Images of Wavelet Transforms

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Notation

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$\int_G f(x)\,dx$: Left Haar integration

$$\int_G f(yx)\,dx = \int_G f(x)\,dx = \Delta(y)\int_G f(xy)\,dx$$

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Then $\varphi^\pi_{\xi,\eta} \in B(G) \subseteq C_b(G)$.

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If λ is left regular representation, then $A_\lambda(G) = A(G)$.

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(b) V_η is a unitary transformation intertwining π and $\lambda_{\mathcal{A}_\eta}$

(c) For all $\xi \in \mathcal{H}_\pi$, we have, weakly in $\mathcal{H}_\pi$,

$$\xi = \int_G V_\eta \xi(x) \pi(x) \eta \, dx.$$

A first example

Let $A \in \mathrm{GL}_k(\mathbb{R})$ have $\delta = |\det(A)| \neq 1$ and form the group

$$G = \mathbb{R}^k \rtimes_A \mathbb{Z} = \{[x, n] : x \in \mathbb{R}^k, n \in \mathbb{Z}\}$$

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Consider $\widehat{\mathbb{R}^k}$ as consisting of row vectors. So, for $f \in L^1(\mathbb{R}^k)$, $\widehat{f}(\gamma) = \int_{\mathbb{R}^k} f(x) e^{2\pi i \gamma x} dx$.

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There exists a measurable $\Omega \subseteq \widehat{\mathbb{R}^k}$ such that

(a) $0 < |\Omega| < \infty$

(b) $\Omega A^n \cap \Omega A^m = \emptyset$ if $n \neq m$

(c) There is a null set $N \subseteq \widehat{\mathbb{R}^k}$ with $\widehat{\mathbb{R}^k} = N \bigcup (\cup_{n \in \mathbb{Z}} \Omega A^n)$.

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Note that π has no irreducible subrepresentations if $k > 1$.

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Note that $\lambda_{\mathcal{K}_\pi}$ is quasi-equivalent to π and $\lambda_{\mathcal{K}_\pi^\perp}$ is disjoint from π

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(f) If G has no nontrivial compact subgroup, then $\mathcal{A}_{\pi(x)\eta} \cap \mathcal{A}_{\pi(y)\eta} = \{0\}$ for any $x \neq y$ in G.

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(a) η is a wavelet iff $\|K^{-1/2}\eta\| = 1$.

(b) If η is a wavelet, then so is $\Delta(x)^{1/2}\pi(x)\eta, \forall x \in G$.

Note: In the important examples, K can be concretely identified.

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Suppose G is a compact group. Then $B(G) \subseteq L^2(G)$ and so every irreducible representation π is square-integrable and every nonzero $\eta \in \mathcal{H}_\pi$ is a multiple of a wavelet.

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Classic orthogonality relations

Let π and σ be irreducible representations of G. Let $\eta, \xi \in \mathcal{H}_\pi, \eta', \xi' \in \mathcal{H}_\sigma$. Then

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That is, $K = d_\pi I_{\mathcal{H}_\pi}$ when G is compact.

Let G be compact and $\pi \in \widehat{G}$. Let $\{\nu_1, \cdots, \nu_{d_\pi}\}$ be an orthonormal basis of $\mathcal{H}_\pi$.

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The rest of Peter-Weyl says $L^2(G) = \oplus_{\pi \in \widehat{G}} \mathcal{K}_\pi$.

How much of this holds when G is no longer compact?

Square-integrable π

Let π be a square integrable representation of a second countable locally compact G.

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Theorem

There exists a countable set $\{\eta_j : j \in J\}$ of wavelets for π such that

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Let $\mathcal{D}_\pi = \mathrm{dom} K^{-1/2}$. Then $K^{-1/2}(\mathcal{D}_\pi)$ is a subspace of $\mathcal{H}_\pi$ and $K^{-1/2}$ is a bijection. By separability of G, we can select a countable and linearly independent subset $\mathcal{J}$ of $\{\pi(x)\eta : x \in G\}$ that is still total in $\mathcal{H}_\pi$. Then perform Gram-Schmidt on $K^{-1/2}(\mathcal{J})$.

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$\lambda_{\mathcal{A}_{\eta_j^\pi}}$ is equivalent to π for each $j \in J$.

Example: The Shearlet group

Fix $c \in \mathbb{R}$, $c \neq 0$. Let

$$H_c = \left\{ \begin{pmatrix} a & 0 \\ b & a^c \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

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Let $G_c = \mathbb{R}^2 \rtimes H_c = \{[x, h] : x \in \mathbb{R}^2, h \in H_c\}$. Group product $[x, h][y, k] = [x + hy, hk]$. When $c = 1/2$, this is the shearlet group.

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G_c is an [AR]-group and $\widehat{G_c}^r = \{\pi_+, \pi_-\}$. We look at π_+.

$\mathcal{H}_{\pi_+} = \{f \in L^2(\mathbb{R}^2) : \mathrm{supp}\widehat{f} \subseteq U^+\}$, where U^+ is the upper half plane and

$$\pi_+ \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & a^c \end{pmatrix} \right] f \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{a^{c+1}}} f \begin{pmatrix} a^{-1}(y_1 - x_1) \\ \frac{y_2 - x_2 - a^{-1} b(y_1 - x_1)}{a^c} \end{pmatrix}$$

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$$L^2(G_c) = \mathcal{K}_{\pi_+} \oplus \mathcal{K}_{\pi_-}$$

THANK YOU!