The Quotient Algebra A/I is Isomorphic to a Subalgebra of A^{**}

(This is a part of a joint work with Prof. A. To-Ming Lau)

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Abstract

Abstract. Let A be an arbitrary Banach algebra with a bounded approximate identity. We consider A^{**} as a Banach algebra under one of the Arens multiplications. The main result of this talk is the following theorem.

Theorem. Let I be a closed ideal of A with a bounded right approximate identity. Then there is an idempotent element u in A^{**} such that the space Au is a closed subalgebra of A^{**} and the quotient algebra A/I is isomorphic to Au.

$\boldsymbol{Notation.}$ Let A be a Banach algebra.

A. First Arens Product on A^{**}

We equip A^{**} with the first Arens multiplication, which is defined in three steps as follows.

A. First Arens Product on A^{**}

1- For a in A and f in A^*, the element $f.a$ of A^* is defined by

$$< f.a, b >=< f, ab > \quad (b \in A).$$

2- For m in A^{**} and $f \in A^*$, the element $m.f$ of A^* is defined by

$$< m.f, a >=< m, f.a > \quad (a \in A).$$

3- For n, m in A^{**} the product nm in A^{**} is defined by

$$< nm, f > = < n, m.f > \quad (f \in A^*).$$

For m fixed, the mapping $n \mapsto nm$ is weak*−weak* continuous.

Introduction

B. Bounded Right Approximate Identity

$\boldsymbol{B.BRAI}$ (=Bounded Right Approximate Identity). Let (e_i) be a BRAI in A. That is, this is a bounded net and, for $a \in A$, $||ae_i - a|| \to 0$. Then every weak* cluster point of the net (e_i) in A^{**} is a right identity. That is,

For $m \in A^{**}$, $\;me = m$.

Introduction

C. Right Identity

C. Let I be a closed ideal of A with a BRAI (ε_i).
Then any weak* cluster point of this net is a right identity in I^{**}.

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From Now On

A is a Banach algebra with a BAI, e is a fixed right identity in A^{**}, I is a closed ideal of A with a BRAI and $\varepsilon \in I^{**}$ is a right identity of I^{**}. We let

$$u = e - e\varepsilon.$$

Lemma 1

$\boldsymbol{Lemma - 1.}$ u is an idempotent and, for $a \in A$,

a is in I iff $au = 0$.

Proof.

i) $u^2 = (e - e\varepsilon)(e - e\varepsilon)$

$= e - e\varepsilon - e\varepsilon e + e\varepsilon e\varepsilon$

$= e - e\varepsilon - e\varepsilon + e\varepsilon = e - e\varepsilon = u.$

ii) Let $a \in A$. If $a \in I$ then $a\varepsilon = a$ so that $au = a(e - e\varepsilon) = 0$. Conversely, if $au = 0$ then $a = a\varepsilon$ so that $a \in A \cap I^{**} \subseteq I$.

Lemma 2

$\boldsymbol{Lemma - 2.}$ Let $u.A^* = \{u.f : f \in A^*\}$. The set $u.A^*$ is a weak* closed subspace of A^* and $u.A^* = I^\perp$.

Lemma 2

Proof

Proof. It is enough to prove the last assertion: $u.A^* = I^\perp$.

For $a \in I$ and $f \in A^*$,
$< a, u.f >=< au, f >= 0$. So $u.A^* \subseteq I^\perp$.
To prove the reverse inclusion, let $g \in I^\perp$.
Then, for any $a \in I$, $< a, g >= 0$.

Lemma 2

Proof

Let $a \in A$. As $a\varepsilon \in I^{\perp\perp}$, $< a\varepsilon, g >= 0$. Hence $< a, u.g >=< au, g >=$

$< a - a\varepsilon, g >=< a, g >$

so that $u.g = g$. Hence g is in $u.A^*$ and $u.A^* = I^{\perp}$.■

Thus $(A/I)^* = u.A^*$.

Theorem 3

***Theorem* – 3.** The space Au is a closed subalgebra of A^{**} and the quotient algebra A/I is isomorphic to Au

Theorem 3
Proof

Proof. Let a and b be in A. Since $u = e - e\varepsilon$, as one can see easily, $aubu = abu$ so that Au is a subalgebra of A^{**}.

Theorem 3

Proof

Let now $\varphi : A/I \to A^{**}$ be the mapping defined by $\varphi(a+I) = au$. This is a well-defined one-to-one linear operator since $au = 0$ iff $u \in I$. It is also a homomorphism.

The range of φ is Au. For the moment we do not know whether Au is closed or not in A^{**}.

Proof

Our aim is to see that both φ and φ^{-1} are continuous. From this it will follow that the space Au is closed in A^{**} and φ is a Banach algebra isomorphism.

Theorem 3
Proof

Since $(A/I)^* = I^\perp$ and $I^\perp = u.A^*$, for any $a \in A$,

$$||a + I|| = Sup_{||u.f|| \leq 1} | < a + I, u.f > | =$$

$$Sup_{||u.f|| \leq 1} | < au, f > |.$$

Theorem 3

Proof

Since $u.A^*$ is closed in A^*, by the open mapping theorem applied to the linear operator $f \mapsto u.f$, there is a $\beta > 0$ such that

$$u.A^*_1 \supseteq \beta.(u.A^*)_1.$$

Proof

Hence

$$Sup_{||u.f||\leq 1}| < au, f > |$$
$$\leq \frac{1}{\beta} Sup_{||f||\leq 1}| < au, f > | = \frac{1}{\beta}||au||$$

so that $||a + I|| \leq \frac{1}{\beta}||au||$.

Theorem 3

Proof

That is,

$$||au|| = ||\varphi(a + I)|| \geq \beta.||a + I||.$$

This shows that φ^{-1} is continuous.

Proof

Now, since $||u.f|| \leq ||u||.||f||$,

$$||au|| = Sup_{||f|| \leq 1}| < au, f > | = Sup_{||f|| \leq 1}| < a + I, u.f > |$$

$$\leq ||a + I||.||u.f|| \leq ||u||.||a + I||$$

so that

$$||au|| = ||\varphi(a + I)|| \leq ||u||.||a + I||.$$

Proof

This proves that φ is continuous. Hence φ is and isomorphism, Au is closed in A^{**} and the Banach algebras A/I and Au are isomorphic.

$\boldsymbol{Remark - 1.}$ If I is a closed left ideal of A and has a BRAI then the spaces A/I and Au are still isomorphic but as Banach spaces.

Remarks

***Remark* – 2.** As is well-known, every separable Banach space X is isomorphic to a quotient space of ℓ^1. This result shows that the hereditary properties of ℓ^1 do not pass to its quotient spaces.

For the same reason, it is not realistic to expect that the quotient algebra A/I be isomorphic to a subalgebra of A.

Remarks

Actually, if A is commutative and semisimple and if the Gelfand spectrum of A is connected then A has no proper idempotent so that, even if I is complemented in A, the quotient algebra A/I has no chance to be isomorphic to a subalgebra of the form Au of A.

Remarks

On the other hand, even if A has no proper idempotent, in general there are lots of idempotent elements in the second dual of A.

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Consequence 1.

1. For any closed ideal I of A with a BRAI, the algebra A/I has all the hereditary properties of the Banach space A^{**}. For instance, if the space A^{**} is weakly sequentially complete then so is A/I.

Consequence 1.

Recall that the dual space of any von Neumann algebra is weakly sequentially complete. In particular, the spaces $L^1(G)^{**}$ and $A(G)^{**}$ are weakly sequentially complete.

Consequence 2.

2. Let $\varphi : A \to B$ be an onto homomorphism from A onto some Banach algebra B. If the ideal $Ker(\varphi)$ has a BRAI then B is isomorphic to a subalgebra of A^{**}.

Consequence 3.

3. Let $q : A \to A/I$ be the quotient mapping. Let K be a subset of A. Then the set $q(K)$ is closed (or compact, or weakly compact) in A/I iff Ku is closed/compact/weakly compact in Au.

Checking these properties in Au might be easier than checking the same properties in the quotient space A/I.

Consequence 4.

4. Suppose that A is commutative. Determining the multiplier algebra of A/I is equivalent to determining the multiplier algebra of Au.

Thank You For Listening