorphic map $\nu: \Sigma \longrightarrow \mathbb{C} \cup\{\infty\}$ given by

$$
\begin{equation*}
\nu:=\frac{d F}{d G} \tag{18}
\end{equation*}
$$

which will be called Lagrangian Gauss map of $\psi$.
Using Theorem 1, Theorem 2 and (16) we have the following holomorphic representation which is a generalization of Theorem 3:

## Theorem 6 (Complex representation:)

- Let $\psi=(x, u): \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}$ be an IA-map. Then there exists a regular planar complex curve $\alpha:=(F, G): \Sigma \longrightarrow \mathbb{C}^{2}$ such that,

$$
\begin{equation*}
\psi=\left(G+\bar{F}, \frac{1}{2}\left(|G|^{2}-|F|^{2}\right)+\Re\left(G F-2 \int F d G\right)\right) \tag{19}
\end{equation*}
$$

- Conversely, given a Riemann surface $\Sigma$ and a complex curve $\alpha:=(F, G): \Sigma \longrightarrow \mathbb{C}^{2}$, then (19) gives an IA-map which is well defined if and only if $\int F d G$ does not have real periods.
The pair $(F, G)$ is called the Weierstrass data of $\psi$.
Remark 3 From (17) and (19), the singular points of $\psi$ correspond with the points where $|d F|=|d G|$, that is, with the points where the flat fundamental form and the BerwaldBlaschke metric $\sigma$ degenerate.

Remark 4 Let $a, b, \mu \in \mathbb{C}$, such that $|a|^{2}-|b|^{2}=1$. Then the Weierstrass data $\alpha:=(F, G)$ and $\hat{\alpha}:=(a F+b G+\mu, \bar{b} F+\bar{a} G+\bar{\mu})$, give affinely equivalent improper affine maps.

### 2.2 IA-maps as relatives of flat fronts in $\mathrm{H}^{3}$

Kokubu, Umehara and Yamada have shown, see Proposition 2.5 in [17], that any flat front $f: \Sigma \longrightarrow \mathbb{H}^{3}$ is the projection, $f=E_{f} E_{f}^{\star}$, of a holomorphic legendrian immersion $E_{f}$ : $\widetilde{\Sigma} \longrightarrow P S L(2, \mathbb{C})$ defined on the universal cover $\widetilde{\Sigma}$ of $\Sigma$. Moreover, if we set

$$
E_{f}^{-1} d E_{f}=\left(\begin{array}{cc}
0 & d G \\
d F & 0
\end{array}\right),
$$

the first and the second fundamental forms are represented as:

$$
\begin{aligned}
I & =|d F|^{2}+|d G|^{2}+2 \Re(d F d G) \\
I I & =|d G|^{2}-|d F|^{2} .
\end{aligned}
$$

As for flat surfaces, the pair $(d F, d G)$ will be the Weierstrass data of $f$.
Using this holomorphic resolution and the conformal representation of IA-maps described in Theorem 6, we have that IA-maps can be considered as relatives of flat fronts in $\mathbb{H}^{3}$. In fact, every flat front $f: \Sigma \longrightarrow \mathbb{H}^{3}$ with Weirestrass data $f \equiv(d F, d G)$ is intimately related to the IA-map $\psi: \widetilde{\Sigma} \longrightarrow \mathbb{R}^{2} \times \mathbb{R}$ with W-data $(F, G)$. The first and second fundamental form of $f$ coincide, respectively, with the flat fundamental form and the affine metric of $\psi$.

### 2.3 Embeddeness and Completeness

For our global considerations we are going to consider the same definition of completeness as in [17]:

Definition 2 An IA-map $\psi=(x, u): \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}$ is called complete if there exists a compact support bilinear symmetric form $T_{e}$ s.t. $d \bar{s}^{2}:=T_{e}+d s^{2}$ is a complete Riemannian metric on $\Sigma$, where $d s^{2}=<d x, d x>$.

From Huber's Theorem (see Theorem 13 in [15]), one has
Proposition 1 Let $\psi$ be a complete IA-map. Then $\Sigma$ is conformallly equivalent to the complement of a a finite pointset $\left\{p_{1}, p_{2}, \cdots, p_{n}\right\}$ in a compact Riemann surface $\bar{\Sigma}$. Moreover, the $W$-data $(F, G)$ of $\psi$ extend meromorphically on $\bar{\Sigma}$.

The points $p_{1}, \cdots, p_{k}$ are the ends of $\psi$.
Consider, $\psi=(x, u): \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}$ a complete IA-map with Weierstrass data $(F, G)$.
Let $p_{j}$ be an end s.t. $|d F|<|d G|$. Then, on a neighborhood $W_{j}$ of $p_{j}$,

$$
G(z)=\left(z-p_{j}\right)^{-m_{j}} G_{0}(z), \quad F(z)=\left(z-p_{j}\right)^{-n_{j}} F_{0}(z)
$$

where, $G_{0}\left(p_{j}\right), F_{0}\left(p_{j}\right) \neq 0, m_{j} \in \mathbb{N}, m_{j} \geq 1, n_{j} \in \mathbb{Z}, n_{j} \leq m_{j}$, and

$$
\begin{aligned}
x(z) & =\left(z-p_{j}\right)^{-m_{j}} G_{0}(z)+\left(\overline{z-p_{j}}\right)^{-n_{j}} \overline{F_{0}(z)} \\
& \left.=\left(z-p_{j}\right)^{-m_{j}}\left[G_{0}(z)+\overline{z-p_{j}}\right)^{m_{j}-n_{j}} e^{i \theta_{j}(z)} \overline{F_{0}(z)}\right],
\end{aligned}
$$

for some $\theta_{j}(z) \in \mathbb{R}$. Hence, $x$ twists $m_{j}$-times around $p_{j}$ and we have proved, [20],
Theorem 7 An end $p$ of $\psi$ is embedded if and only if $G$ and $F$ have at most a single pole at $p$.

### 2.4 Examples of complete IA-maps with embedded ends:

1. The elliptic paraboloid: It can be obtained by taking $\bar{\Sigma}$ the Riemann sphere $\mathbb{S}^{2} \equiv$ $\mathbb{C} \cup\{\infty\}, \Sigma=\mathbb{C}$ and Weierstrass data $(z, k z)$, where $k$ is constant. (see Figure 6). It is clear that its Lagrangian Gauss map is constant.
2. Rotational improper affine maps: They are obtained by considering $\bar{\Sigma}$ to be the Riemann sphere $\mathbb{S}^{2}, \Sigma=\mathbb{C} \backslash\{0\}$ and Weierstrass data $\left(z, \pm r^{2} / z\right), r \in \mathbb{R} \backslash\{\underline{\{0\}}$, (see Figure 1). In this case, the Lagrangian Gauss map has symmetric ends, i.e. $\overline{\nu(0)}=1 / \nu(\infty)$ (see Figure 6).


Figure 6: Rotational IA-maps.
3. Non-rotational improper affine maps with two embedded ends. More complete examples with only two embedded ends can be obtained by taking $\bar{\Sigma}=\mathbb{S}^{2}, \Sigma=\mathbb{C} \backslash\{0\}$ and Weierstrass data: $(z, a z+b / z+c)$, where $b \in \mathbb{R}, a, c \in \mathbb{C},|a| \neq 1$. In these examples the Lagrangian Gauss map does not have symmetric ends (see Figure 7).


Figure 7: Non-rotational IA-maps with two ends
4. Some multivalued improper affine map: By considering $\bar{\Sigma}$ as the Riemann sphere $\mathbb{S}^{2}$, $\Sigma=\mathbb{C} \backslash\{0\}$ and Weierstrass data $\left(z, \pm \sqrt{-1} r^{2} / z\right), r \in \mathbb{R} \backslash\{0\}$ we obtain multivalued complete improper affine maps with a vertical period and two embedded ends. (see Figure 8)


Figure 8: A multivaluted IA-map
5. Improper affine maps of genus 0 and $n$-ends: Consider $n$-arbitrary points $p_{1}, \cdots p_{n} \in \mathbb{S}^{2}$. It is not a restriction to assume that $p_{n}=\infty$. We set $\Sigma=\mathbb{C} \backslash\left\{p_{1}, \cdots, p_{n-1}\right\}$ and the Weierstrass data $(F, G)$ given by

$$
G(z)=z, \quad F(z)=\frac{r_{1}}{z-p_{1}}+\cdots+\frac{r_{n-1}}{z-p_{n-1}},
$$

for some real numbers $r_{1}, \cdots, r_{n}$. Then (19) defines a well-defined improper affine map with embedded ends $p_{1}, \cdots, p_{n}$.
6. Complete improper affine map of genus one with three embedded ends: Let $\bar{\Sigma}=\mathbb{C} / L$ represent the torus obtained as the quotient space of the complex plane by the lattice $L=\{m+\sqrt{-1} n \mid m, n \in \mathbb{Z}\}$. Consider $\Pi$ the canonical projection from $\mathbb{C}$ onto $\bar{\Sigma}$ and $\wp$ the Weierstrass function associated with the lattice $L$. Then, $\wp$ induces a well-defined meromorphic function (also denoted by $\wp$ ) on the torus $\bar{\Sigma}$ such that,

$$
\left(\wp^{\prime}\right)^{2}=4 \wp\left(\wp^{2}-a^{2}\right),
$$

where $a=\wp(1 / 2)=-\wp(-i / 2)$. If we set $\Sigma=\bar{\Sigma} \backslash\{\Pi(0), \Pi(1 / 2), \Pi(i / 2)\}$ and the Weierstrass data $(F, G)$ given by

$$
F=r \frac{\wp^{\prime}}{\wp+a}, \quad G=\frac{\wp^{\prime}}{\wp-a},
$$

for some real number $r \in \mathbb{R}, r \neq 1$, then it is not difficult to see that (19) gives a well-defined complete improper affine map with genus one and three embedded ends (see Figure 9).


Figure 9: A genus one IA-map with three ends

### 2.5 Global results

The holomorphic representation for IA-maps let us characterize some of the above examples, [20].

First, the elliptic paraboloid can be characterized as follows,
Theorem 8 A complete IA-map is an elliptic paraboloid if and only if one of the following assertions holds:

- Its lagrangian map is constant.
- It has only one end which is embedded.

Remark 5 The assumption of embeddeness in the above result is essential. Indeed, by taking $\bar{\Sigma}=\mathbb{S}^{2}, \Sigma=\mathbb{C}$ and the Weierstrass data $\left(z, z+z^{2}\right)$ we obtain a complete improper affine map with only one end which is non-embedded and non affinely equivalent to the elliptic paraboloid. (see Figure 10)


Figure 10: Non-embededed IA-map with only one end

Finally, complete complete IA-maps with two embedded ends can be also classified thus:
Theorem 9 [20] A complete improper affine map $\psi: \Sigma \longrightarrow \mathbb{R}^{2} \times \mathbb{R}$ with exactly two ends, which are embedded, is affinely equivalent to either a rotational improper affine map or to one of the examples of improper affine maps described in Example 3.

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