

Weak amenability of the Fourier algebra

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joint work with Y. Choi

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\mathcal{A} : Ban. alg.

X : Ban. \mathcal{A} -bimodule

Example

- \mathcal{A} Ban. \mathcal{A} -bimodule (algebra multiplication=module action).
- \mathcal{A}^* Ban. \mathcal{A} -bimodule via $a \cdot f(x) = f(xa)$, $f \cdot a(x) = f(ax)$.

$D : \mathcal{A} \rightarrow X$ derivation if $D(ab) = D(a) \cdot b + a \cdot D(b)$.

Definition

comm. Ban. alg. \mathcal{A} is weakly amenable (w. amen.) if every bndd der $D : \mathcal{A} \rightarrow \mathcal{A}^*$ is zero.

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Some results

[Johnson] $L^1(G)$ w. amen. for all G .

[Haagerup] C^* -algebras are w. amen.

[Dales-Ghahramani-Helemskii] G non-discrete $\Rightarrow M(G)$ not w. amen.

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For which groups G , $A(G)$ is w. amen?

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Fourier-Stieltjes algebra

G : LCG, \mathcal{H} : Hilbert space, $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$: unitary rep.

$\xi *_{\pi} \eta(x) := \langle \pi(x)\xi, \eta \rangle$ **coeff. func.** assoc. with π .

$\Sigma = \{\text{equiv. classes of unitary rep.}\}$

Definition

- $B(G) := \{\xi *_{\pi} \eta : \pi \in \Sigma, \xi, \eta \in \mathcal{H}_{\pi}\}$.
- For $u \in B(G)$, $\|u\|_{B(G)} = \inf\{\|\xi\|_{\mathcal{H}_{\pi}}\|\eta\|_{\mathcal{H}_{\pi}} : u = \xi *_{\pi} \eta\}$.
- $(B(G), +, \cdot, \|\cdot\|_{B(G)})$ Ban alg.

Example

- G : Abelian, $B(G) = \mathcal{F}(M(\hat{G}))$, with $\|\mathcal{F}\mu\|_A = \|\mu\|$.

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Fix $\pi : G \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ unitary rep.

Definition

- $A_\pi(G) := \overline{\text{span}}^{\|\cdot\|_{B(G)}} \{ \xi *_\pi \eta : \xi, \eta \in \mathcal{H}_\pi \}$.
- $A_\pi(G)$ consists of $u = \sum_{n=1}^{\infty} \xi_n *_\pi \eta_n$, $\sum_{i=1}^{\infty} \|\xi_i\| \|\eta_i\| < \infty$.
- $\|u\|_{B(G)} = \inf \{ \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| : u \text{ as above} \}$.
- $A_\pi(G)$ closed subspace of $B(G)$.

Example (Fourier algebra)

$$A(G) = A_\lambda(G).$$

Note. $A(G) := \{ \xi *_\lambda \eta : \xi, \eta \in L^2(G) \}$ ideal in $B(G)$.

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Known results

- [Johnson, 91] $L^1(G)$ is always w. amen.
- [Johnson, 92] $A(\mathrm{SO}_3(\mathbb{R}))$ not w. amen.
- [Johnson, Forrest, 90s] G totally disconnected $\Rightarrow A(G)$ w. amen.

Conjecture

$A(G)$ w. amen. $\Leftrightarrow G_e$ abelian.

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Observation

\mathcal{A}, \mathcal{B} comm. Ban. alg. $\phi : \mathcal{A} \rightarrow \mathcal{B}$ cts hom with dense range.

- If $D : \mathcal{B} \rightarrow \mathcal{B}^*$ cts nonzero der then $\phi^* \circ D \circ \phi : \mathcal{A} \rightarrow \mathcal{A}^*$ cts nonzero der.
- \mathcal{B} **not** w. amen. $\Rightarrow \mathcal{A}$ **not** w. amen.

Proposition

G LCG, H closed subgroup.

- Restriction map $r : A(G) \rightarrow A(H)$ is cts onto hom.
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Evidence to support the conjecture

G_e connected component of identity.

Conj. $A(G)$ w. amen. $\Leftrightarrow G_e$ abelian.

Known results

- [Forrest-Runde, 04] G_e abelian $\Rightarrow A(G)$ w. amen.
- [Johnson, 92] $A(\mathrm{SO}_3(\mathbb{R}))$ not w. amen.
- [Plymen, 01] G cpt Lie gp and $A(G)$ w. amen. $\Rightarrow G_e$ abelian.
- [Forrest-Samei-Spronk, 09] G cpt gp and $A(G)$ w. amen. $\Rightarrow G_e$ abelian.

Theorem (Choi-G., 13)

- $A(ax + b)$ is not w. amen.
- G non-cpt semisimple Lie gp $\Rightarrow A(G)$ is not w. amen.

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Thm. $A(\mathrm{SO}_3(\mathbb{R}))$ not w. amen. [Johnson, 92].

Proof.

$A(\mathrm{SO}(3)) \rightarrow A(\mathrm{SU}(2))$ isometric inclusion.

Find $D : A(\mathrm{SU}(2)) \rightarrow A(\mathrm{SU}(2))^*$ s.t. $D|_{A(\mathrm{SO}(3))} \neq 0$.

For $\phi \in \mathbb{R}$, $s_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$.

$\partial_\phi : C^1(\mathrm{SU}(2)) \rightarrow C(\mathrm{SU}(2))$, $\partial_\phi f(p) = \left. \frac{\partial}{\partial \phi} f(ps_\phi) \right|_{\phi=0}$.

$D_b : C^1(\mathrm{SU}(2)) \times C(\mathrm{SU}(2)) \rightarrow \mathbb{C}$, $D_b(f, g) = \int_{\mathrm{SU}(2)} (\partial_\phi f) g d\mu$.

$|D_b(f)(g)| \leq C \|f\|_A \|g\|_A$. □

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What is $ax + b$ group?

An important non-compact group

Definition

- $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}_+^*, b \in \mathbb{R} \right\}$.
- $\mathbb{R}_+^* := (\mathbb{R}_+, *)$ with Haar measure $t^{-1} dt$.
 \mathbb{R}_+^* acts on \mathbb{R} by mult.
- $G \simeq \mathbb{R} \rtimes \mathbb{R}_+^* = \{(b, a) : b \in \mathbb{R}, a \in \mathbb{R}_+^*\}$.

Properties of $G = ax + b$

Left Haar measure: $d\mu(a, b) = a^{-2} da db$.

Modular function: $\Delta(a, b) = \frac{1}{a}$.

What is $ax + b$ group?

Representation theory of $ax + b$

∞ -dim irred rep

Mackey machine for induced rep:

$$\pi_{\pm} : G \rightarrow \mathcal{U}(L^2(\mathbb{R}_+^*, dt/t)),$$

- $\pi_+(b, a)\xi(t) = e^{-2\piibt}\xi(at),$
- $\pi_-(b, a)\xi(t) = e^{+2\piibt}\xi(at).$

coefficient functions

- $(\xi *_{\pi_+} \eta)(b, a) = \int_0^{\infty} e^{-2\piibt}\xi(at)\overline{\eta(t)} t^{-1} dt,$
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$$\mathcal{H} = L^2(\mathbb{R}_+^*, dt/t).$$

\mathcal{F} : Fourier transform on \mathbb{R} .

- $K : \mathcal{H} \rightarrow \mathcal{H}, K\xi(t) = t\xi(t).$
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ax + b is "nice".

Definition

Irred rep π is **square integrable** if $\exists 0 \neq \xi *_{\pi} \eta \in L^2(G)$.

Note. π sq. integ. iff $\pi < \lambda$.

Orthogonality rel in ax + b

$\eta_1, \eta_2, \xi_1, \xi_2 \in C_c^2(\mathbb{R}_+^*)$. Then

$$\langle \xi_1 *_{\pi_+} \eta_1, \xi_2 *_{\pi_+} \eta_2 \rangle_{L^2(G)} = \langle \eta_2, \eta_1 \rangle_{\mathcal{H}} \langle K^{-\frac{1}{2}} \xi_1, K^{-\frac{1}{2}} \xi_2 \rangle_{\mathcal{H}}.$$

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Corollary

π_+ and π_- are sq. integ.

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Norm on $A_{\pi_{\pm}}$

$$\|\xi *_{\pi_{\pm}} \eta\|_{A(G)} = \|\xi\| \|\eta\|.$$

Proposition

$$A(G) = A_{\pi_+} \oplus_1 A_{\pi_-}.$$

Proof.

- $\{\xi *_{\pi_{\pm}} \eta : \xi, \eta \in C_c(\mathbb{R}_+)\}$ is dense in $L^2(G)$.
- $\langle \lambda(\mathbf{x})(\xi *_{\pi_{\pm}} \eta), \xi' *_{\pi_{\pm}} \eta' \rangle = \alpha \langle \pi_{\mp}(\mathbf{x}) \bar{\eta}, \bar{\eta}' \rangle$.
- $\langle \lambda(\mathbf{x})(\xi *_{\pi_{\pm}} \eta), \xi' *_{\pi_{\mp}} \eta' \rangle = 0$.



Defining the derivation

- $\mathcal{C} = \text{Span}\{\xi *_{\pi_{\pm}} \eta : \xi, \eta \in \mathcal{C}_c^{\infty}(\mathbb{R}_+^*)\}$.
- $D_1 : \mathcal{C} \rightarrow \mathcal{C}, D_1(f)(b, a) = \frac{-a}{2\pi i} \frac{\partial}{\partial b} f(b, a)$.
- $D_1(\xi *_{\pi_{\pm}} \eta) = \pm(K\xi *_{\pm} \eta)$.
- $D : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{C}, D(f, g) = \int D_1(f)g d\mu$.
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$$\begin{aligned} D(\xi *_{\pi_+} \eta, \xi' *_{\pi_-} \eta') &= \langle K\xi *_{\pi_+} \eta, \overline{\xi'} *_{\pi_+} \overline{\eta'} \rangle \\ &= \langle K^{-\frac{1}{2}} K\xi, K^{-\frac{1}{2}} \overline{\xi'} \rangle \langle \overline{\eta'}, \eta \rangle. \end{aligned}$$

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Definition

$$\mathbb{H}_r := \{(p, q, e^{2\pi i\theta}) : p, q \in \mathbb{R}, \theta \in [0, 1)\}.$$

$$(p, q, e^{2\pi i\theta}) \cdot (p', q', e^{2\pi i\theta'}) = (p + p', q + q', e^{2\pi i(\theta + \theta')}) e^{\pi i(pq' - qp')}.$$

Representations

For $n \in \mathbb{Z} \setminus \{0\}$, $\text{Sch}_n : \mathbb{H}_r \rightarrow \mathcal{U}(L^2(\mathbb{R}))$ is defined by

$$\text{Sch}_n(p, q, e^{2\pi i\theta})\xi(x) = e^{2\pi inq(-x + \frac{p}{2})} e^{2\pi in\theta} \xi(-p + x).$$

Proposition

- $\langle \xi_1 *_{\pi_n} \eta_1, \xi_2 *_{\pi_n} \eta_2 \rangle_{L^2(G)} = \frac{1}{n} \langle \eta_2, \eta_1 \rangle_{\mathcal{H}} \langle \xi_1, \xi_2 \rangle_{\mathcal{H}}.$
- $\langle \xi_1 *_{\pi_n} \eta_1, \xi_2 *_{\pi_m} \eta_2 \rangle_{L^2(G)} = 0,$ whenever $n \neq m.$
- $A(\mathbb{H}_r) = \bigoplus_{n \neq 0} A_{\text{Sch}}(\mathbb{H}_r) \oplus A(\mathbb{H}_r : \mathbb{T}).$

Question

Let $G = \mathbb{R}^2 \rtimes \text{So}(2)$. Is $A(G)$ weakly amenable?