The dual space of precompact groups

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If E is a metrizable locally convex space, the precompact-open topology on its dual E' coincides with the topology of $\mathfrak{N}$-convergence, where $\mathfrak{N}$ is the collection of all compact subsets of E each of which is the set of points of a sequence converging to 0.

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So far, this result had been extended to metrizable abelian groups by several authors: Banaszczyk (1991) for metrizable vector groups, Aussenhofer (1999) and, independently, Chasco (1998) for metrizable abelian groups.

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So far, this result had been extended to metrizable abelian groups by several authors: Banaszczyk (1991) for metrizable vector groups, Aussenhofer (1999) and, independently, Chasco (1998) for metrizable abelian groups.

I'm going to report on our findings concerning the extension of the Banach - Dieudonné Theorem to non necessarily abelian, metrizable, precompact groups.

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Notation and basic facts

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- When $\widehat{G}$ is neither Abelian nor compact, $\widehat{G}$ usually is non-Hausdorff.

In general, little is known about the properties of the Fell topology.

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- If G is a dense subgroup of a compact group H, the precompact-open topology on $\widehat{G}$ coincides with the compact-open topology on $\widehat{H}$. Since the dual space of a compact group is discrete, in order to prove that a precompact group G satisfies the Banach - Dieudonné Theorem, it suffices to verify that $\widehat{G}$ is discrete.
- Thus, we look at the following question: for what precompact groups G is $\widehat{G}$ discrete?

- Two unitary representations $\rho : G \to U(\mathcal{H}_1)$ and $\psi : G \to U(\mathcal{H}_2)$ are equivalent if there exists a Hilbert space isomorphism $M : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\rho(x) = M^{-1}\psi(x)M$ for all $x \in G$.

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- The dual object of G is the set $\widehat{G}$ of equivalence classes of irreducible unitary representations of G.
- If G is a compact group, all irreducible unitary representation of G are finite-dimensional and the Peter-Weyl Theorem determines an embedding of G into the product of unitary groups $\mathbb{U}(n)$.

Functions of positive type

- If $\rho : G \to U(\mathcal{H})$ is a unitary representation, a complex-valued function f on G is called a function of positive type associated with ρ if there exists a vector $v \in \mathcal{H}$ such that $f(g) = (\rho(g)v, v) \; \forall \, g \in G$

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- We denote by P'_ρ be the set of all functions of positive type associated with ρ. Let P_ρ be the convex cone generated by P'_ρ.
- If ρ_1 and ρ_2 are equivalent representations, then $P'_{\rho_1} = P'_{\rho_2}$ and $P_{\rho_1} = P_{\rho_2}$.

Fell topology

- Let G be a topological group, $\mathcal{R}$ a set of equivalence classes of unitary representations of G. The Fell topology on $\mathcal{R}$ is defined as follows: a typical neighborhood of $[\rho] \in \mathcal{R}$ has the form

$$W(f_1, \cdots, f_n, C, \epsilon) = \{[\sigma] \in \mathcal{R} : \exists g_1, \cdots, g_n \in P_\sigma \, \forall x \in C \, |f_i(x) - g_i(x)| < \epsilon\},$$

where $f_1, \cdots, f_n \in P_\rho$ (or P'_ρ), C is a compact subspace of G, and $\epsilon > 0$.

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  where $f_1, \cdots, f_n \in P_\rho$ (or P'_ρ), C is a compact subspace of G, and $\epsilon > 0$.
- In particular, the Fell topology is defined on the dual object $\widehat{G}$.

Kazhdan's property (T)

- The group G has property (T) if the trivial representation 1_G is isolated in $\mathcal{R} \cup \{1_G\}$ for every set $\mathcal{R}$ of equivalence classes of unitary representations of G without non-zero invariant vectors.

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- Let π be a unitary representation of a topological group G on a Hilbert space $\mathcal{H}$. Let $F \subseteq G$ and $\epsilon > 0$. A unit vector $v \in \mathcal{H}$ is called (F, ϵ)-invariant if $\|\pi(g)v - v\| < \epsilon$ for every $g \in F$.

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Proposition

A topological group G has property (T) if and only if there exists a pair (Q, ϵ) (called a Kazhdan pair), where Q is a compact subset of G and $\epsilon > 0$, such that for every unitary representation ρ having a unit (Q, ϵ)-invariant vector there exists a non-zero invariant vector

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Precompact metrizable groups

Theorem 1

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Lemma 1

Let X be compact space, D a dense subset of X, and N a compact subset of $C(X)$. If $g \in C(X)$ is at the distance $> \epsilon$ from N, there exists a finite subset $F \subseteq D$ such that the distance from $g|_F$ to $N|_F$ in $C(F)$ is $> \epsilon$.

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Lemma 2

The space $\widehat{G}$, equipped with the Fell topology, is T_1.

Idea of the proof

- Since G is metrizable, it follows that $\widehat{G} = \{[\rho_i]) : i \in \mathbb{N}\}$. Therefore, taking into account that $\widehat{G}$ is T_1, in order to prove that $\widehat{G}$ is discrete, it suffices to show that for every point $[\rho] \in \widehat{G}$ there is a a neighborhood W of $[\rho]$ which for some integer i_0 does not contain any $[\rho_i]$ with $i \geq i_0$.

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- Our neighborhood is of the form $W = W(h, F, \epsilon)$, where h is the normalized character of $[\rho]$ and $F = \{e\} \cup \bigcup_{i \geq i_0} F_i$ is a compact subset of G, where (F_i) is a sequence of finite sets which converges to e and the finite set F_i ensures that the neighborhood W does not contain $[\rho_i]$.

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- Our neighborhood is of the form $W = W(h, F, \epsilon)$, where h is the normalized character of $[\rho]$ and $F = \{e\} \cup \bigcup_{i \geq i_0} F_i$ is a compact subset of G, where (F_i) is a sequence of finite sets which converges to e and the finite set F_i ensures that the neighborhood W does not contain $[\rho_i]$.
- We derive the existence of F_i from the orthogonality of characters. If V is a neighborhood of e on which h is close to 1, we have that $\int_V \chi_i \to 0$ as $i \to \infty$, which forces $\mathrm{Re}\chi_i$ to be close to 0 somewhere on V for $i \geq i_0$. This implies that h and h_i are not close to each other on V.

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- With a little more work we can show that h is not close to any element of P_i and, using Lemma 1, that this is witnessed by a certain finite subset F_i of V.

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- We remark that there exists a single null sequence $C \subseteq G$ such that for every $[\rho_i] \in \widehat{G}$ the neighborhood $W(h_i|_G, C, 1/6)$ of $[\rho_i]$ in $\widehat{G}$ is finite.

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Corollary

If G is a metrizable precompact group, there is a null sequence C that topologically generates the group and defines the discrete topology on $\widehat{G}$.

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Discrete metrics

Let G and L be a topological group and a compact Lie group, respectively, and let $C(G, L)$ denote the group of all continuous functions of G into L. If $K \subseteq G$, $E \subseteq C(G, L)$ and d is an invariant metric defined on L, then we can define a pseudometric d_K^L on E in terms of d as follows

$$d_K^L(\varphi, \psi) = \sup\{d(\varphi(x), \psi(x)) : \ x \in K\}$$

for all φ, ψ in E. Furthermore, if K separate the points in E, then d_K^L is in fact a metric on E.

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In the case that $L = \mathbb{U}(n)$ and $E = \mathrm{irrep}_n(G)$, we denote by d_K^n the pseudometric associated to $K \subseteq G$ and the unitary group $\mathbb{U}(n)$ as above.

It is possible to equip $\mathrm{irrep}(G)$ with a single pseudometric d_K that "includes canonically" the pseudometrics $\{d_K^n : n \in \mathbb{N}\}$ as follows:

Discrete metrics

$$d_K(\phi,\psi) = d_K^n(\phi,\psi)$$

if $\{\phi,\psi\} \subseteq \mathrm{irrep}_n(G)$ for some $n \in \mathbb{N}$ and

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Furthermore, if $\pi : \mathrm{irrep}(G) \longrightarrow \widehat{G}$ is the canonical quotient mapping, then the dual object $\widehat{G}$ is equipped with a pseudometric $\widehat{d}_K$, inherited from $\mathrm{irrep}(G)$, as follows:

$$\widehat{d}_K([\varphi], [\psi]) = \inf\{d_K(\rho, \mu) : \rho \in [\varphi], \mu \in [\psi]\}.$$

When G is compact, d_G equips $\widehat{G}$ with the discrete topology. The so-called *(pre)compact open topology on* $\widehat{G}$ is the topology generated by the collection of pseudometrics $\{\widehat{d}_K : K$ is a (pre)compact subset of $G\}$.

Discrete metrics

Theorem

If G is a metrizable precompact group, there is a null sequence C that satisfies the following properties:

- C topologically generates the group G;
- C defines the discrete topology on $\widehat{G}$; and
- for all $n \in \mathbb{N}$ and $[\varphi] \in \widehat{G}_n$ there is $\delta_n > 0$ such that if $\psi \in \widehat{G}$ and $d_C([\phi], [\psi]) < \delta_n$ then $[\phi] = [\psi]$.

  As a consequence, the metric d_C defines the discrete topology on $\widehat{G}$ and, furthermore, it is equivalent to the $\{0, 1\}$-valued discrete metric on the subspaces $\widehat{G}_n$.

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Non-metrizable precompact groups

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- In the Abelian case, this question has been clarified in the work of several authors. If G is an Abelian topological group, $\widehat{G}$ can be viewed as the group of all continuous homomorphisms $G \to \mathbb{U}(1)$ equipped with the compact-open topology, where $\mathbb{U}(1) = \{z \in \mathbb{C} : |z| = 1\}$.

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Subsequently, it was shown that the result also holds without assuming the continuum hypothesis (H., Macario, and Trigos-Arrieta, 2008) and (Dikranjan, Shakhmatov, 2009). Therefore, a compact abelian group is determined iff it is metrizable.

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Our goal in this section is to extend this result to compact groups that are not necessarily Abelian.

Non-metrizable precompact groups

Theorem 2

If G is a countable precompact non-metrizable group, then 1_G is not an isolated point in $\widehat{G}$.

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Theorem 3

If H is a non-metrizable compact group, then H has a dense subgroup G such that $\widehat{G}$ is not discrete.

Idea of the proof

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Proposition

Suppose that there exists an integer n such that $w(K) < |\widehat{G}_n|$ for every compact subset K of G. Then 1_G is not an isolated point in $\widehat{G}$.

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- Since countable compact groups are metrizable, Theorem 2 follows from this Proposition.

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- Since countable compact groups are metrizable, Theorem 2 follows from this Proposition.
- As for the proof of Theorem 3, it is enough to replace G by an appropriate quotient of weight ω_1.

Theorem 4

Let H be a compact group. The following conditions are equivalent:

1. H is metrizable.
2. If G is an arbitrary dense subgroup of H, there is a null sequence $C \subseteq G$ that satisfies the following properties:
   - C topologically generates the group G;
   - C defines the discrete topology on $\widehat{G}$; and
   - for all $n \in \mathbb{N}$ and $[\varphi] \in \widehat{G}_n$ there is $\delta_n > 0$ such that if $\psi \in \widehat{G}$ and $d_C([\phi], [\psi]) < \delta_n$ then $[\phi] = [\psi]$.

   As a consequence, the metric d_C defines the discrete topology on $\widehat{G}$ and, furthermore, it is equivalent to the $\{0, 1\}$-valued discrete metric on the subspaces $\widehat{G}_n$.

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Question

Does there exist a non-compact precompact Abelian group with property (T)?