Compact complete proper minimal immersions in strictly convex bounded regular domains of $\mathbb{R}^3$

Antonio Alarcón

Departamento de Matemática Aplicada, Universidad de Murcia, E-30100 Espinardo, Murcia, Spain

Abstract. For any strictly convex bounded regular domain $C$ of $\mathbb{R}^3$ we find a compact Riemann surface $\mathcal{M}$, an open domain $M \subset \mathcal{M}$ with arbitrary finite topological type, and a conformal complete proper minimal immersion $X : M \to C$ which can be extended to a continuous map $X : \mathcal{M} \to C$.

Keywords: Complete minimal surface, Proper immersion, Plateau problem, Limit set
PACS: 02.40.-k

INTRODUCTION AND PRELIMINARIES

One of the central questions in the global theory of complete minimal surfaces in $\mathbb{R}^3$ has been the Calabi-Yau problem, which consists of determining the existence or not of a complete bounded minimal surface in $\mathbb{R}^3$. The most important result in this line is the construction of a complete minimal disk in a ball [14]. After this discovery, new questions related to the embeddedness, properness and topology of surfaces of this type were posed [17].

Concerning the embedded question, complete embedded minimal surfaces in $\mathbb{R}^3$ with either finite genus and countably many ends or positive injectivity radius are proper in $\mathbb{R}^3$ [4, 12, 13]. In particular, they must be unbounded.

Regarding the properness of the examples, a domain $\mathcal{O}$ in $\mathbb{R}^3$ which is either convex or smooth and bounded, admits a complete proper minimal immersion of any open surface [9, 10, 2, 6]. In contrast to this result, any Riemannian three-manifold contains many nonsmooth domains with compact closure which do not admit any complete properly immersed surfaces with at least one annular end and bounded mean curvature [7, 8].

The study of the Calabi-Yau problem gave rise to new lines of work and techniques. Among other things, these new ideas established a surprising relationship between the theory of complete minimal surfaces in $\mathbb{R}^3$ and the Plateau problem. This problem consists of finding a minimal surface spanning a given family of closed curves in $\mathbb{R}^3$, and it is solved for any Jordan curve [5, 16]. The link between complete minimal surfaces and the Plateau problem is the existence of compact complete minimal immersions in $\mathbb{R}^3$, according to the following definition [3, 1].

Definition 1 By a compact minimal immersion we mean a minimal immersion $X : M \to \mathbb{R}^3$, where $M$ is an open region of a compact Riemann surface $\mathcal{M}$, and such that $X$ can be extended to a continuous map $X : \mathcal{M} \to \mathbb{R}^3$.

Let $\mathbb{D}$ denote the unit disk in the complex plane. Compact complete conformal minimal immersions $X : \mathbb{D} \to \mathbb{R}^3$ such that $X|_{\partial \mathbb{D}}$ is an embedding and $X(S^1)$ is a Jordan
curve with Hausdorff dimension 1 exist [11]. Moreover, there exist compact complete conformal minimal immersions of Riemann surfaces of arbitrary finite topology [1]. In addition, the set of closed curves given by the limit sets of these immersions is dense in the space of finite families of closed curves in $\mathbb{R}^3$ which admit a solution to the Plateau problem. In spite of this density result, there are some requirements for the limit set of a compact complete minimal immersion. For instance, given $\mathcal{D} \subset \mathbb{R}^3$ a regular domain and $X : \mathcal{D} \to \mathcal{D}$ a compact complete conformal proper minimal immersion, then the second fundamental form of the surface $\partial \mathcal{D}$ at any point of the limit set of $X$ must be non-negatively definite [3, 15].

The aim of the present paper is to join the techniques used in the construction of compact complete minimal immersions in convex domains of $\mathbb{R}^3$, and those used to construct compact complete minimal immersions, in order to prove the following result.

**Theorem 2** For any $C$ strictly convex bounded regular domain of $\mathbb{R}^3$, there exist compact complete proper minimal immersions $X : M \to C$ with arbitrary finite topology.

Moreover, for any finite family $\Sigma$ of closed curves in $\partial C$ which admits a solution to the Plateau problem, and for any $\xi > 0$, there exists a minimal immersion $X : M \to C$ in the above conditions and such that $\delta^H(\Sigma, X(\partial M)) < \xi$, where $\delta^H$ means the Hausdorff distance.

**Convex domains and Hausdorff distance**

Given $E$ a bounded regular convex domain of $\mathbb{R}^3$, and $p \in \partial E$, we let $\kappa_2(p) \geq \kappa_1(p) \geq 0$ denote the principal curvatures of $\partial E$ at $p$ associated to the inward pointing unit normal. Moreover, we write $\kappa_1(\partial E) := \min\{\kappa_1(p) \mid p \in \partial E\} \geq 0$. If $E$ is in addition strictly convex, then $\kappa_1(\partial E) > 0$. If we consider $\mathcal{N} : \partial E \to \mathbb{S}^2$ the outward pointing unit normal or Gauss map of $\partial E$, then there exists a constant $a > 0$ (depending on $E$) such that $\partial E_t = \{p + t \cdot \mathcal{N}(p) \mid p \in \partial E\}$ is a regular (convex) surface $\forall t \in [-a, +\infty[$.

We label $E_t$ as the convex domain bounded by $\partial E_t$.

The set of convex bodies of $\mathbb{R}^3$, i.e. convex compact sets of $\mathbb{R}^3$ with nonempty interior, can be made into a metric space by using the Hausdorff distance. Recall that given two compact subsets $C, D \subset \mathbb{R}^3$, the Hausdorff distance between $C$ and $D$ is defined by $\delta^H(C, D) = \max\{\sup_{x \in C} \inf_{y \in D} \|x - y\|, \sup_{y \in D} \inf_{x \in C} \|x - y\|\}$.

**Riemann surfaces**

Throughout the paper we consider $M'$ a fixed but arbitrary compact Riemann surface of genus $\sigma \in \mathbb{N} \cup \{0\}$, and $d\sigma^2$ a Riemannian metric in $M'$.

Consider a subset $W \subset M'$, and a Riemannian metric $d\tau^2$ in $W$. Given $U, V \subset W$, we denote by $\text{dist}_{(W,d\tau^2)}(U, V)$ the distance in $W$ between $U$ and $V$ with the metric $d\tau^2$. Given a conformal minimal immersion $Y : \overline{W} \to \mathbb{R}^3$, by $d\nu_Y^2$ we mean the Riemannian metric induced by $Y$ in $\overline{W}$. Moreover, we write $\text{dist}_{(\overline{W},Y)}$ instead of $\text{dist}_{(\overline{W},d\nu_Y)}$. 

Let $E \in \mathbb{N}$, and let $D_1, \ldots, D_E \subset M'$ be open disks such that \{\gamma := \partial D_i\}_{i=1}^E$ are analytic Jordan curves and $D_i \cap D_j = \emptyset$ for all $i \neq j$.

**Definition 3** Each curve $\gamma$ is called a cycle on $M'$ and the family $\mathcal{J} = \{\gamma_1, \ldots, \gamma_E\}$ is called a multicyle on $M'$. We denote by $\text{Int}(\gamma_i)$ the disk $D_i$, for $i = 1, \ldots, E$. We also define $M(\mathcal{J}) = M' \setminus \bigcup_{i=1}^E \text{Int}(\gamma_i)$. Notice that $M(\mathcal{J})$ is a hyperbolic Riemann surface with genus $\sigma$ and $E$ ends.

Given $\mathcal{J} = \{\gamma_1, \ldots, \gamma_E\}$ and $\mathcal{J}' = \{\gamma'_1, \ldots, \gamma'_E\}$ two multicycles on $M'$ we write $\mathcal{J}' < \mathcal{J}$ if $\text{Int}(\gamma'_i) \subset \text{Int}(\gamma_i)$ for $i = 1, \ldots, E$. Notice that $\mathcal{J}' < \mathcal{J}$ implies $M(\mathcal{J}) \subset M(\mathcal{J}')$.

Let $\mathcal{J} = \{\gamma_1, \ldots, \gamma_E\}$ be a multicyle on $M'$. If $\epsilon > 0$ is small enough, we can consider the multicyle $\mathcal{J}^\epsilon = \{\gamma_1^\epsilon, \ldots, \gamma_E^\epsilon\}$, where by $\gamma^\epsilon$ we mean the cycle satisfying $\text{Int}(\gamma_i) \subset \text{Int}(\gamma_i^\epsilon)$ and $\text{dist}_{(M', d_{\mathcal{J}})}(q, \gamma) = \epsilon$ for all $q \in \gamma_i^\epsilon$. Notice that $\mathcal{J}^\epsilon < \mathcal{J}$.

### A Preliminary Lemma

Although next lemma can be found in [2], its usefulness in the construction of compact complete minimal immersions has been entirely exploited in this paper.

**Lemma 4** Let $\mathcal{J}$ be a multicyle on $M'$, $X : M(\mathcal{J}) \to \mathbb{R}^3$ a conformal minimal immersion, and $p_0 \in M(\mathcal{J})$ with $X(p_0) = 0$. Consider $E$ a strictly convex bounded regular domain, and $E'$ a convex bounded regular domain, with $0 \in E \subseteq E' \subseteq E'$. Let $a$ and $\epsilon$ be positive constants satisfying that $p_0 \in M(\mathcal{J}^\epsilon)$, $E_{-a}$ makes sense and

$$X(M(\mathcal{J})) \cap M(\mathcal{J}^\epsilon) \subset E \setminus \overline{E_{-a}}. \quad (1)$$

Consider also $b > 0$ such that $E_{-2b-a}$ and $E'_{-b}$ make sense.

Then there exist a multicyle $\mathcal{J}'$ and a conformal minimal immersion $\hat{X} : M(\mathcal{J}') \to \mathbb{R}^3$ with the following properties:

L1. $\hat{X}(p_0) = 0$.

L2. $\mathcal{J}^\epsilon < \mathcal{J}' < \mathcal{J}$.

L3. $1/\epsilon < \text{dist}_{(M(\mathcal{J}'), \hat{X})}(p, \mathcal{J}^\epsilon)$, $\forall p \in \mathcal{J}'$.

L4. $\hat{X}(\mathcal{J}') \subset E' \setminus \overline{E'_{-b}}$.

L5. $\hat{X}(M(\mathcal{J}) \setminus M(\mathcal{J}^\epsilon)) \subset \mathbb{R}^3 \setminus \overline{E_{-2b-a}}$.

L6. $\|\hat{X} - X\| < \epsilon$ in $M(\mathcal{J}^\epsilon)$.

L7. $\|\hat{X} - X\| < m(a, b, \epsilon, E, E')$ in $M(\mathcal{J}')$, where

$$m(a, b, \epsilon, E, E') := \epsilon + \sqrt{\frac{2(\delta^H(E, E') + a + 2b)}{\kappa_1(\partial E)}} + (\delta^H(E, E') + a)^2.$$

For a detailed proof of this lemma we refer the reader to [2, Lemma 5]. We have stated it here just to make this paper self-contained.
THE THEOREM

The Theorem stated in the introduction trivially follows from the following one.

**Theorem 5** Let $C$ be a strictly convex bounded regular domain of $\mathbb{R}^3$. Consider $\mathcal{J}$ a multicyle on the Riemann surface $M'$ and $\phi : M(\mathcal{J}) \to C$ a conformal minimal immersion satisfying $\phi(\mathcal{J}) \subset \partial C$.

Then, for any $\mu > 0$ there exist a domain $M_\mu$ and a complete proper conformal minimal immersion $\phi_\mu : M_\mu \to C$ such that:

(i) $M(\mathcal{J}_\mu) \subset M_\mu \subset M_\mu \subset M(\mathcal{J})$, and $M_\mu$ has the topological type of $M(\mathcal{J})$.

(ii) $\phi_\mu$ admits a continuous extension $\Phi_\mu : \overline{M_\mu} \to C$ and $\Phi_\mu(\partial M_\mu) \subset \partial C$.

(iii) $\|\phi_\mu - \Phi_\mu\| < \mu$ in $\overline{M_\mu}$.

(iv) $\delta^H(\phi(M(\mathcal{J})), \Phi_\mu(M_\mu)) < \mu$.

(v) $\delta^H(\phi(\mathcal{J}), \Phi_\mu(\partial M_\mu)) < \mu$.

The proof of the above Theorem consists roughly of the following. First we look for an exhaustion sequence $\{E^n\}_{n \in \mathbb{N}}$ of strictly convex bounded regular domains covering $C$. Then we use Lemma 4 in a recursive way in order to construct a sequence of minimal immersions $\{X_n\}_{n \in \mathbb{N}}$, starting at $X_1 = \phi$. To construct the immersion $X_{n+1}$ we apply the lemma to the data $X = X_n$, $E = E^n$, $E' = E^{n+1}$ and constants $a = b_n$, $b = b_{n+1}$ and $\varepsilon = \varepsilon_{n+1}$. These constants and the convex domains $\{E^n\}_{n \in \mathbb{N}}$ are suitably chosen so that the sequence $\{X_n\}_{n \in \mathbb{N}}$ has a limit immersion $\phi_\mu$ which satisfies the conclusion of Theorem 5.

**Proof.** Assume $\mathcal{J} = \{\gamma_1, \ldots, \gamma_e\}$. First of all, we define a positive constant $\varepsilon < \mu/2$. In order to do it, consider $\mathcal{T}(\gamma_i)$ a tubular neighborhood of $\gamma_i$ in $\overline{M(\mathcal{J})}$, and denote by $P_i : \mathcal{T}(\gamma_i) \to \gamma_i$ the natural projection, $i = 1, \ldots, e$. Choose $\varepsilon > 0$ small enough so that $\overline{M(\mathcal{J})} \setminus M(\mathcal{J}_\varepsilon) \subset \bigcup_{i=1}^e \mathcal{U}(\gamma_i)$, and

$$\|\phi(p) - \phi(P_i(p))\| < \frac{\mu}{2}, \quad \text{for any } p \in (\overline{M(\mathcal{J})} \setminus M(\mathcal{J}_\varepsilon)) \cap \mathcal{T}(\gamma_i), i = 1, \ldots, e. \quad (2)$$

This choice is possible since the uniform continuity of $\phi$. The definition of $\varepsilon$ is nothing but a trick to obtain statements (iv) and (v) from statement (iii).

Let us describe how to define the family $\{E^n\}_{n \in \mathbb{N}}$ of convex sets. Consider $t_0 > 0$ small enough so that, for any $t \in [0, t_0]$, $C_{-t}$ is a well defined strictly convex bounded regular domain, and $\Gamma_{-t} := \phi^{-1}(\partial C_{-t} \cap \phi(M(\mathcal{J})))$ is a multicyle on $M'$.

Let $c_1$ be a positive constant (which will be specified later) small enough so that $c_1^2 \cdot \sum_{k \geq 1} \frac{1}{k^2} < \min\{t_0, \varepsilon\}$, and define, for any natural $n$, $t_n := c_1^2 \cdot \sum_{k \geq n} \frac{1}{k^2}$. Then, $\forall n \in \mathbb{N}$, we consider the strictly convex bounded regular domain $E^n := C_{-t_n}$. Notice that $\bigcup_{n \in \mathbb{N}} E^n = C$, $E^n \subset E^{n+1}$, and

$$\delta^H(E^{n-1}, E^n) = \frac{c_1^2}{n^2}, \quad \forall n \in \mathbb{N}. \quad (3)$$
Finally, consider a decreasing sequence of positives \( \{b_n\}_{n \in \mathbb{N}} \) satisfying

\[
b_1 < 2(t_0 - t_1), \quad \text{and} \quad b_n < \frac{c_1^2}{n^2}. \tag{4}\]

These numbers will take the role of the constants \( a \) and \( b \) of Lemma 4 in the recursive process. Now we use Lemma 4 to construct, for any \( n \in \mathbb{N} \), a family \( \chi_n = \{J_n, X_n, \epsilon_n, \xi_n\} \), where \( J_n \) is a multicycle on \( M' \), \( X_n : M(J_n) \to C \) is a conformal minimal immersion and \( \{\epsilon_n\}_{n \in \mathbb{N}} \) are decreasing sequences of positive real numbers with

\[
\xi_n < \epsilon_n < \frac{c_1}{n^2}. \tag{5}\]

Moreover, the sequence \( \{\chi_n\}_{n \in \mathbb{N}} \) must satisfy the following list of properties:

\begin{itemize}
  \item [(A_n)] \( J^\epsilon < J^{\xi_n-1} < J^{\epsilon_n-1} < J^{\xi_n} < J_n < J_{n-1} \).
  \item [(B_n)] \( \epsilon_n < \text{dist}(M(J^{\xi_n-1}), X_n) \).
  \item [(C_n)] \( ||X_n - X_{n-1}|| < \epsilon_n \) in \( M(J^{\epsilon_n}) \).
  \item [(D_n)] \( ds_{X_n} \geq \alpha_n \cdot ds_{X_{n-1}} \in M(J^{\xi_n-1}) \), where the sequence \( \{\alpha_k\}_{k \in \mathbb{N}} \) is given by \( \alpha_1 := \frac{1}{2} e^{1/2} \), \( \alpha_k := e^{-1/2^k} \) for \( k > 1 \). Notice that \( 0 < \alpha_k < 1 \) and \( \prod_{m=1}^{k} \alpha_m \to 1/2 \).
  \item [(E_n)] \( X_n(p) \in E^n \setminus (E^n)_{-b_n} \), for any \( p \in J_n \).
  \item [(F_n)] \( X_n(p) \in \mathbb{R}^3 \setminus (E^{n-1})_{-b_{n-1}-2b_n} \), for any \( p \in M(J_n) \setminus M(J^{\epsilon_n}) \).
  \item [(G_n)] \( ||X_n - X_{n-1}|| < m(b_{n-1}, b_n, \epsilon_n, E^{n-1}, E^n) \) in \( M(J_n) \), where \( m \) is defined in Lemma 4.
\end{itemize}

To define \( \chi_1 \), we choose \( X_1 = \emptyset \) and \( J_1 = \Gamma_{-t_1-b_{1/2}} \). The first inequality of (4) guarantees that \( J_1 \) is well defined. From this choice we conclude that \( X_1(J_1) \subset \partial C_{-t_1-b_{1/2}} \subset E^1 \setminus (E^1)_{-b_1} \), and so property (E_1) holds. Then, we take \( \epsilon_1 \) and \( \xi_1 \) satisfying (5) and being \( \xi_1 \) small enough so that \( J^\epsilon \subset J^{\xi_1} \). The remainder properties of the family \( \chi_1 \) do not make sense. The definition of \( \chi_1 \) is done.

Now, assume that we have constructed the families \( \chi_1, \ldots, \chi_n \) satisfying the desired properties. Let us show how to construct \( \chi_{n+1} \). First of all, notice that property (E_n) guarantees the existence of a positive constant \( \lambda \) such that \( X_n(M(J_n) \setminus M(J^\lambda_n)) \subset E^n \setminus (E^n)_{-b_n} \). Then, Lemma 4 can be applied to the data

\[
J = J_n, \quad X = X_n, \quad E = E^n, \quad E' = E^{n+1}, \quad a = b_n, \quad \epsilon, \quad b = b_{n+1},
\]

for any \( 0 < \epsilon < \lambda \). Now, consider a sequence of positives \( \{\tilde{\epsilon}_m\}_{m \in \mathbb{N}} \) decreasing to zero and such that

\[
\hat{\epsilon}_m < \min\{\lambda, \xi_n, c_1/(n+1)^2\}, \quad \text{for any } m \in \mathbb{N}. \tag{6}\]

Let \( J_m \) and \( Y_m : M(J_m) \to \mathbb{R}^3 \) be the multicycle and the conformal minimal immersion given by Lemma 4 for the above data and \( \epsilon = \tilde{\epsilon}_m \). Statement (L2) and (6) imply that

\[
J^\xi_n \subset J^\tilde{\epsilon}_m < J_m, \quad \text{for any } m \in \mathbb{N}. \tag{7}\]
Taking (7) into account, (L6) guarantees that \( \{Y_m\}_{m \in \mathbb{N}} \) converges to \( X_n \) uniformly in \( M(\mathcal{F}_n^{\infty}) \). In particular, \( \{dsY_m\}_{m \in \mathbb{N}} \) converges to \( dsX_n \) uniformly in \( M(\mathcal{F}_n^{\infty}) \). Therefore, there exists \( m_0 \in \mathbb{N} \) large enough so that

\[
    dsY_{m_0} \geq \alpha_{n+1} \cdot dsX_n \quad \text{in} \quad M(\mathcal{F}_n^{\infty}).
\]

(8)

Define \( \mathcal{J}_{n+1} := \mathcal{J}_{m_0}, X_{n+1} := Y_{m_0} \), and \( \epsilon_{n+1} := \hat{\epsilon}_{m_0} \). From (7) and statement (L3) in Lemma 4 we deduce that \( 1/\epsilon_{n+1} < \text{dist}_{M(\mathcal{F}_{n+1})}(X_{n+1}, \mathcal{F}_n) \). Then, taking into account (7), the existence of a positive \( \xi_{n+1} \) small enough so that (5), (A_{n+1}) and (B_{n+1}) hold is guaranteed. Properties (C_{n+1}), (D_{n+1}), (E_{n+1}), (F_{n+1}) and (G_{n+1}) follow from (L6), (8), (L4), (L5) and (L7), respectively. The definition of \( \chi_{n+1} \) is done.

Define

\[
    M_\mu := \bigcup_{n \in \mathbb{N}} M(\mathcal{F}_n^{\infty}) = \bigcup_{n \in \mathbb{N}} M(\mathcal{F}_n^{\xi}).
\]

Since \( (A_n) \) holds, \( n \in \mathbb{N} \), the set \( M_\mu \) is an expansive union of domains with the same topological type as \( M(\mathcal{F}) \). Therefore, elementary topological arguments give that \( M_\mu \) is a domain with the same topological type as \( M(\mathcal{F}) \). Furthermore, \( (A_n), n \in \mathbb{N}, \) also imply that

\[
    \overline{M_\mu} = \bigcap_{n \in \mathbb{N}} M(\mathcal{F}_n).
\]

(9)

This is the moment of specifying \( c_1 \). Take it small enough so that

\[
    \sum_{n=2}^{\infty} m(b_{n-1}, b_n, E_n, E_n^{n-1}, E_n) < \epsilon.
\]

(10)

Taking into account (9), (10) and properties \( (G_n) \), \( n \in \mathbb{N} \), we infer that \( \{X_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence uniformly in \( \overline{M_\mu} \) of continuous maps. Hence, it converges to a continuous map \( \Phi_\mu : \overline{M_\mu} \to \mathbb{R}^3 \). Define \( \phi_\mu := (\Phi_\mu)_{|M_\mu} : M_\mu \to \mathbb{R}^3 \). Let us check that \( \phi_\mu \) satisfies the conclusion of the theorem.

- Properties \( (D_n) \), \( n \in \mathbb{N} \), guarantee that \( \phi_\mu \) is a conformal minimal immersion.
- The completeness of \( \phi_\mu \) follows from properties \( (B_n), (D_n) \), \( n \in \mathbb{N} \), and the fact that the sequence \( \{1/\epsilon_n\}_{n \in \mathbb{N}} \) diverges.
- The properness of \( \phi_\mu \) in \( C \) is equivalent to the fact that \( \Phi_\mu(\partial M_\mu) \subset \partial C \). Let us check it. Consider \( p \in \partial M_\mu \). For any \( n \in \mathbb{N} \), let \( p_n \) be a point in \( M(\mathcal{F}_n^{\xi}) \) such that the sequence \( \{p_n\}_{n \in \mathbb{N}} \) converges to \( p \). Fix \( k \in \mathbb{N} \). The convex hull property for minimal surfaces and \( (E_n) \) imply that \( X_n(p_k) \in E^n \), for any \( n \geq k \). Taking limits as \( n \to \infty \) we obtain that \( \Phi_\mu(p_k) \in \overline{C} \). Now, taking limits as \( k \to \infty \), we get that \( \Phi_\mu(p) \in \overline{C} \). On the other hand, \( p \in \partial M_\mu \subset M(\mathcal{F}_n^{\xi}) \setminus M(\mathcal{F}_n^{\xi_{n-1}}) \), \( \forall n \in \mathbb{N} \). Fix \( k \in \mathbb{N} \). Properties \( (F_n) \), \( n \in \mathbb{N} \), imply that \( X_n(p) \in \mathbb{R}^3 \setminus (E^{k-1})_{-b_{k-1}-2b_k} \), for any \( n > k \). Taking limits as \( n \to \infty \) we have that \( \Phi_\mu(p) \in \overline{C} \setminus (E^{k-1})_{-b_{k-1}-2b_k} \). Hence, \( \Phi_\mu(p) \in \overline{C} \setminus (\cup_{k \in \mathbb{N}} (E^{k-1})_{-b_{k-1}-2b_k}) = \overline{C} \setminus C = \partial C \).
- Statement (i) follows from \( (A_n), n \in \mathbb{N} \).
- Statement (ii) trivially holds.
• Taking into account (9), (10) and properties (G\textsubscript{n}), n ∈ N, we conclude that

\[\|\phi - \Phi_\mu\| < \varepsilon \quad \text{in } M_{\mu}.\]  \hspace{1cm} (11)

This inequality implies statement (iii).

• From (2) follows \(\delta^H(\phi(M(\mathcal{J}^\varepsilon)), \phi(M(\mathcal{J}))) < \mu/2\). Then, to prove (iv) we use (11), the fact that \(M(\mathcal{J}^\varepsilon) \subset M_{\mu} \subset M(\mathcal{J})\) and the above inequality in the following way:

\[\delta^H(\Phi_{\mu}(M_{\mu}), \phi(M(\mathcal{J}))) < \delta^H(\Phi_{\mu}(M_{\mu}), \phi(M(\mathcal{J}))) + \delta^H(\phi(M_{\mu}), \phi(M(\mathcal{J}))) < \varepsilon + \delta^H(\phi(M(\mathcal{J}^\varepsilon)), \phi(M(\mathcal{J}))) < \varepsilon + \frac{\mu}{2} < \mu.\]

Finally, let us check statement (v). Consider \(p \in \partial M_{\mu}\). Let \(i \in \{1, \ldots, E\}\) such that \(p \in \mathcal{F}(\gamma_i)\) and label \(q = P_i(p) \in \mathcal{F}\). Then \(\|\Phi_{\mu}(p) - \phi(q)\| < \|\Phi_{\mu}(p) - \phi(p)\| + \|\phi(p) - \phi(q)\| < \varepsilon + \frac{\mu}{2} < \mu\), where we have used (11) and (2). On the other hand, given \(q \in \mathcal{F}\) we can find a point \(p \in \partial M_{\mu}\) such that \(q = P_i(p)\) for some \(i \in \{1, \ldots, E\}\). The above computation gives \(\|\Phi_{\mu}(p) - \phi(q)\| < \mu\). In this way we have proved (v).

The proof is done. □

REFERENCES