Spectral Synthesis in Fourier Algebras of Double Coset Hypergroups

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Some basic Notation and Definitions

A a regular and semisimple commutative Banach algebra

- $\Delta(A) = \{\varphi : A \to \mathbb{C} \text{ surjective homomorphism}\} \subseteq A_1^*$, equipped with the w^*-topology
- Gelfand transformation $a \to \widehat{a}, A \to C_0(\Delta(A))$, $\widehat{a}(\varphi) = \varphi(a)$
- hull of $M \subseteq A$: $h(M) = \{\varphi \in \Delta(A) : \varphi(M) = \{0\}\}$

For a closed subset E of $\Delta(A)$, let

- $k(E) = \{a \in A : \widehat{a} = 0 \text{ on } E\}$
- $j(E) = \{a \in A : \widehat{a} \text{ has compact support disjoint from } E\}$

If I is any ideal of A with $h(I) = E$, then $j(E) \subseteq I \subseteq k(E)$.

Synthesis Notions

Definition

A closed subset E of $\Delta(A)$ is called a

- *set of synthesis* or*spectral set* if $k(E) = \overline{j(E)}$
- *Ditkin set* if $a \in \overline{aj(E)}$ for every $a \in k(E)$.

We say that

- *spectral synthesis holds for A* if every closed subset of $\Delta(A)$ is a set of synthesis.
- *A satisfies Ditkin's condition at infinity* if $\emptyset$ is a Ditkin set, i.e. given any $a \in A$ and $\epsilon > 0$, there exists $b \in A$ such that $\widehat{b}$ has compact support and $\|a - ab\| \leq \epsilon$.

Remark

If A satisfies Ditkin's condition at infinty and $\Delta(A)$ is discrete, then every subset of $\Delta(A)$ is a Ditkin set. In particular, spectral synthesis holds for A.

$L^1(G)$, G locally compact abelian

$\Delta(L^1(G)) = \widehat{G}$, the dual group of G
$\widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}dx, \; f \in L^1(G), \gamma \in \widehat{G}$.

Example

(1) For $n \geq 3$, $S^{n-1} \subseteq \mathbb{R}^n = \Delta(L^1(\mathbb{R}^n))$ fails to be a set of synthesis (L. Schwartz, 1948)

(2) $S^1 \subseteq \mathbb{R}^2$ is a set of synthesis for $L^1(\mathbb{R}^2)$ (C. Herz, 1958).

Theorem

(P. Malliavin, 1959)
Let G be any locally compact abelian group. Then spectral synthesis holds for $L^1(G)$ (if and) only if G is compact.

A more constructive proof than Malliavin's was given by Varopoulos (1967), using tensor product methods.

Further examples

- Every closed set in the coset ring of $\widehat{G}$ is a set of synthesis (and the ideal $k(E)$ has a bounded approximate identity)
- Every closed convex set in $\mathbb{R}^n$ is set of synthesis
- If $\partial(E)$ is compact and countable, then E is a spectral set
- If $E, F \subseteq \widehat{G}$ are Ditkin sets, then $E \cup F$ is a Ditkin set

Problems

(1) E, F sets of synthesis $\Rightarrow$ $E \cup F$ set of synthesis? (Union problem)
(2) E set of synthesis $\Rightarrow$ E Ditkin set? (C-set/S-set problem)

Fourier and Fourier-Stieltjes Algebras

Definition

Let G be a locally compact group. Let $B(G)$ denote the linear span of the set of all continuous positive definite functions on G. Then $B(G)$ can be identified with the dual space of the group C^*-algebra $C^*(G)$ through the duality

$$\langle u, f\rangle = \int_G f(x)u(x)dx,\ \ f \in L^1(G), u \in B(G).$$

With pointwise multiplication and the dual norm, $B(G)$ is a semisimple commutative Banach algebra, the *Fourier-Stieltjes algebra* of G.

The *Fourier algebra* $A(G)$ of G is the closed ideal of $B(G)$ generated by all functions in $B(G)$ with compact support. Note that $A(G) \subseteq C_0(G)$.

P. Eymard, *L'algebre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181-236.

Remark

- The spectrum $\sigma(A(G))$ of $A(G)$ can be canonically identified with G: the map
$$x \to \varphi_x, \quad \varphi_x(u) = u(x), \; u \in A(G),$$
is a homeomorphism from G onto $\sigma(A(G))$.
- Suppose that G is an abelian locally compact group with dual group $\widehat{G}$. Then the Fourier-Stieltjes transform gives isometric isomorphisms
$$M(G) \to B(\widehat{G}) \quad \text{and} \quad L^1(G) \to A(\widehat{G}).$$

Theorem

Let G be an arbitrary locally compact group. Then spectral synthesis holds for $A(G)$ if and only if G is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.

E. Kaniuth and A.T. Lau, Spectral synthesis for $A(G)$ and subspaces of $VN(G)$, Proc. Amer. Math. Soc. **129** (2001), 3253-3263.

This result was later, but independently, also shown by Parthasarathy and Prakash.

Definition

A closed subset E of $\Delta(A)$ is called a *weak spectral set* or *set of weak synthesis* if there exists $n \in \mathbb{N}$ such that

$$a^n \in \overline{j(E)} \quad \text{for every } a \in k(E).$$

The smallest such n is called the *characteristic*, $\xi(E)$, of E.
Weak spectral synthesis holds for A if every closed $E \subseteq \Delta(A)$ is a weak spectral set.

Remark

If E and F are weak spectral sets in $\Delta(A)$, then so is $E \cup F$ and $\xi(E \cup F) \leq \xi(E) + \xi(F)$.

C.R. Warner, *Weak spectral synthesis.* Proc. Amer. Math. Soc. **99** (1987), 244-248.

Examples

(1) For each $n \in \mathbb{N}$, $S^{n-1} \subseteq \mathbb{R}^n = \Delta(L^1(\mathbb{R}^n))$ is a weak spectral set with $\xi(S^{n-1}) = \lfloor \frac{n+1}{2} \rfloor$.

N.Th. Varopoulos, *Spectral synthesis on spheres*. Math. Proc. Cambr. Phil. Soc. **62** (1966), 379-387.

(2) For each $n \in \mathbb{N}$, $\mathbb{T}^\infty = \Delta(L^1(\widehat{\mathbb{T}^\infty}))$ contains a weak spectral set E with $\xi(E) = n$.
(Warner)

(3) $C^n[0,1] =$ algebra of n-times continuously differentiable functions on $[0,1]$; identify $\Delta(C^n[0,1])$ with $[0,1]$. Then, for a closed subset E of $[0,1]$,

- E is a spectral set if and only if E has no isolated points.
- $\xi(E) = n + 1$ otherwise.

(4) (X,d) a compact metric space, $0<\alpha\leq 1$. A function $f:X\to\mathbb{C}$ belongs to $\mathrm{Lip}_\alpha(X)$ if

$$p_\alpha(f)=\sup\left\{\frac{|\ f(x)-f(y)\ |}{d(x,y)^\alpha}:x,y\in X,x\neq y\right\}<\infty.$$

$\mathrm{Lip}_\alpha(X)$: $\|\ f\ \|=\|\ f\ \|_\infty+p_\alpha(f)$, $\quad\Delta(\mathrm{Lip}_\alpha(X))=X$. Then $E\subseteq X$ closed

- is a spectral set if and onyl if E is open in X
- $\xi(E)=2$ otherwise

(5) The *Mirkil algebra*

$$M=\{f\in L^2(\mathbb{T}):f\text{ is continuous on }I=[-\pi/2,\pi/2]\}$$

with convolution and $\|\ f\ \|=\|\ f\ \|_2+\|\ f\ |_I\|_\infty$. Then $\Delta(M)=\mathbb{Z}$ and

- $\xi(E)\leq 2$ for every $E\subseteq\mathbb{Z}$
- $E=4\mathbb{Z}$ and $F=4\mathbb{Z}+2$ are sets of synthesis, but $2\mathbb{Z}=E\cup F$ is not.

A. Atzmon, *On the union of sets of synthesis and Ditkin's condition in regular Banach algebras*. Bull. Amer. Math. Soc. **2** (1980), 317-320.

Theorem

Let G be a locally compact abelian group. If weak spectral synthesis holds for $L^1(G)$, then G is compact. Thus weak spectral synthesis holds for $A(G)$ only if G is discrete.

K. Parthasarathy and S. Varma, *On weak spectral synthesis*. Bull. Austral. Math. Soc. **43** (1991), 279-282.

Theorem

Let G be an arbitrary locally compact group. Then weak spectral synthesis holds for the Fourier algebra $A(G)$ if and only if G is discrete.

E. Kaniuth, *Weak spectral synthesis in commutative Banach algebras*, J. Funct. Anal. **254** (2008), 987-1002.

Hypergroups

Definition

Let H be a locally compact Hausdorff space. Suppose that $M^b(H)$ admits a multiplication $*$, under which it is an algebra, and which satisfies the following conditions:

- For $x, y \in H$, $\delta_x * \delta_y$ is a probability measure with compact support
- $(x, y) \to \delta_x * \delta_y, H \times H \to M^1(H)$ is continuous
- $(x, y) \to \mathrm{supp}(\delta_x * \delta_y), H \times H \to \mathcal{K}(H)$ is continuous
- There exists $e \in H$ such that $\delta_x * \delta_e = \delta_e * \delta_x$ for all $x \in H$
- There exists an involution $x \to \tilde{x}$ such that $(\delta_x * \delta_y)^\sim = \delta_{\tilde{y}} * \delta_{\tilde{x}}$ for all $x, y \in H$
- For $x, y \in H$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $y = \tilde{x}$

Then $(H, *)$ is called a locally compact *hypergroup*

Double Coset Hypergroups

- G locally compact group
- K a compact subgroup of G, with normalized Haar measure μ_K
- $G//K = \{KxK : x \in G\}$, equipped with the quotient topology
- For $x, y \in G$, define a probability measure on $G//K$ by

$$\delta_{KxK} * \delta_{KyK} = \int_K \delta_{KxtyK} d\mu_K(t)$$

This mapping $G//K \times G//K \to M^1(G//K)$ and the involution $KxK \to Kx^{-1}K$ turn $G//K$ into a locally compact hypergroup, a *double coset hypergroup*. A left Haar measure on $G//K$ is given by

$$\int_{G//K} f(\dot{x})d\dot{x} = \int_G f \circ q(x)dx,$$

the image of left Haar measure on G under the quotient map $q : G \to G//K$, $x \to \dot{x} = KxK$.

Spherical Hypergroups

Definition

Let G be a locally compact group. A map $\pi : C_c(G) \to C_c(G)$ is called a *spherical projector* if π and its adjoint $\pi^* : M(G) \to M(G)$ satisfy the following conditions:

- $\pi^2 = \pi$ and $\pi(f) \geq 0$ if $f \geq 0$
- $\pi(\pi(f)g) = \pi(f)\pi(g)$
- $\langle \pi(f), g \rangle = \langle f, \pi(g) \rangle$
- $\int_G \pi(f)(x)dx = \int_G f(x)dx$
- $\pi(\pi(f) * \pi(g)) = \pi(f) * \pi(g)$
- For $x, y \in G$, either $\operatorname{supp} \pi^*(\delta_x) \cap \operatorname{supp} \pi^*(\delta_y) = \emptyset$ or $\operatorname{supp} \pi^*(\delta_x) = \operatorname{supp} \pi^*(\delta_y)$
- $x \to O_x = \operatorname{supp} \pi^*(\delta_x)$, $G \to \mathcal{K}(G)$ is continuous
- For $x, y \in G$, $x \in O_y \Rightarrow x^{-1} \in O_{y^{-1}}$ and $O_{xy} = O_e \Rightarrow O_y = O_{x^{-1}}$

Definition

The set $H = \{O_x : x \in G\}$, equipped with the quotient topology and the product

$$\delta_{\dot{x}} * \delta_{\dot{y}} = \pi^*(\pi^*(\delta_x) * \pi^*(\delta_y))$$

becomes a hypergroup, the *spherical hypergroup* associated to (G, π).

A Haar measure on H is given by

$$\int_H f(\dot{x}) d\dot{x} = \int_G (f \circ q)(x) dx.$$

V. Muruganandam, *Fourier algebra of a hypergroup II. Spherical hypergroups.* Math. Nachr. **281** (2008), 1590-1603.

A similar notion, called *average projector*, appears in work of Damek and Ricci.

Definition

A function f on G is called π*-radial* if $\pi(f) = f$. H is called an *ultraspherical hypergroup* if the modular function on H is π-radial.

The Fourier space of a hypergroup

H a locally compact hypergroup with left Haar measure

- $C^*(H)$ enveloping C^*-algebra of $L^1(H)$
- $B(H)$ space of all coefficient functions of representations of $L^1(H)$ (or $C^*(H)$)
- $B(H) = C^*(H)^*$, eqipped with the dual space norm

Definition

The *Fourier space* $A(H)$ of H is defined to be the closure in $B(G)$ of all functions of the form $f * \widetilde{f}$, $f \in C_c(H)$ where

- $\widetilde{f}(x) = \overline{f(\tilde{x})}$, $f(x * y) = \langle f, \delta_x * \delta_y \rangle$
- $f * g(x) = \int_H f(x * y) g(\widetilde{y}) dy$

When is the Fourier space a Banach algebra?

Theorem

Let H be the ultraspherical hypergroup defined by (G, π) and let

$$A_\pi(G) = \{u \in A(G) : \pi(u) = u\}.$$

Then

- *$A(H)$ is isometrically isomorphic to the subalgebra $A_\pi(G)$ of $A(G)$.*
- *The map $\dot{x} \to \varphi_{\dot{x}}$, where $\varphi_{\dot{x}}(u) = u(\dot{x})$ for $u \in A(H)$, is a homeomorphism from H onto $\Delta(A(H))$.*
- *$A(H)$ is regular, semisimple and Tauberian.*

V. Muruganandam, *Fourier algebra of a hypergroup. I*, J. Austral. Math. Soc. **82** (2007), 59-83.

V. Muruganandam, *Fourier algebra of a hypergroup II. Spherical hypergroups.* Math. Nachr. **281** (2008), 1590-1603.

Theorem

Let H be the ultraspherical hypergroup associated with (G, π), $p : G \to H$ the projection and E a closed subset of H.

- *If $p^{-1}(E)$ is a set of weak synthesis for $A(G)$, then E is a set of weak synthesis for $A(H)$, and $\xi(E) \leq \xi(p^{-1}(E))$*
- *If $p^{-1}(E)$ is a Ditkin set for $A(G)$, then E is Ditkin set for $A(H)$*

In particular, every closed subhypergroup of H is a set of synthesis.

Example

$G = SO(d)$, $d \geq 3$, $H = SO(d)//SO(d-1)$. Homeomorphism

$$[-1, 1] \to H, \quad x \to SO(d-1)a(x)SO(d-1).$$

For any $x \in]-1, 1[$, $\{x\}$ is a weak spectral set with $\xi(x) = \lfloor \frac{d+1}{2} \rfloor$, and hence $\xi(SO(d-1)a(x)SO(d-1)) \geq \lfloor \frac{d+1}{2} \rfloor$.

M. Vogel, *Spectral synthesis on algebras of orthogonal polynomial series*, Math. Z. **194** (1987), 99-116.

Theorem

Let G be a noncompact connected semisimple Lie group with finite centre and K a maximal compact subgroup of G. Let $G = KAN$ denote the Iwasawa decomposition of G, and assume that $\dim A = 1$ and $\dim(G/K) \geq 3$. Then KaK fails to be a set of synthesis for $A(G)$ for almost all $a \in A$.

C. Meaney, *Spherical functions and spectral synthesis*, Compos. Math. **54** (1985), 311-329.

Example

G a compact connected semisimple Lie group, $\mathrm{Inn}(G)$ the group of inner automorphisms of G. Let $\widetilde{G} = G \rtimes \mathrm{Inn}(G)$ and $H = \widetilde{G}//\mathrm{Inn}(G)$. Then $\Delta(A(H))$ equals the space of conjugacy classes, and C_x is not a set of synthesis for almost all $x \in G$.

C. Meaney, *On the failure of spectral synthesis for compact semisimple Lie groups*, J. Funct. Anal. **48** (1982), 43-57.

When does (weak) spectral synthesis hold for $A(G//K)$?

Clearly, if K is open in G. Does the converse hold?

Theorem

Let G be a nilpotent locally compact group and K a compact subgroup of G. Then the following are equivalent.

1. *Spectral synthesis holds for $A(G//K)$.*
2. *Weak spectral synthesis holds for $A(G//K)$.*
3. *K is open in G.*

Later: This theorem does not remain true for solvable G!

Lemma

Let K and N be compact subgroups of G with N normal. If (weak) spectral synthesis holds for $A(G//K)$, then (weak) spectral synthesis also holds for $A((G/N)//(KN/N))$.

Lemma

Let K and L be compact subgroups of G such that $K \subseteq L$ and let

$$q : G//K \to G//L, \quad KxK \to LxL.$$

Then, for any closed subset E of $G//L$, $\xi(E) \leq \xi(q^{-1}(E))$. In particular, if (weak) spectral synthesis holds for $A(G//K)$, then it also holds for $A(G//L)$.

Lemma

Let G be a nilpotent compact group and K a closed subgroup of G. If weak spectral synthesis holds for $A(G//K)$, then K has finite index in G.

Proof

Show by induction on j that $Z_j \cap K$ has finite index in Z_j.

- $[Z_j : (Z_j \cap K)] \leq [Z_j : (Z_j \cap Z_{j-1}K)] \cdot [Z_{j-1} : (Z_{j-1} \cap K)]$
- have to show that $[Z_j : (Z_j \cap Z_{j-1}K)] < \infty$
- weak spectral synthesis for $A(G//K)$ implies weak spectral synthesis for $A(Z_jK//K)$
- then weak spectral synthesis holds for $A(Z_jK//Z_{j-1}K)$ by Lemma 2
- $Z_{j-1}K$ is normal in Z_jK and $Z_jK/Z_{j-1}K$ is abelian, since Z_j/Z_{j-1} is contained in the centre of G/Z_{j-1}
- it follows that $Z_jK/Z_{j-1}K = Z_j/(Z_j \cap Z_{j-1}K)$ is finite.

Lemma

Let G be a nilpotent locally compact group such that G_0, the connected component of the identity, has finite index in G. Suppose that there exists a compact subgroup K of G such that weak spectral synthesis holds for $A(G//K)$. Then G is compact.

Proof

Assume first that $G = G_0$ and prove by induction on j that $Z_j \subseteq K$.

- if $Z_{j-1} \subseteq K$, then K is normal in $Z_j K$
- since weak spectral synthesis holds for $A(G//K)$, it also holds for $A(Z_j K//K)$
- since $Z_j K//K$ is a group, it folllows that $Z_j K/K$ is discrete
- Z_j is connected, since G is connected, hence $Z_j \subseteq K$.

proof continued

Now assume that $[G : G_0] < \infty$ and consider G^c, the set of all compact elements of G

- G^c is a compact (normal) subgroup of G (since G is nilpotent and compactly generated)
- G/G^c is a Lie group and compact-free
- G/G_0G^c is discrete, trosion-free and finite, so that $G = G_0G^c$ and G/G^c is connected
- by Lemma 1, weak spectral synthesis holds for $A(G//KG^c) = A((G/G^c)//(KG^c/G^c))$
- the first part of the proof shows that $KG^c = G$.

G is solvable if there exists $n \in \mathbb{N}$ such that

$$G \supseteq G_1 = [G, G] \supseteq \ldots \supseteq G_n = [G_{n-1}, G] = \{e\}.$$

Theorem

Let G be a solvable locally comopact group such that G_0 is abelian. If K is a compact subgroup of G such that weak spectral synthesis holds for $A(G//K)$, then $K \supseteq G_0$.

Theorem

Let G be a solvable locally compact group and F a finite group of topological automorphisms of G. If weak spectral synthesis holds for $A(G \rtimes F//F)$, then G is totally disconnected.

The Counterexample

C.F. Dunkl and D.E. Ramirez, *A family of countably compact P_*-hypergroups*, Trans. Amer. Math. Soc. **202** (1975), 339-356.

Let p be a prime number. The p-adic norm $\|\cdot\|_p$ on $\mathbb{Q}$ is defined by $\|0\|_p = 0$ and $\|x\|_p = p^{-m}$ if $x = p^m y$, where the nominator and the denominator of y are both not divisible by p

Ω_p = completion of $\mathbb{Q}$ with respect to $\|\cdot\|_p$ is a locally compact field, and Ω_p is totally disconnected since, for each $x \in \Omega_p$ and $r > 0$, the closed ball

$$K(x,r) = \{y \in \Omega_p : \|y - x\|_p \leq r\}$$

is also open in Ω_p.

- $K(0,r)$ is an additive subgroup of Ω_p
- $\Delta_p = K(0,1)$ is a compact subring, the ring of p-adic integrers

$K =$ multiplicative group of all $x \in \Omega_p$ with $\|x\|_p = 1$.
K is compact and acts on Δ_p through multiplication and
$K \cdot x = \{y \in \Delta_p : \|y\|_p = \|x\|_p\}$, which is open and closed in Δ_p for every $x \neq 0$

Let $G = \Delta_p \rtimes K$, the semidirect product of two abelian compact groups
$H = G//K$ is topologically isomorphic to $\mathbb{Z}_+ \cup \{infty\}$ the one-point compactification of $\mathbb{Z}_+$:
$n \in \mathbb{Z}_+ \to \{x \in \Delta_p : \|x\|_p = p^{-n}\} \times K$ and $\infty \to \{0\} \times K$.

Theorem

Eevery closed subset of H is a set of synthesis.

Slightly better: Every closed subset E of H is a Ditkin set, and the ideal $k(E)$ has a bounded approximate identity if and only if either E is finite or $H \setminus E$ is finite.