FIXED POINT AND RELATED GEOMETRIC PROPERTIES ON THE FOURIER AND FOURIER STIELTJES ALGEBRAS OF LOCALLY COMPACT GROUPS

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Outline of Talk

- Historical remarks

- Weak fixed point property and Radon Nikodym property on preduals of von Neumann algebras

- Weak fixed point property of the Fourier algebra

- Fixed point property of the Fourier algebra

- Weak* fixed point property for the Fourier Stieltjes algebra:

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Let K be a bounded closed convex subset of a Banach space. A mapping $T : K \to K$ is called **non-expansive** if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad x, y \in K.$$

In general, K need **NOT** contain a fixed point for T :

Example 1. $E = c_0$: all sequences (x_n), $x_n \in \mathbb{R}$, such that $x_n \to 0$

$$\|(x_n)\| = \sup\{|x_n|\}.$$

Define: $T(x_1, x_2, \dots) = (1, x_1, x_2, \dots)$

$$K = \text{ unit ball of } c_0.$$

Then T is a non-expansive mapping $K \to K$ without a fixed point.

Example 2. $E = \ell^1$: all sequences (x_n) such that $\sum |x_n| < \infty$

$$\|x_n\|_1 = \sum |x_n|.$$

Let $S : \ell^1 \to \ell^1$ be the shift operator:

$$S(x_n) = (0, x_1, x_2, \ldots)$$

$$K = \{(x_n) : x_n \geq 0,\ \|x_n\|_1 = 1\}.$$

Then S is a non-expansive mapping $K \to K$ without a fixed point.

Proposition. *Let* K *be a bounded closed convex subset of a Banach space, and* $T : K \to K$ *is non-expansive, then* T *has an approximate fixed point, i.e.* $\exists$ *a sequence* $x_n \in K$ *such that* $\|T(x_n) - x_n\| \to 0$.

Proof: We assume $0 \in K$. For each $1 > \lambda > 0$, define

$$T_\lambda(x) = T(\lambda x).$$

Then

$$\begin{aligned} \|T_\lambda(x) - T_\lambda(y)\| &= \|T(\lambda x) - T(\lambda y)\| \\ &\le \|\lambda x - \lambda y\| = \lambda \|x - y\| \end{aligned}$$

so by the Banach Contractive Mapping Theorem, $\exists\, x_\lambda \in K$ such that $T_\lambda(x_\lambda) = x_\lambda$.

Now

$$
\begin{aligned}
\|T(x_\lambda) - x_\lambda\| &= \|T(x_\lambda) - T_\lambda(x_\lambda)\| \\
&= \|T(x_\lambda) - T(\lambda x_\lambda)\| \\
&\leq \|x_\lambda - \lambda x_\lambda\| \\
&= (1 - \lambda)\|x_\lambda\| \to 0.
\end{aligned}
$$

Example 3 (Alspach, PAMS 1980)

$$E = L^1[0,1] \qquad \|f\|_1 = \int_0^1 |f(t)|dt$$

$$K = \left\{ f \in L^1[0,1], \quad \int_0^1 f(x)dx = 1, \quad 0 \le f \le 2 \right\}.$$

Then K is **weakly** compact and convex.

$$T : K \to K$$

$$(Tf)(t) = \begin{cases} \min\{2f(2t), 2\}, & 0 \le t \le \frac{1}{2} \\ \max\{2f(2t-1) - 2, 0\}, & \frac{1}{2} < t \le 1. \end{cases}$$

Then T is non-expansive, and fixed point free.

Theorem (T. Dominguez-Benavides, M.A. Japon, and S. Prus, J. of Functional Analysis, 2004)**.** *Let* C *be a nonempty closed convex subset of a Banach space. Then* C *is weakly compact if and only if* C *has the* **generic** *fixed point property for continuous affine maps i.e. if* $K \subseteq C$ *is a nonempty closed convex subset of* C, *and*

$T : K \to K$

T *is continuous and affine, then* T *has a fixed point in* K.

A map $T : K \to K$ is **affine** if for any $x, y \in K$, $0 \leq \lambda \leq 1$, $T\big(\lambda x + (1-\lambda)y\big) = \lambda T\big(x + (1-\lambda)Ty\big)$.

Let X be a bounded closed convex subset of a Banach space E. A point x in X is called a **diametral point** if

$$\sup\{\|x-y\| : y \in X\} = \operatorname{diam}(X).$$

The set X is said to have **normal structure** if every nontrivial (i.e. contains at least two points) convex subset K of X contains a non-diametral point.

Theorem (Kirk, 65)**.** *If X is a weakly compact convex subset of E, and X has normal structure, then every non-expansive mapping $T : X \mapsto X$ has a fixed point.*

Remark:

1. compact convex sets always have normal structure.
2. Alspach's example shows that weakly compact convex sets need **not** have normal structure.

A Banach space E is said to have the **weak fixed point property** (weak-f.p.p.) if for each weakly compact convex subset $X \subseteq E$, and $T : X \to X$ a non-expansive mapping, X contains a fixed point for T.

Theorem (F. Browder, 65)**.** *If E is uniformly convex, then E has the weak fixed point property.*

Theorem (B. Maurey, 81)**.** *c_0 has the weak fixed point property.*

Theorem (T.C. Lim, 81)**.** *ℓ_1 has the weak* fixed point property and hence the weak fixed point property.*

Theorem (Llorens - Fusta and Sims, 1998)**.**

- *Let* C *be a closed bounded convex subset of* c_0. *If the set* C *has an interior point, then* C *fails the weak* f.p.p.

- *There exists non-empty convex bounded subset which is compact in a locally convex topology slightly coarser than the weak topology and fails the weak* f.p.p.

Question: Does weak f.p.p. for a closed bounded convex set in c_0 characterize the set being weakly compact?

Theorem (Dowling, Lennard, Turrett, Proceedings A.M.S. 2004)**.** *A non-empty closed bounded convex subset of* c_0 *has the weak* f.p.p. *for non-expansive mapping* $\Longleftrightarrow$ *it is weakly compact.*

Radon Nikodym Property and Weak Fixed Point Property

Banach space E is said to have **Radon Nikodym property** (RNP) if each closed bounded convex subset D of E is dentable i.e. for any $\varepsilon > 0$, there exists and $x \in D$ such that $x \notin \overline{co}\big(D \backslash B_\varepsilon(x)\big)$, where

$$B_\varepsilon(x) = \{y \in E; \|y - x\| < \varepsilon\}.$$

Theorem (M. Rieffel)**.** *Every weakly compact convex subset of a Banach space is dentable.*

Note: 1. $L^1[0,1]$ does not have f.p.p and R.N.P.

2. ℓ^1 has the f.p.p. and R.N.P.

Question: Is there a relation between f.p.p. and R.N.P.?

Theorem 1 (Mah-Ülger-Lau, PAMS 1997). *Let* M *be a von Neumann algebra. If* M_* *has the* RNP, *then* M_* *has the weak* f.p.p.

Problem 1: Is the converse of Theorem 1 true?

Note: c_0 has the weak f.p.p. but not the R.N.P.

However $c_0 \ncong M_*$, M a von Neumann algebra.

$$M = \text{ von Neumann algebra}$$

$$\subseteq B(H)$$

$$M^+ = \text{ all positive operators in } M$$

$\tau : M^+ \to [0, \infty]$ be a trace i.e. a function on M^+ satisfying:

(i) $\tau(\lambda A) = \lambda \tau(A), \quad \lambda \geq 0, \quad A \in M^+$

(ii) $\tau(A + B) = \tau(A) + \tau(B), \quad A,\ B \in M^+$

(iii) $\tau(A^* A) = \tau(AA^*)$ for all $A \in M$

τ is **faithful** if $\tau(A) = 0, \quad A \in M^+,$ then $A = 0$.

τ is **semifinite** if $\tau(A) = \sup\{\tau(B);\, B \in M^+,\ B \leq A, \quad \tau(B) < \infty\}$.

τ is **normal** if for any increasing net $(A_\alpha) \subseteq M^+$,

$A_\alpha \uparrow A$ in the weak*-topology, then $\tau(A_\alpha) \uparrow \tau(A)$.

Theorem 2 (Leinert - Lau, TAMS 2008)**.** *Let M be a von Neumann algebra with a faithful normal semi-finite trace, then M_* has RNP $\Longleftrightarrow$ M_* has the weak* f.p.p.

$G =$ locally compact group with a fixed left Haar measure λ.

- A continuous unitary representation of G is a pair: $\{\pi, H\}$, where $H =$ Hilbert space and π is a continuous homomorphism from G into the group of unitary operators on H such that for each ξ, $n \in H$,

  $x \longrightarrow \langle \pi(x)\xi, n \rangle$

  is continuous.

- $\{\pi, H\}$ is *irreducible* if $\{0\}$ and H are the only $\pi(G)$-invariant subspaces of H.

- $\{\pi, H\}$ is *atomic* if $\{\pi, H\} \cong \sum \oplus \{\pi_\alpha, H_\alpha\}$ where each π_α is a irreducible representation.

$$L^2(G) = \quad \text{all measurable} \quad f : G \to \mathbb{C}$$

$$\int |f(x)|^2 d\lambda(x) < \infty$$

$$\langle f, g \rangle = \int f(x)\, \overline{g(x)}\, d\lambda(x)$$

$$L^2(G) \quad \text{is a Hilbert space.}$$

Left regular representation:

$$\{\rho, L^2(G)\},$$

$$\rho : G \mapsto B\big(L^2(G)\big),$$

$$\rho(x)h(y) = h(x^{-1}y),\ x \in G,\ h \in L^2(G).$$

$$
\begin{aligned}
G = &\quad \text{locally compact group} \\
A(G) = &\quad \text{Fourier algebra of} \quad G \\
= &\quad \text{subalgebra of} \quad C_0(G) \\
&\quad \text{consisting of all functions} \quad \phi : \\
\phi(x) =& \langle \rho(x)h, k \rangle, \quad h, k \in L^2(G) \\
\rho(x)h(y) =& h(x^{-1}y) \\
\|\phi\| =& \sup \left\{ \left| \sum_{i=1}^{n} \lambda_i \phi(x_i) \right| : \left\| \sum_{i=1}^{n} \lambda_i \rho(x_i) \right\| \leq 1 \right\} \\
\geq & \|\phi\|_\infty .
\end{aligned}
$$

P. Eymard (1964):

$$
\begin{aligned}
A(G)^* &= VN(G) \\
&= \text{ von Neumann algebra in } \quad \mathcal{B}\big(L^2(G)\big) \\
&\qquad \text{generated by} \quad \{\rho(x) : x \in G\} \\
&= \overline{\langle \rho(x) : x \in G \rangle}^{\text{WOT}}
\end{aligned}
$$

If G is abelian and $\widehat{G} =$ dual group of G, then

$$
A(G) \cong L^1(\widehat{G}), \qquad VN(G) \cong L^\infty(\widehat{G})
$$

When G is **abelian**, $\widehat{G} =$ dual group

$$\mathbb{T} = \{\lambda \in \mathbb{C},\ |\lambda| = 1\}$$

$$\widehat{\mathbb{T}} = (\mathbb{Z}, +), \quad \widehat{\mathbb{Z}} = \mathbb{T}.$$

Hence $A(\mathbb{Z}) \cong L^1(\mathbb{T})$.

Theorem (Alspach)**.** *If* $G = (\mathbb{Z}, +)$, *then* $A(\mathbb{Z})$ *does* **not** *have weak* f.p.p.

Question: Given a locally compact group G, when does $A(G)$ have the weak f.p.p.?

Theorem (Mah - Lau, TAMS 1988). *If* G *is a compact group, then* $A(G)$ *has the weak* f.p.p.

Theorem (Mah - Ülger - Lau, PAMS 1997).

a) *If* G *is abelian, then* $A(G)$ *has the weak* f.p.p. $\iff G$ *is compact.*

b) *If* G *is discrete and* $A(G)$ *has the weak* f.p.p., *then* G *cannot contain an infinite abelian subgroup. In particular, each element in* G *must have finite order.*

Example: $G =$ all 2×2 matrices

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \longleftrightarrow (x, y)$$

with $x, y \in R, \quad x \neq 0.$ ("$ax + b$"-group).

Topologize G as a subset of $\mathbb{R}^2$ with multiplication

$$(x, y) \circ (u, v) = (xu, xv + y).$$

Then G is a non-compact group. But $A(G)$ has Radon Nikodym Property (K. Taylor). Hence it must have weak f.p.p.

A locally compact group G is called an [IN]-group if there is a compact neighborhood U of the identity e such that $x^{-1}Ux = U$ for all $x \in G$.

Example: compact groups

discrete groups

abelian groups

Theorem 3 (Leinert - Lau, TAMS 2008)**.** *Let G be an* [IN]*-group.* TFAE:

(a) *G is compact*

(b) *$A(G)$ has weak* f.p.p.

(c) *$A(G)$ has* RNP

Corollary. *Let* G *be a discrete group. Then* $A(G)$ *has the weak* f.p.p. $\Longleftrightarrow G$ *is finite.*

Proof: If G is a [SIN]-group, then $VN(G)$ is finite. Apply Theorems 1 and 2.

- A (discrete) semigroup group S is left reversible if $aS \cap bS \neq \emptyset$ for any $a, b, \in S$.

- S commutative $\Longrightarrow S$ is left reversible.

- We say that a Banach space E has the weak f.p.p. for commutative (left reversible) semigroup if whenever S is a commutative (resp. left reversible) semigroup and K is a weakly compact convex subset of E for on K and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as non-expansive mappings from K into K, then K has a common fixed point for $\mathcal{S}$.

Theorem (R. Bruck, 74)**.** *If a Banach space* E *has the weak* f.p.p., *then* E *has the weak* f.p.p. *for commutative semigroup.*

Corollary. *If* G *is a locally compact group such that* $A(G)$ *has the* RNP, *then* $A(G)$ *has the weak fixed point property for commutative semigroups.*

Theorem 4 (Lau-Mah, JFA 2010)**.** *Let* G *be an* [IN]-*group* TFAE.

(a) G *is compact.*

(b) $A(G)$ *has the weak* f.p.p. *for left reversible semigroup.*

Theorem (Garcia-Falset)**.** *If* H *is a Hilbert space* $\mathcal{K}(H) = C^*$*-algebra of compact operators on* H *has the weak fixed point property.*

- If G is a compact group, then

$$C^*(G) = \overline{\{\rho(f);\ f \in L^1(G)\}} \subseteq \mathcal{K}\big(L^2(G)\big).$$

Hence $C^*(G)$ has the weak fixed point property. Consequently the weak fixed point property for commutative semigroups.

Problem 2. If G is a compact group, does $C^*(G)$ have the weak f.p.p. for left reversible semigroups?

Proposition (Lau-Mah-Ülger, PAMS 1997)**.** $VN(G)$ *has the weak* f.p.p. *for left reversible semigroup if and only if* G *is finite.*

Problem 3 (Bruck)**:** If a Banach space E has the weak f.p.p., does it always have the weak f.p.p. for left reversible (or amenable) semigroup?

Fixed Point Property

Let E be a Banach space, and K be a non-empty bounded closed convex subset of E. We say that K has the **fixed point property (f.p.p.)** if every nonexpansive mapping $T : K \to K$ has a fixed point. We say that E has the fixed point property if every bounded closed convex subset K of E has the fixed point propety.

- ℓ^p, $1 < p < \infty$, has the fixed point property

- ℓ^1 has the weak fixed point property but not the fixed point property

- A closed subspace of $L^1[0,1]$ has the fixed point property if and only if it is reflexive.

Theorem 5 (Leinert and Lau, TAMS 2008)**.** *For* G *locally compact, if a nonzero closed ideal of* A *has the* f.p.p., *then* G *is discrete.*

Corollary. $A(G)$ *has the* f.p.p. $\Longleftrightarrow G$ *is finite.*

Proof. By above, G must be discrete. Since f.p.p. $\Longrightarrow$ weak f.p.p., it follows that $A(G)$ has the weak f.p.p. Consequently, it must be finite.

Theorem (P.K. Lin, Nonlinear Analysis 2008)**.** ℓ^1 *can be renormed to have the* f.p.p.

Theorem (C. Hernandez Lineares and M.A. Japon, JFA 2010)**.** *If* G *is a separable compact group, then* $A(G)$ *can be renormed to have the* f.p.p.

Remark (Dowling, Lennard and Turett, TMAA 1996): This theorem is not true for non-separable groups.

Weak* Fixed Point Property

A **dual Banach space** E is said to have **weak*-f.p.p.** if every weak*-compact convex subset K of E has the fixed point property.

E is said to have the **weak* Kadec-Klee property** if the weak*-topology and norm topology agree on the unit sphere.

Theorem (T.C. Lim, Pacific J. Math. 1980)**.** $\ell_1 = c_0^*$ *has the weak*-f.p.p. property.*

Theorem (C. Lennard, PAMS 1990)**.** *Let* H *be a Hilbert space. Then* $B(H)_*$ *has the weak*-f.p.p.*

G–locally compact group

$$
\begin{aligned}
P(G) = \quad & \text{continuous positive definite} \\
& \text{functions on} \quad G \\
\text{i.e.} \quad & \text{all continuous} \quad \phi : G \to \mathbb{C} \quad \text{such that} \\
& \sum \lambda_i \overline{\lambda}_j \phi(x_i x_j^{-1}) \geq 0, \quad \begin{array}{l} x_1, \ldots, x_n \in G, \\ \lambda_i, \ldots, \lambda_n \in \mathbb{C} \end{array} \\
\text{i.e.} \quad & \text{the } n \times n \text{ matrix } \big(\phi(x_i x_j^{-1})\big) \text{ is positive} \\
\phi \in P(G) \iff & \text{there exists a continuous} \\
& \text{unitary representation} \quad \{\pi, \mathcal{H}\} \\
& \text{of} \quad G, \ \eta \in \mathcal{H}, \quad \text{such that} \\
& \phi(x) = \langle \pi(x)\eta, \eta \rangle, \quad x \in G.
\end{aligned}
$$

Let $B(G) = \langle P(G) \rangle \subseteq CB(G)$ (**Fourier Stieltjes algebra** of G)

Equip $B(G)$ with norm $\|u\| = \sup \left\{ \left| \int f(t)u(t)dt \right| ; f \in L^1(G) \text{ and } |||f||| \leq 1 \right\}$ where

$$|||f||| = \sup\{\|\pi(f)\|; \{\pi, H\} \quad \text{continuous unitary representation of } G\}$$

Let $C^*(G)$ denote the completion of $\left(L^1(G), |||\cdot|||\right)$. Then $C^*(G)$ is a C^*-algebra (the **group C^*-algebra** of G), and $B(G) = C^*(G)^*$.

- When G is *amenable*, then $|||f||| = \|\rho(f)\|$, where ρ is the left regular representation of G.

- When G is abelian, $B(G) \cong M(\widehat{G})$ (measure algebra of $\widehat{G}$), and $C^*(G) \cong C_0(\widehat{G})$.

A dual Banach space E is said to have the *weak* * *-Kadec-Klee property* if the norm and weak * -topology agree on $E_1 = \{x \in E; \|x\| = 1\}$.

Theorem (Lau-Mah, TAMS 88)**.** (a) *For a locally compact group* G, *the measure algebra* $M(G)$ *has the weak** fpp $\iff G$ *is discrete* $\iff M(G)$ *has the weak*-Kadec-Klee* property.*

(b) *If* G *is compact, then* $B(G) = C^*(G)^*$ *has the weak**-fpp.

Theorem (Lau-Mah, TAMS 88/Bekka-Kaniuth-Lau-Schlichting, TAMS 1998)**.** *Let* G *be a locally compact group. Then* G *is compact* $\iff B(G)$ *has the weak* Kadec Klee property.*

Theorem 6 (Fendler-Lau-Leinert, JFA 2013)**.** *If* G *is a locally compact group and* $B(G)$ *has the* w^**-f.p.p. then* G *is compact.*

Theorem (T.C. Lim, Pacific J. Math. 1980)**.** *The dual Banach space* $B(\mathbb{T}) \cong \ell^1(\mathbb{Z})$ *has the* *weak** f.p.p. *for left reversible semigroup.*

Theorem 7 (Fendler-Lau-Leinert, JFA 2013)**.** *For any compact group* G, $B(G)$ *has the* *weak** f.p.p. *for left reversible semigroups.*

When G is separable, Theorem 6 and Theorem 7 were proved by Lau and Mah (JFA, 2010).

Key Lemma

Lemma A. *Let* G *be a compact group, and let* $\{D_\alpha : \alpha \in \Lambda)$ *be a decreasing net of bounded subsets of* $B(G)$, *and* $\{\phi_m : m \in M\}$, *be a* weak* *convergent bounded net with* weak* *limit* ϕ. *Then*

$$\limsup_m \lim_\alpha \sup\{\|\phi_m - \psi\| : \psi \in D_\alpha\} = \lim_\alpha \sup\{\|\phi - \psi\| : \psi \in D_\alpha\} + \limsup_m \|\phi_m - \phi\|.$$

Let C be a nonempty subset of a Banach space X and $\{D_\alpha : \alpha \in \Lambda\}$ be a decreasing net of bounded nonempty subsets of X. For each $x \in C$, and $\alpha \in \Lambda$, let

$$r_\alpha(x) = \sup\{\|x - y\| : y \in D_\alpha\},$$

$$r(x) = \lim_\alpha r_\alpha(x) = \inf_\alpha r_\alpha(x),$$

$$r = \inf\{r(x) : x \in C\}.$$

The set (possibly empty)

$$\mathcal{AC}(\{D_\alpha : \alpha \in \Lambda\}) = \{x \in C : r(x) = r\}$$

is called the *asymptotic center* of $\{D_\alpha : \alpha \in \Lambda\}$ with respect to C and r is called the *asymptotic radius* of $\{D_\alpha : \alpha \in \Lambda\}$ with respect to C.

Theorem 8 (Fendler-Lau-Leinert, JFA 2013)**.** *Let* G *be a compact group. Let* C *be a nonempty* *weak** *closed convex subset of* $B(G)$ *and* $\{D_\alpha : \alpha \in \Lambda\}$ *be a decreasing net of nonempty bounded subsets of* C. *Let* $r(x)$ *be as defined above. Then for each* $s \geq 0$, $\{x \in C : r(x) \leq s\}$ *is* *weak** *compact and convex, and the asymptotic center of* $\{D_\alpha : \alpha \in \Lambda\}$ *with respect to* C *is a nonempty norm compact convex subset of* C.

Theorem (Narcisse Randrianantoania, JFA 2010)**.** *For any* G :

(a) $A(G)$ *has the weak* f.p.p. $\iff$ $A(G)$ *has the* R.N.P. $\iff$ *The left regular representation of* G *is atomic. In this case* $A(G)$ *has the weak* f.p.p. *for left reversible semigroups.*

(b) $B(G)$ *has the weak* f.p.p. $\iff$ $B(G)$ *has* R.N.P. $\iff$ *every continuous unitary representation of* G *is atomic. In this case* $B(G)$ *has the weak* f.p.p. *for left reversible semigroups.*

Theorem 8 answers the following problem: For any locally compact group G does R.N.P. on $B(G)$ imply weak* f.p.p.?

Open problem 5. Let G be a locally compact group. Let $B_\rho(G)$ denote the reduced Fourier-Stieltjes algebra of $B(G)$, i.e. $B_\rho(G)$ is the weak* closure of $C_{00}(G) \cap B(G)$. Then $B_\rho(G) = C_\rho(G)^*$. Does the weak* fixed point property on $B_\rho(G)$ imply G is compact? This is true when G is amenable by Theorem 6, since $B(G) = B_\rho(G)$ in this case.

Open problem 6. Let G be a locally compact group. Does the asymptotic centre property on $B_\rho(G)$ imply that G is compact?

Problem: When G is a topological group,

$P(G) =$ continuous positive definite functions on G

$B(G) =$ linear span of $P(G)$.

Theorem (Lau-Ludwig, Advances of Math 2012). $B(G)^*$ *is a von Neumann algebra.*

Problem 5: When does $B(G)$ have the weak fixed point property?

APPENDIX A

A Banach space X is said to be **uniformly convex** if for each $0 < \varepsilon \leq 2$, $\exists\, \delta > 0$ such that for any $x, y \in X$,

$$\left.\begin{array}{r} \|x\| \leq 1 \\ \|y\| \leq 1 \\ \|x - y\| > \varepsilon \end{array}\right\} \left\|\frac{x+y}{2}\right\| \leq \delta$$

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