Fourier transform for nilpotent Lie groups
Granada
June 22 2013
Nilpotent Lie algebras and nilpotent Lie groups

Let $g$ be a nilpotent Lie algebra over $\mathbb{R}$, i.e., the sequence of ideals $g_0 = g$, $g_j = [g, g_{j-1}]$ stops with $g_d = \{0\}$ for some $d > 0$. Let $G = \exp(g)$ be the corresponding simply connected connected (nilpotent) Lie group.

Jordan-Hölder basis of $g$: $Z = \{Z_1, \cdots, Z_n\}$, i.e., $g_j := \text{span}\{Z_j, \cdots, Z_n\}$ ideal of $g$, $j = 1, \cdots, n$. 

The dual space of a nilpotent Lie group

Index sets and representations

Index sets and representations

Index sets and representations

Index sets and representations

Index sets and representations

Index sets and representations

An example

Variable groups

Fourier Transform

Un-sufficient data

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds
Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{R}$, i.e; the sequence of ideals

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}^j = [\mathfrak{g}, \mathfrak{g}^{j-1}]$$

stops with $\mathfrak{g}^d = \{0\}$ for some $d > 0$. 
Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{R}$, i.e; the sequence of ideals

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}^j = [\mathfrak{g}, \mathfrak{g}^{j-1}]$$

stops with $\mathfrak{g}^d = \{0\}$ for some $d > 0$. Let $G = \exp(\mathfrak{g})$ be the corresponding simply connected connected (nilpotent) Lie group.
Nilpotent Lie algebras and nilpotent Lie groups

Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{R}$, i.e; the sequence of ideals

$$g_0 = g, \quad g^j = [g, g_{j-1}]$$

stops with $g^d = \{0\}$ for some $d > 0$.

Let $G = \exp(g)$ be the corresponding simply connected connected (nilpotent) Lie group.

Jordan-Hölder basis of $\mathfrak{g}$:

$$\mathcal{Z} = \{Z_1, \cdots, Z_n\}$$

i.e.

$$g_j := \text{span}\{Z_j, \cdots, Z_n\} \quad \text{ideal of } g, \quad j = 1, \cdots, n.$$
Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, let $H = \exp(\mathfrak{h})$. 
Let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{g} \), let \( H = \exp(\mathfrak{h}) \).

A Malcev basis \( \mathcal{Y} = \{ Y_1, \cdots, Y_s \} \) of \( \mathfrak{g} \) modulo \( \mathfrak{h} \) is a basis of \( \mathfrak{g} \) modulo \( \mathfrak{h} \) such that

\[
\sum_{i=j}^{s} \mathbb{R} Y_i + \mathfrak{h}
\]

is a subalgebra for \( j = 1, \cdots, s \).
The dual space of a nilpotent Lie group

Index sets and representations

Index sets and representations

Index sets and representations

An example

Variable groups

Fourier Transform

Un-sufficient data

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, let $H = \exp(\mathfrak{h})$.

A Malcev basis $\mathfrak{Z} = \{Y_1, \cdots, Y_s\}$ of $\mathfrak{g}$ modulo $\mathfrak{h}$ is a basis of $\mathfrak{g}$ modulo $\mathfrak{h}$ such that

$$\sum_{i=j}^{s} \mathbb{R} Y_i + \mathfrak{h}$$

is a subalgebra for $j = 1, \cdots, s$.

The mapping

$$E_{\mathfrak{Z}} : \mathbb{R}^s \times \mathfrak{h} \ni (t_1, \cdots, t_s, U) \mapsto \exp(t_1 Y_1) \cdots \exp(t_s Y_s) \cdot H$$

is a diffeomorphism.
The dual space of a nilpotent Lie group

Index sets and representations

An example

Variable groups

Fourier Transform

Un-sufficient data

Fourier inversion for sub-manifolds

Polarization at \( \ell \) is a subalgebra \( p \) of \( g \) of dimension \( \frac{1}{2}(\dim(g) + \dim(g(\ell))) \) such that

\[
\langle \ell, [p, p] \rangle = \{0\}.
\]
$\ell \in g^*$,

$g(\ell) := \{ U \in g, \langle \ell, [U, g] \rangle = \{0\} \}$,
\( \ell \in g^* , \)

\[
g(\ell) := \{ U \in g, \langle \ell, [U, g] \rangle = \{0\}\},
\]

\[
a(\ell) = \bigcap_{g \in G} g(\text{Ad}^*(g)\ell) = \text{largest ideal of } g \text{ contained in } g(\ell).
\]
\( \ell \in g^* \),
\[
g(\ell) := \{ U \in g, \langle \ell, [U, g] \rangle = \{0\} \},
\]
\[
a(\ell) = \bigcap_{g \in G} g(\text{Ad}^* (g) \ell) = \text{largest ideal of } g \text{ contained in } g(\ell).
\]
\[
\langle \text{Ad}^* (g) \ell, V \rangle := \langle \ell, \text{Ad} (g^{-1}) V \rangle, \ V \in g.
\]
\( \ell \in \mathfrak{g}^* \),

\[ \mathfrak{g}(\ell) := \{ U \in \mathfrak{g}, \langle \ell, [U, g] \rangle = \{0\} \}, \]

\[ a(\ell) = \bigcap_{g \in G} \mathfrak{g}(\operatorname{Ad}^*(g)\ell) = \text{largest ideal of } \mathfrak{g} \text{ contained in } \mathfrak{g}(\ell). \]

\[ \langle \operatorname{Ad}^*(g)\ell, V \rangle := \langle \ell, \operatorname{Ad}(g^{-1})V \rangle, \ V \in \mathfrak{g}. \]

A \textit{polarization} at \( \ell \) is a subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \) of dimension

\[ \frac{1}{2}(\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell))) \] such that
\[ \ell \in g^*, \]
\[ g(\ell) := \{ U \in g, \langle \ell, [U, g] \rangle = \{0\}\}, \]
\[ a(\ell) = \bigcap_{g \in G} g(\text{Ad}^*(g)\ell) = \]
\[ \text{largest ideal of } g \text{ contained in } g(\ell). \]

\[ \langle \text{Ad}^*(g)\ell, V \rangle := \langle \ell, \text{Ad}(g^{-1})V \rangle, \quad V \in g. \]

A \textit{polarization} at \( \ell \) is a subalgebra \( p \) of \( g \) of dimension \( \frac{1}{2}(\text{dim}(g) + \text{dim}(g(\ell))) \) such that
\[ \langle \ell, [p, p] \rangle = \{0\}. \]
Vergne polarisation

Let \( \ell \in g^* \). Let \( \mathcal{Z} = \{Z_1, \cdots, Z_n\} \) be a Jordan-Hölder basis of \( g \): 

Vergne polarization at \( \ell \):

\[
p^{\mathcal{Z}}(\ell) := \sum_{j=1}^{n} g_j(\ell|_{g_j})
\]
Monomial representation:

Let $H = \exp(h) \subset G$ be a closed connected subgroup.
Monomial representation:

Let $H = \exp(h) \subset G$ be a closed connected subgroup. $G/H$ admits a $G$-invariant Borel measure $dx$. 
Monomial representation:

Let $H = \exp(\mathfrak{h}) \subset G$ be a closed connected subgroup. $G/H$ admits a $G$-invariant Borel measure $dx$. Let $\ell \in \mathfrak{g}^*$ with $\langle \ell, [\mathfrak{h}, \mathfrak{h}] \rangle = \{0\}$. 

Monomial representation:

Let $H = \exp(\mathfrak{h}) \subset G$ be a closed connected subgroup. $G/H$ admits a $G$-invariant Borel measure $dx$. Let $\ell \in \mathfrak{g}^*$ with $\langle \ell, [\mathfrak{h}, \mathfrak{h}] \rangle = \{0\}$.

$$\chi_\ell(h) := e^{-2i\pi \langle \ell, \log(h) \rangle}, \quad h \in H.$$
Monomial representation:

Let $H = \exp(\mathfrak{h}) \subset G$ be a closed connected subgroup. $G/H$ admits a $G$-invariant Borel measure $dx$. Let $\ell \in \mathfrak{g}^*$ with $\langle \ell, [\mathfrak{h}, \mathfrak{h}] \rangle = \{0\}$.

$$\chi_\ell(h) := e^{-2i\pi \langle \ell, \log(h) \rangle}, \ h \in H.$$
Definition

\[ \mathcal{H}_{\ell, h} = L^2(G/H, \chi_\ell) = \{ \xi : G \to \mathbb{C}, \text{measurable} \} \]
Definition

\[ \mathcal{H}_{\ell,h} = L^2(G/H, \chi_\ell) = \{ \xi: G \to \mathbb{C} \text{ measurable} , \]
\[ \xi(gh) = \chi_\ell(h^{-1})\xi(g) , g \in G , h \in H \}
\[ \int_{G/H} |\xi(g)|^2 \, dg < \infty. \]
Definition

\[ \mathcal{H}_{\ell,h} = L^2(G/H, \chi_{\ell}) \]

\[ = \{ \xi : G \to \mathbb{C}, \text{ measurable}, \]

\[ \xi(gh) = \chi_{\ell}(h^{-1})\xi(g), g \in G, h \in H \} \]

\[ \int_{G/H} |\xi(g)|^2 dg < \infty. \]

Let

\[ \sigma_{\ell,h}(g) \xi(s) := \xi(g^{-1}s), g, s \in G, \xi \in L^2(G/H, \chi_{\ell}). \]
Definition

\[ \mathcal{H}_{\ell, h} = L^2(G/H, \chi_\ell) \]
\[ = \{ \xi: G \to \mathbb{C}, \text{ measurable} \, , \]
\[ \xi(gh) = \chi_\ell(h^{-1})\xi(g), g \in G, h \in H \} \]
\[ \int_{G/H} |\xi(g)|^2 \, dg < \infty. \]

Let

\[ \sigma_{\ell, h}(g)\xi(s) := \xi(g^{-1}s), g, s \in G, \xi \in L^2(G/H, \chi_\ell). \]
Proposition

For $F \in L^1(G)$:

$$\sigma_{\ell, \hbar}(F)\xi(s) = \int_{G/H} F_{\ell, \hbar}(s, t)\xi(t)dt,$$

where $F_{\ell, \hbar}(s, t) = \int_H F(sht^{-1})\chi_\ell(h)dh$. 
Theorem

Let $\ell \in g^*$ and let $p$ be a polarization at $\ell$. Then $\sigma_{\ell, p}$ is irreducible.
Orbit picture

**Theorem**

- Let $\ell \in g^*$ and let $p$ be a polarization at $\ell$. Then $\sigma_{\ell, p}$ is irreducible.

- Let $\ell_i \in g^*$ and let $p_i, i = 1, 2$ be a polarization at $\ell_i, i = 1, 2$. Then

  $$\sigma_{\ell_1, p_1} \simeq \sigma_{\ell_2, p_2} \iff \text{Ad}^* (G) \ell_2 = \text{Ad}^* (G) \ell_1.$$ 

  Write:

  $$[\pi_{\ell}] := [\sigma_{\pi, p}]$$
The dual space of a nilpotent Lie group

Index sets and representations

An example

Variable groups

Fourier Transform

Un-sufficient data

Fourier inversion for sub-manifolds

Orbit picture

**Theorem**

- Let $\ell \in g^*$ and let $p$ be a polarization at $\ell$. Then $\sigma_{\ell,p}$ is irreducible.

- Let $\ell_i \in g^*$ and let $p_i, i = 1, 2$ be a polarization at $\ell_i, i = 1, 2$. Then

\[ \sigma_{\ell_1,p_1} \simeq \sigma_{\ell_2,p_2} \iff \text{Ad}^*(G)\ell_2 = \text{Ad}^*(G)\ell_1. \]

Write:

\[ [\pi_\ell] := [\sigma_{\pi,p}] \]

- Let $(\pi, H_{\pi}) \in \hat{G} \Rightarrow \exists \ell \in g^*$ such that

\[ [\pi] = [\pi_\ell] \]
A homeomorphism

**Theorem**

The mapping $\mathcal{K}: g^* / G \to \hat{G}$ defined by

$$\mathcal{K}(\operatorname{Ad}^*(G)\ell) := [\pi_\ell]$$

is a homeomorphism
A partition of the orbit space

Index sets:
A partition of the orbit space

Index sets: Let $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ be a Jordan-Hölder basis of $g$ and let $\ell \in g^*$. 
A partition of the orbit space

Index sets: Let \( \mathcal{Z} = \{Z_1, \ldots, Z_n\} \) be a Jordan-Hölder basis of \( g \) and let \( \ell \in g^* \). The index set \( I(\ell) = I^\mathcal{Z}(\ell) \) of \( \ell \in g^* \) is defined by:
A partition of the orbit space

Index sets: Let $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ be a Jordan-Hölder basis of $\mathfrak{g}$ and let $\ell \in \mathfrak{g}^*$. The index set $I(\ell) = I^\mathcal{Z}(\ell)$ of $\ell \in \mathfrak{g}^*$ is defined by:

$I(\ell) = \emptyset$ if $\ell$ is a character.
Index sets: Let $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ be a Jordan-Hölder basis of $\mathfrak{g}$ and let $\ell \in \mathfrak{g}^*$. The index set $I(\ell) = I^\mathcal{Z}(\ell)$ of $\ell \in \mathfrak{g}^*$ is defined by:

$I(\ell) = \emptyset$ if $\ell$ is a character. Otherwise, let

$$j_1 = j_1(\ell) = \max\{j \in \{1, \ldots, n\} \mid Z_j \notin \mathfrak{a}(\ell)\}$$

$$k_1 = k_1(\ell) = \max\{k \in \{1, \ldots, n\} \mid \langle l, [Z_{j_1(\ell)}, Z_k] \rangle \neq 0\}.$$
We let

\[ \nu_1(\ell) : = \langle \ell, [Z_{k_1}, Z_{j_1}] \rangle \]

\[ S_1 = S_1(\ell) : = \frac{1}{\nu_1(\ell)} [Z_{k_1}, Z_{j_1}] \]

\[ Y_1 = Y_1(\ell) : = Z_{j_1} - \frac{\langle \ell, Y_1 \rangle}{\nu_1(\ell)} S_1 \]

\[ X_1 = X_1(\ell) : = Z_{k_1} - \frac{\langle \ell, Z_{k_1} \rangle}{\nu_1(\ell)} S_1. \]
We let

\[ \nu_1(\ell) : = \langle \ell, [Z_{k_1}, Z_{j_1}] \rangle \]

\[ S_1 = S_1(\ell) : = \frac{1}{\nu_1(\ell)}[Z_{k_1}, Z_{j_1}], \]

\[ Y_1 = Y_1(\ell) : = Z_{j_1} - \frac{\langle \ell, Y_1 \rangle}{\nu_1(\ell)} S_1 \]

\[ X_1 = X_1(\ell) : = Z_{k_1} - \frac{\langle \ell, Z_{k_1} \rangle}{\nu_1(\ell)} S_1. \]

Then we have that:

\[ \langle \ell, X_1 \rangle = \langle \ell, Y_1 \rangle = 0, \]

\[ \langle \ell, [X_1, Y_1] \rangle = 1. \] (0.1)

We consider

\[ g^1(\ell) := \{ U \in \mathfrak{g} \mid \langle \ell, [U, Y_1(\ell)] \rangle \geq 0 \} \] (0.2)

which is an ideal of codimension one in \( \mathfrak{g} \).
A Jordan-Hölder basis of \((g^1(\ell), [\cdot, \cdot])\) is given by 
\[
\{ Z_i^1(\ell) \mid i \neq k_1(\ell) \}
\] defined by 
\[
Z_i^1(\ell) = Z_i - \frac{< l, [Z_i, Y_1(\ell)] >}{\nu_1(\ell)} X_1(\ell), i \neq k_1(\ell). \tag{0.3}
\]
A Jordan-Hölder basis of \((g^1(\ell), [\cdot, \cdot])\) is given by 
\[ \{ Z^1_i(\ell) \mid i \neq k_1(\ell) \} \] defined by

\[ Z^1_i(\ell) = Z_i - \frac{\langle l, [Z_i, Y_1(\ell)] \rangle}{\nu_1(\ell)} X_1(\ell), \quad i \neq k_1(\ell). \quad (0.3) \]

As previously we may now compute the indices 
\( j_2(\ell), k_2(\ell) \) of \( l_1 := l|_{g^1(\ell)} \) with respect to this new basis and construct the corresponding subalgebra \( g^2(\ell) \) with its associated basis \( \{ Z^2_i(\ell) \mid i \neq k_1(\ell), k_2(\ell) \} \).
A Jordan-Hölder basis of \((g^1(\ell), [\cdot, \cdot])\) is given by 
\[\{Z_i^1(\ell) \mid i \neq k_1(\ell)\}\] defined by 
\[Z_i^1(\ell) = Z_i - \frac{< l, [Z_i, Y_1(\ell)] >}{\nu_1(\ell)} X_1(\ell), \ i \neq k_1(\ell). \quad (0.3)\]

As previously we may now compute the indices 
\(j_2(\ell), k_2(\ell)\) of \(l_1 := l|_{g^1(\ell)}\) with respect to this new basis 
and construct the corresponding subalgebra \(g^2(\ell)\) with its 
associated basis \(\{Z_i^2(\ell) \mid i \neq k_1(\ell), k_2(\ell)\}\). 
This procedure stops after a finite number \(r_\ell = r\) of steps. Let 
\[l_Z(\ell) = l(\ell) = ((j_1(\ell), k_1(\ell)), \ldots, (j_r(\ell), k_r(\ell)))\]
is called the index of \(\ell\) in \(g\) with respect to the basis 
\(Z = \{Z_1, \ldots Z_n\}\).
It is known that the last subalgebra $g_r(\ell)$ obtained by this construction coincides with the Vergne polarization of $\ell$ in $g$ with respect to the basis $\mathcal{Z}$. 
It is known that the last subalgebra $\mathfrak{g}_r(\ell)$ obtained by this construction coincides with the Vergne polarization of $\ell$ in $\mathfrak{g}$ with respect to the basis $\mathcal{Z}$. The length $|I| = 2r$ of the index set $I(\ell)$ gives us the dimension of the coadjoint orbit $\text{Ad}^*(G)\ell$. 
Partition of $g^*/G$

For an index set $I \in \mathbb{N}^{2j}, j = 0, \ldots, \dim(g/2)$:

$$g_i^* := \{ \ell \in g^*, I(\ell) = I, \langle \ell, X_i(\ell) \rangle = 0, \langle \ell, Y_i(\ell) \rangle = 0, i = 1, \ldots, r \}. $$

Let

$$\mathcal{I} := \left\{ I \in \bigcup_{j=0}^{\dim(g/2)} \mathbb{N}^j, g_i^* \neq \emptyset \right\}. $$
Partition of $g^*/G$

For an index set $I \in \mathbb{N}^{2j}, j = 0, \cdots, \dim(g/2)$:

$g_i^* := \{\ell \in g^*, I(\ell) = I, \langle I, X_i(\ell) \rangle = 0, \langle I, Y_i(\ell) \rangle = 0, i = 1, \cdots, r\}.$

Let

$$\mathcal{I} := \{I \in \bigcup_{j=0}^{\dim(g/2)} \mathbb{N}^j, g_i^* \neq \emptyset\}.$$

Then:

$$g^*/G \simeq g_{\mathcal{I}}^* := \bigcup_{I \in \mathcal{I}} g_i^*$$
Properties of the $\mathfrak{g}_i^*$:

There exists an index $I^{gen} \in \mathcal{I}$ such that

$$\mathfrak{g}^{gen} := \{ \ell \in \mathfrak{g}^*, I(\ell) = I^{gen} \}$$

is $G$-invariant and Zariski open in $\mathfrak{g}^*$. 
Properties of the $\mathfrak{g}^*_I$:

There exists an index $I^{\text{gen}} \in \mathcal{I}$ such that

$$
\mathfrak{g}^{\text{gen}} := \{ \ell \in \mathfrak{g}^*, I(\ell) = I^{\text{gen}} \}
$$

is $G$-invariant and Zariski open in $\mathfrak{g}^*$. There exists an order on $\mathcal{I}$ such that

- $I^{\text{gen}}$ is maximal for this order,
- such that

$$
\mathfrak{g}^*_I := \bigcup_{I' \leq I} \mathfrak{g}^*_{I'}
$$

is Zariski closed in $\mathfrak{g}^*$. 
Realization on $L^2(\mathbb{R}^r)$

Proposition

For every $l \in \mathcal{I}$ the mappings

$$g^*_j \ni l \mapsto X_j(l), \ l \mapsto Y_j(l), \ l \mapsto p^Z(l)$$

are smooth.
Realization on $L^2(\mathbb{R}^r)$

Proposition

- For every $I \in \mathcal{I}$ the mappings
  $$g^*_j \ni \ell \mapsto X_j(\ell), \ell \mapsto Y_j(\ell), \ell \mapsto \mathfrak{p}^Z(\ell)$$
  are smooth.

- The family of vectors $\mathfrak{M}(\ell) = \{X_j(\ell), j = 1, \cdots, r\}$ form a Malcev-basis of $\mathfrak{g}$ modulo $\mathfrak{p}^Z(\ell)$, the vectors $\{Y_j(\ell), j = 1, \cdots, r\}$ form a Malcev basis of $\mathfrak{p}^Z(\ell)$ modulo $\mathfrak{g}(\ell)$. 

Realization on $L^2(\mathbb{R}^r)$

**Proposition**

- For every $I \in \mathcal{I}$ the mappings
  $$\mathfrak{g}_I^* \ni \ell \mapsto X_j(\ell), \ell \mapsto Y_j(\ell), \ell \mapsto p^Z(\ell)$$

  are smooth.

- The family of vectors $\mathfrak{X}(\ell) = \{X_j(\ell), j = 1, \cdots, r\}$ form a Malcev-basis of $\mathfrak{g}$ modulo $p^Z(\ell)$, the vectors $\{Y_j(\ell), j = 1, \cdots, r\}$ form a Malcev basis of $p^Z(\ell)$ modulo $\mathfrak{g}(\ell)$.

- We identify the Hilbert space $L^2(G/P^Z(\ell), \chi_\ell)$ with $L^2(\mathbb{R}^{r_\ell})$ using the unitary operator:
  $$U_\ell(\eta) = \eta \circ E^Z_\ell \in L^2(\mathbb{R}^{r_\ell}), \eta \in L^2(G/P^Z(\ell), \chi_\ell).$$
An example

Let $g = \text{span} \{ A, B, C, D, U, V \}$.


($s \in R^*$).
Let $\ell \in g^*$

$$\mu = \langle \ell, U \rangle, \langle \ell, V \rangle = \nu.$$

$\triangleright \nu \neq 0 \Rightarrow$

$$g^1(\ell) = \text{span}\{A, B - \frac{s\mu}{\nu} C, D, U, V\},$$

$$j_1(\ell) = 4, k_1(\ell) = 3$$
Let $\ell \in g^*$

$$
\mu = \langle \ell, U \rangle, \langle \ell, V \rangle = \nu.
$$

$\nu \neq 0 \Rightarrow$

$$
g^1(\ell) = \text{span}\{A, B - \frac{s\mu}{\nu} C, D, U, V\},
$$

$j_1(\ell) = 4$, $k_1(\ell) = 3$

$$
Z_1^1 = A, Z_2^1 = B - \frac{s\mu}{\nu} C,
$$

$$
Z_4^1 = D, Z_5^1 = U, Z_6^1 = V.
$$
\([Z_1^1, Z_2^1]_{s,\mu,\nu} = Z_5^1 - \frac{s\mu}{\nu} Z_6^1\),

\([Z_2^1, Z_4^1]_{s,\mu,\nu} = sZ_5^1 - \frac{s\mu}{\nu} Z_6^1\).

\(j_2(\ell) = 2, k_2(\ell) = 1, \text{ if } s \neq 1\).
If \( \nu = 0, \mu \neq 0 \Rightarrow g^1(\ell) = \text{span}\{A, C, D, U, V\}\)
If $\nu = 0$, $\mu \neq 0 \Rightarrow g^1(\ell) = \text{span}\{A, C, D, U, V\}$ and $j_1(\ell) = 4$, $k_1(\ell) = 2$. 
Variable groups.

**Definition**

A variable locally compact group is a pair 

$$(B, G)$$

where $B$ and $G$ are locally compact topological spaces, such that for every $\beta \in B$ there exists a group multiplication $\cdot_{\beta}$ on $G$, which turns $(G, \cdot_{\beta})$ into a topological group, such that
Variable groups.

**Definition**
A variable locally compact group is a pair 

$$(B, G)$$

where $B$ and $G$ are locally compact topological spaces, such that for every $\beta \in B$ there exists a group multiplication $\cdot_\beta$ on $G$, which turns $(G, \cdot_\beta)$ into a topological group, such that

$$B \times (G \times G) \mapsto G, (\beta, (s, t)) \mapsto s \cdot_\beta t$$

is continuous.
Definition

A variable nilpotent Lie algebra is a triple

\[(g, \mathcal{Z}, \mathcal{B})\]

of a real finite dimensional vector space \(g\), of a basis \(\mathcal{Z} = \{Z_1, \cdots, Z_n\}\) of \(g\) and a smooth manifold \(\mathcal{B}\), such that

- for every \(\beta \in \mathcal{B}\) there is a Lie algebra product \([., .]_\beta\) on \(g\),
- \([Z_i, Z_j]_\beta = \sum_{k=j+1}^n c_{k}^{ij}(\beta)Z_k, 1 \leq i < j \leq n\)
- and such that the functions \(\beta \rightarrow c_{k}^{ij}(\beta)\) are all smooth.
Definition

\[ l^\infty(\hat{G}) := \{ (\varphi(\ell) \in \mathcal{K}(\mathcal{H}_\ell)_{\ell \in g_I^*}, \| \varphi \|_\infty := \sup_{\ell \in g_I^*} \| \varphi(\ell) \|_{op} < \infty \} , \]

The dual space of a nilpotent Lie group
Index sets and representations
An example
Variable groups

Fourier Transform
Fourier inversion for sub-manifolds
Fourier inversion for sub-manifolds
Fourier inversion for sub-manifolds
**Fourier transform**

**Definition**

\[ \mathcal{L}^\infty(\hat{G}) := \{ (\varphi(\ell) \in \mathcal{K}(\mathcal{H}_\ell)_{\ell \in \mathfrak{g}_I^*}, \| \varphi \|_\infty := \sup_{\ell \in \mathfrak{g}_I^*} \| \varphi(\ell) \|_{op} < \infty \}, \]

Write for \( \ell \in \mathfrak{g}_I^* \), \((\pi_\ell, \mathcal{H}_\ell) = (\sigma_\ell, p^\mathcal{Z}(\ell), L^2(\mathbb{R}^{r_\ell})).\)
The dual space of a nilpotent Lie group

Index sets and representations

Definition

\[ l^\infty(\hat{G}) := \{ (\varphi(\ell) \in \mathcal{K}(\mathcal{H}_\ell)_{\ell \in g^*_I}, \| \varphi \|_\infty := \sup_{\ell \in g^*_I} \| \varphi(\ell) \|_{op} < \infty \}, \]

Write for \( \ell \in g^*_I \), \((\pi_\ell, \mathcal{H}_\ell) = (\sigma_\ell, p^Z(\ell), L^2(\mathbb{R}^{r_\ell}) ). \)

For \( F \in L^1(G) \), let

\[ \mathcal{F}(F)(\ell) = \hat{F}(\ell) := \pi_\ell(F), \ \ell \in g^*_I. \]
Fourier transform

Definition

\[ l^\infty(\hat{G}) := \{(\varphi(\ell) \in K(\mathcal{H}_\ell)_{\ell \in \mathfrak{g}^*_I}, \|\varphi\|_\infty := \sup_{\ell \in \mathfrak{g}^*_I} \|\varphi(\ell)\|_{op} < \infty\}. \]

Write for \( \ell \in \mathfrak{g}^*_I \), \((\pi_\ell, \mathcal{H}_\ell) = (\sigma_\ell, p^Z(\ell), L^2(\mathbb{R}^{r_\ell})\).

For \( F \in L^1(G) \), let

\[ \mathcal{F}(F)(\ell) = \hat{F}(\ell) := \pi_\ell(F), \ \ell \in \mathfrak{g}^*_I. \]

For \( u \in \mathcal{U}(g) \) let

\[ \hat{u}(\ell) = d\pi_\ell(u) \in \mathcal{P}\mathcal{D}(\mathbb{R}^{r_\ell}), \ell \in \mathfrak{g}^*_I \]
Properties of $\hat{u}$

- For every $u \in \mathcal{U}(g)$, for $\ell \in g_I$,

$$d\sigma_{\ell, pZ}(\ell)(u) = \hat{u}(\ell) = \sum_{\alpha \in \mathbb{R}^r} p^u_\alpha(\ell) \partial^\alpha$$

with polynomial coefficients $p^u_\alpha(\ell)$ which depend smoothly on $\ell \in g^*_I$.

Let

$$d\mu(u) := (d\sigma_{\ell, pZ}(\ell)(u))_{\ell \in I^{\text{gen}}}$$

- For every $D = \sum_{\alpha \in \mathbb{N}^r} p_\alpha \partial^\alpha$ there exists a smooth mapping $\rho_{D, I} : g^*_I \rightarrow \mathcal{U}(g)$, such that

$$d\sigma_{\ell, pZ}(\ell)(\rho_{D, I}(\ell)) = D, \ell \in g^*_I.$$
Properties of $\hat{F}, F \in S(G)$

- With respect to the basis $\mathfrak{X}(\ell) = \{X_1(\ell), \cdots, X_r(\ell)\}$

  the kernel functions of the operators $\sigma_{\ell,pZ(\ell)}(F) :$

  $$F_{Z}(\ell, x, x') := \int_{PZ(\ell)} F(E_{\mathfrak{X}(\ell)}(x)hE_{\mathfrak{X}(\ell)}(x')^{-1})\chi_{\ell}(h)dh$$

  defined on $g^* \times \mathbb{R}^r \times \mathbb{R}^r$ are smooth and Schwartz in $x, x'$. 
Properties of $\hat{F}, F \in S(G)$

- With respect to the basis $\mathcal{X}(\ell) = \{X_1(\ell), \cdots, X_r(\ell)\}$ the kernel functions of the operators $\sigma_{\ell, p}(F)$:

  $$F_Z(\ell, x, x') := \int_{P_Z(\ell)} F(E_{\mathcal{X}(\ell)}(x)hE_{\mathcal{X}(\ell)}(x')^{-1})\chi_\ell(h)dh$$

defined on $g^*_I \times \mathbb{R}^r \times \mathbb{R}^r$ are smooth and Schwartz in $x, x'$.

- Let $Q \in \mathbb{C}[g]$. For every $I = I^{\text{gen}}$, there exists a partial differential operator $D_Q(I)$ on $g^*_I \times \mathbb{R}^{r_I}$ with polynomial coefficients in the variable $(x, x') \in \mathbb{R}^{r_I} \times \mathbb{R}^{r_I}$ and smooth coefficients in $\ell \in g^*_I$, such that for every $F \in S(G)$:

  $$(QF)_Z(\ell, x, x') = D_Q(\ell)(F_Z)(\ell, x, x').$$
Properties of $\hat{F}, F \in S(G)$

- With respect to the basis $\mathcal{X}(\ell) = \{X_1(\ell), \cdots, X_r(\ell)\}$ the kernel functions of the operators $\sigma_{\ell,p\mathcal{Z}(\ell)}(F)$:
  $$F_{\mathcal{Z}}(\ell, x, x') := \int_{P_{\mathcal{Z}(\ell)}} F(E_{\mathcal{X}(\ell)}(x)hE_{\mathcal{X}(\ell)}(x')^{-1})\chi_{\ell}(h)dh$$

defined on $g^*_i \times \mathbb{R}^r \times \mathbb{R}^r$ are smooth and Schwartz in $x, x'$.

- Let $Q \in \mathbb{C}[g]$. For every $l = l^\text{gen}$, there exists a partial differential operator $D_Q(l)$ on $g^*_i \times \mathbb{R}^{r_i}$ with polynomial coefficients in the variable $(x, x') \in \mathbb{R}^{r_i} \times \mathbb{R}^{r_i}$ and smooth coefficients in $\ell \in g^*_i$, such that for every $F \in S(G)$:
  $$(QF)_{\mathcal{Z}}(\ell, x, x') = D_Q(\ell)(F_{\mathcal{Z}})(\ell, x, x').$$

Let
  $$\delta(Q) := (D_Q(\ell))_{\ell \in l^\text{gen}}$$
Properties of $\hat{F}, F \in L^1(G)$:

1. the operator field $\hat{F}$ is contained in $l^\infty(\hat{G})$.
2. on the subsets $g^*_i, i \in \mathcal{I}$, the mappings
   
   $$\ell \mapsto \hat{F}(\ell) \in \mathcal{K}(L^2(\mathbb{R}^{r_i}))$$

   are operator norm continuous.

3. For every sequence $(\text{Ad}^*(G)\ell_k)_{k \in \mathbb{N}}$ which goes to infinity in $g^*/G$, we have that

   $$\lim_{k \to \infty} \|\hat{F}(\ell_k)\|_{op} = 0.$$
Questions:

- Characterize the image of $C^*(G)$ in $l^\infty(\hat{G})$ under the Fourier transform, i.e. understand how $\pi_\ell(F)$ varies if $\ell \in g_i^*$ approaches the boundary of $g_i^*$. 

The dual space of a nilpotent Lie group

Index sets and representations

Index sets and representations

Index sets and representations

Index sets and representations

An example

Variable groups

Fourier Transform

Un-sufficient data

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds

Fourier inversion for sub-manifolds
Questions:

- Characterize the image of $C^*(G)$ in $l^\infty(\hat{G})$ under the Fourier transform, i.e. understand how $\pi_\ell(F)$ varies if $\ell \in g^*_i$ approaches the boundary of $g^*_i$.

- Characterize the image of $S(G)$ in $l^\infty(\hat{G})$ under the Fourier transform.
Properly converging sequences in $\hat{G}$

Let $I \in \mathcal{I}$ and let $\overline{O} = (\pi_{O_k})$ be a properly converging sequence in $\hat{G}_I$ with limit set $L(\overline{O})$ contained in $\hat{G}_{<I}$, then the elements $\rho \in L(\overline{O})$ are “entangled” by $\overline{O}$.
Properly converging sequences in $\hat{G}$

Let $I \in \mathcal{I}$ and let $\overline{O} = (\pi_{O_k})$ be a properly converging sequence in $\hat{G}_I$ with limit set $L(\overline{O})$ contained in $\hat{G}_{<I}$, then the elements $\rho \in L(\overline{O})$ are “entangled” by $\overline{O}$: For instance if for some $F \in C^*(G)$ we have that $\pi_{O_k}(F) = 0$ for an infinity of $k$’s then

$$\rho(F) = 0, \forall \rho \in L(\overline{O}).$$
Properly converging sequences in $\hat{G}$

Let $I \in \mathcal{I}$ and let $\mathcal{O} = (\pi_{O_k})$ be a properly converging sequence in $\hat{G}_I$ with limit set $L(\mathcal{O})$ contained in $\hat{G}_{< I}$, then the elements $\rho \in L(\mathcal{O})$ are “entangled ” by $\mathcal{O}$: For instance if for some $F \in C^*(G)$ we have that $\pi_{O_k}(F) = 0$ for an infinity of $k$’s then

$$\rho(F) = 0, \forall \rho \in L(\mathcal{O}).$$

**Question:** What is the relation between the sequence of operators

$$(\pi_{O_k}(F) \in \mathcal{B}(L^2(\mathbb{R}^{r_l})))_k$$

and the operator field

$$(\rho(F))_{\rho \in L(\mathcal{O})}?$$
\( S(\widehat{G}) \)

**Definition**

Let

\[
L^2(\widehat{G}) = \{ (\varphi(l))_{l \in g^*_\text{gen}}, l \rightarrow \varphi(l) \text{ measurable, } \int_{\widehat{G}} \|\varphi(l)\|^2_{H-S} d\mu(l) < \infty \}\]
Definition

Let

\[ L^2(\hat{G}) = \{ (\varphi(l))_{l \in g_i^*}, l \to \varphi(l) \text{ measurable}, \int_{\hat{G}} \| \varphi(l) \|^2_{H-S} d\mu(l) < \infty \} \]

Let

\[ S(\hat{G}) = \{ \varphi \in L^2(\hat{G}), \]
The dual space of a nilpotent Lie group

Definition

Let

\[ L^2(\hat{G}) = \left\{ (\varphi(l))_{l \in \mathfrak{g}_{\text{gen}}} : l \rightarrow \varphi(l) \text{ measurable,} \right. \]

\[ \int_{\hat{G}} \| \varphi(l) \|^2_{H-S} d\mu(l) < \infty \} \]

Let

\[ S(\hat{G}) = \left\{ \varphi \in L^2(\hat{G}), \right. \]

\[ d\mu(u)(\varphi) \in L^2(\hat{G}), u \in \mathcal{U}(\mathfrak{g}), \]
Definition
Let

\[ L^2(\hat{G}) = \{ (\varphi(\ell))_{\ell \in \mathfrak{g}^*}, \ell \to \varphi(\ell) \text{ measurable}, \int_{\hat{G}} \| \varphi(\ell) \|^2_{H-S} d\mu(\ell) < \infty \} \]

Let

\[ S(\hat{G}) = \{ \varphi \in L^2(\hat{G}), d\mu(u)(\varphi) \in L^2(\hat{G}), u \in \mathcal{U}(\mathfrak{g}), \delta(Q)\varphi \in L^2(\hat{G}), Q \in \mathbb{C}[\mathfrak{g}] \}. \]

Theorem
The Fourier transform maps \( S(G) \) onto \( S(\hat{G}) \).
Inverse Fourier transform

**Theorem**

There exists a $G$-invariant polynomial function $P_{gen}$ on $\mathfrak{g}^*$ such that for every $F \in S(G)$:

$$F(g) = \int_{\mathfrak{g}_{gen}^*} \text{tr} (\pi_\ell(g^{-1}) \circ \hat{F}(\ell)) |P_{gen}(\ell)| d\ell,$$

$$= \int_{\hat{G}} \text{tr} (\pi(g^{-1}) \circ \pi(F)) d\mu(\pi), g \in G.$$
Smooth compactly supported operator fields

**Definition**

Let

\[ C_c^\infty(\hat{G}) = \{(\varphi(\ell) \in \mathcal{K}(\mathbb{R}^{r_{1gen}})), \ell \in \mathfrak{g}^\text{gen}\}; \]
Smooth compactly supported operator fields

**Definition**

Let

$$C_c^\infty(\hat{G}) = \{(\varphi(\ell) \in \mathcal{K}(\mathbb{R}^{r_{\text{gen}}}) \text{, } \ell \in \mathfrak{g}_{\text{gen}}^* \text{; support } (\varphi) \text{ compact in } \mathfrak{g}_{\text{gen}}^*)\}$$
Smooth compactly supported operator fields

**Definition**
Let

\[ C_c^\infty(\hat{G}) = \left\{ (\varphi(\ell) \in \mathcal{K}(\mathbb{R}^{r_{gen}})), \ell \in \mathfrak{g}_{\text{gen}}^*; \right. \]

support (\varphi) compact in \( \mathfrak{g}_{\text{gen}}^* \),
the function \( (\ell, x, x') \rightarrow \varphi(\ell)(x, x') \)
is smooth in \( \ell \)
and Schwartz in \( (x, x') \in \mathbb{R}^{r_{gen}} \times \mathbb{R}^{r_{gen}} \).

**Theorem**
*For every \( \varphi \in C_c^\infty(\hat{G}) \) there exists a unique \( F \in S(G) \), such that*

\[ \hat{F} = \varphi. \]
Un-sufficient data

What can we do, if we have only a smooth field 
$(\varphi(\ell) \in \mathcal{K}(L^2(\mathbb{R}^r)))_{\ell \in \mathcal{M}}$ defined on a smooth submanifold of $G$?
What can we do, if we have only a smooth field $(\varphi(\ell) \in \mathcal{K}(L^2(\mathbb{R}^r)))_{\ell \in M}$ defined on a smooth submanifold of $G$?

Example: $M$ is the one point set $\{\pi_\ell\}$
What can we do, if we have only a smooth field $(\varphi(\ell) \in \mathcal{K}(L^2(\mathbb{R}^r)))_{\ell \in M}$ defined on a smooth submanifold of $G$?

Example: $M$ is the one point set $\{\pi_\ell\}$

Let $p$ be a polarization at $\ell$, $\mathfrak{X} = \{X_1, \cdots, X_r\}$ Malcev basis with respect to $p$.

**Theorem**

*(R. Howe)* For every $\varphi \in S(\mathbb{R}^r \times \mathbb{R}^r)$ there exists $F \in S(G)$ such that

$$F_{\ell,p}(E_\mathfrak{X}(x), E_\mathfrak{X}(x')) = \varphi(x, x'), \ x, x' \in \mathbb{R}^r.$$
What can we do, if we have only a smooth field 
\((\varphi(\ell) \in K(L^2(\mathbb{R}^r)))_{\ell \in M}\) defined on a smooth submanifold of \(G\)?

Example: \(M\) is the one point set \(\{\pi_\ell\}\)

Let \(\mathfrak{p}\) be a polarization at \(\ell\), \(\mathfrak{X} = \{X_1, \cdots, X_r\}\) Malcev basis with respect to \(\mathfrak{p}\).

**Theorem**

*(R. Howe)* For every \(\varphi \in S(\mathbb{R}^r \times \mathbb{R}^r)\) there exists \(F \in S(G)\) such that

\[
F_{\ell,\mathfrak{p}}(E_{\mathfrak{X}}(x), E_{\mathfrak{X}}(x')) = \varphi(x, x'), \quad x, x' \in \mathbb{R}^r.
\]

This means that

\[
\sigma_{\ell,\mathfrak{p}}(S(G)) = \mathcal{B}(\mathcal{H}_{\ell,\mathfrak{p}})^\infty.
\]
Theorem (Currey-L-Molitor-Braun) Let $g^*I$ be a fixed layer of $g^*I$. Let $M$ be a smooth sub-manifold of $g^*I$. There exists an open subset $M_0$ of $M$ such that for any smooth kernel function $\Phi$ with compact support $C \subset M_0$, there is a function $F$ in the Schwartz space $S(G)$ such that $\pi_\ell(F)$ has $\Phi(\ell)$ as an operator kernel for all $\ell \in M_0$. Moreover, the Schwartz function $F$ may be chosen such that $\pi_\ell(F) = 0$ for all $\ell \in M \setminus M_0$ and for any $\ell$ in $g^*<I$ and such that the map $\Phi \mapsto F$ is continuous with respect to the corresponding function space topologies.
Theorem

(Currey-L-Molitor-Braun) Let $g^*_i$ be a fixed layer of $g^*$. Let $M$ be a smooth sub-manifold of $g^*_i$. 
Theorem

(Currey-L-Molitor-Braun) Let $\mathfrak{g}_i^*$ be a fixed layer of $\mathfrak{g}^*$. Let $M$ be a smooth sub-manifold of $\mathfrak{g}_i^*$. There exists an open subset $M^0$ of $M$ such that for any smooth kernel function $\Phi$ with compact support $C \subset M^0$, there is a function $F$ in the Schwartz space $S(G)$ such that $\pi_\ell(F)$ has $\Phi(\ell)$ as an operator kernel for all $\ell \in M^0$. Moreover, the Schwartz function $F$ may be chosen such that $\pi_\ell(F) = 0$ for all $\ell \in M \setminus M^0$ and for any $\ell$ in $\mathfrak{g}^* < I$ and such that the map $\Phi \mapsto F$ is continuous with respect to the corresponding function space topologies.
Theorem

(Currey-L-Molitor-Braun) Let $\mathfrak{g}_I^*$ be a fixed layer of $\mathfrak{g}^*$. Let $M$ be a smooth sub-manifold of $\mathfrak{g}_I^*$. There exists an open subset $M^0$ of $M$ such that for any smooth kernel function $\Phi$ with compact support $C \subset M^0$, there is a function $F$ in the Schwartz space $S(G)$ such that $\pi_\ell(F)$ has $\Phi(\ell)$ as an operator kernel for all $\ell \in M^0$. Moreover, the Schwartz function $F$ may be chosen such that $\pi_\ell(F) = 0$ for all $\ell \in M \setminus M^0$ and for any $\ell$ in $\mathfrak{g}_{<I}^*$. 
Theorem

(Currey-L-Molitor-Braun) Let $g_i^*$ be a fixed layer of $g^*$. Let $M$ be a smooth sub-manifold of $g_i^*$. There exists an open subset $M^0$ of $M$ such that for any smooth kernel function $\Phi$ with compact support $C \subset M^0$, there is a function $F$ in the Schwartz space $S(G)$ such that $\pi_{\ell}(F)$ has $\Phi(\ell)$ as an operator kernel for all $\ell \in M^0$. Moreover, the Schwartz function $F$ may be chosen such that $\pi_{\ell}(F) = 0$ for all $\ell \in M \setminus M^0$ and for any $\ell$ in $g^*_{<i}$ and such that the map $\Phi \mapsto F$ is continuous with respect to the corresponding function space topologies.
An application

Let $A \subset Aut(G)$ be a Lie group of auto-morphisms of $G$ acting smoothly on $G$. 
An application

Let $A \subset Aut(G)$ be a Lie group of auto-morphisms of $G$ acting smoothly on $G$.
For instance if $G$ is connected Lie group containing $G$ as nil-radical and $A = Ad(G)$.
An application

Let $A \subset Aut(G)$ be a Lie group of auto-morphisms of $G$ acting smoothly on $G$. For instance if $G$ is connected Lie group containing $G$ as nil-radical and $A = Ad(G)$. Let $J \subset L^1(G)$ be a closed $A$-prime ideal.
An application

Let $A \subset Aut(G)$ be a Lie group of auto-morphisms of $G$ acting smoothly on $G$.
For instance if $G$ is connected Lie group containing $G$ as nil-radical and $A = \text{Ad}(G)$.
Let $J \subset L^1(G)$ be a closed $A$-prime ideal.
For instance: $(\rho, E)$ an irreducible bounded representation $\rho$ of $G$ on a Banach space $E$ and

$$J = \ker(\rho|_G)_{L^1(G)}.$$
\( \hat{G} \) is Baire space, \( L^1(G) \) has the Wiener property and \( J \) is \( A \)-prime
\( \hat{G} \) is Baire space, \( L^1(G) \) has the Wiener property and \( J \) is \( A \)-prime \( \Rightarrow \) the hull \( h(J) \) of \( J \) in \( \hat{G} \) is the closure of an \( A \)-orbit in \( \hat{G} \):

\[
h(J) = \overline{A \cdot \pi \ell} \text{ for some } \ell \in g^*.
\]
Let

\[ J_S := J \cap S(G). \]

**Theorem**

*The ideal \( J_S \) is a closed \( A \)-prime ideal in \( S(G) \).*
Let

\[ J_S := J \cap S(G). \]

**Theorem**

The ideal \( J_S \) is a closed A-prime ideal in \( S(G) \).

\[ \ker(h(J))_S/j(h(J))_S \text{ is nilpotent} \Rightarrow J_S = \ker(h(J))_S. \]
Problem:
Is $J_S$ dense in $J$?
Problem:
Is $J_S$ dense in $J$? Let $\varphi \in L^\infty(G)$, such that

$$\langle \varphi, J_S \rangle = \{0\}.$$
Problem:
Is $J_S$ dense in $J$? Let $\varphi \in L^\infty(G)$, such that
$$\langle \varphi, J_S \rangle = \{0\}.$$ 
Is $\varphi = 0$ on $J$?
If $A \cdot \pi_\ell$ is closed (or locally closed) in $\hat{G}$, then $A \cdot \pi_\ell$ is a smooth manifold.
If $A \cdot \pi_\ell$ is closed (or locally closed) in $\hat{G}$, then $A \cdot \pi_\ell$ is a smooth manifold and the theorem above tells us that $S(G)/J_S \simeq S(A \cdot \pi_\ell)$.
If $A \cdot \pi_\ell$ is closed (or locally closed) in $\hat{G}$, then $A \cdot \pi_\ell$ is a smooth manifold
and the theorem above tells us that $S(G)/J_S \simeq S(A \cdot \pi_\ell)$
and $\varphi$ defines a tempered distribution $d_\varphi$ on $S(A \cdot \pi_\ell)$. 

<table>
<thead>
<tr>
<th>The dual space of a nilpotent Lie group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index sets and representations</td>
</tr>
<tr>
<td>Index sets and representations</td>
</tr>
<tr>
<td>Index sets and representations</td>
</tr>
<tr>
<td>Index sets and representations</td>
</tr>
<tr>
<td>An example</td>
</tr>
<tr>
<td>Variable groups</td>
</tr>
<tr>
<td>Fourier Transform</td>
</tr>
<tr>
<td>Un-sufficient data</td>
</tr>
<tr>
<td>Fourier inversion for sub-manifolds</td>
</tr>
<tr>
<td>Fourier inversion for sub-manifolds</td>
</tr>
<tr>
<td>Fourier inversion for sub-manifolds</td>
</tr>
</tbody>
</table>
If $A \cdot \pi_\ell$ is closed (or locally closed) in $\hat{G}$, then $A \cdot \pi_\ell$ is a smooth manifold and the theorem above tells us that $S(G)/J_S \simeq S(A \cdot \pi_\ell)$ and $\varphi$ defines a tempered distribution $d\varphi$ on $S(A \cdot \pi_\ell)$.

From this one can show that

$$|\langle \varphi, F \rangle| \leq \sup_{\pi \in A \cdot \pi_\ell} \|\pi(F)\|_{op}, F \in L^1(G).$$
Theorem

Suppose that $J \subset L^1(G)$ is $A$-prime and $h(J) = A \cdot \pi$ is a closed $A$-orbit in $\hat{G}$, then $J = \ker(A \cdot \pi)$. 