The radical of $\ell^1(\beta\mathbb{N})$

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The Jacobson radical of an algebra

Let A be a (complex, associative) algebra with identity e_A.

The Jacobson **radical** of A is denoted by $J(A)$; A is **semisimple** if $J(A) = \{0\}$.

An element $a \in A$ is **quasi-nilpotent** if $ze_A - a$ is invertible for each $z \in \mathbb{C}$ with $z \neq 0$; the set of these is $\mathcal{Q}(A)$.

Fact Let A be a unital algebra. Then

$$J(A) = \{a \in A : ba \in \mathcal{Q}(A) \ \ (b \in A)\} .$$

Thus $J(A) \subset \mathcal{Q}(A)$. $\square$

For non-unital A, we have $J(A) = J(A^\sharp)$.

Banach spaces

The dual and bidual of a Banach space E are E' and E'', respectively.

Let S be a non-empty set. Then $\ell^1(S)$ is the usual Banach space. The characteristic function of $\{s\}$ for an element $s \in S$ is δ_s, and so a generic element of $\ell^1(S)$ is $\sum_{s \in S} f(s)\delta_s$.

The linear space spanned by the functions δ_s is $\mathbb{C}S$; these are the elements of **finite support**. Thus $\mathbb{C}S$ is a dense subspace of $(\ell^1(S), \|\cdot\|_1)$.

The **dual space** of $\ell^1(S)$ is $\ell^\infty(S)$ with the usual duality.

Semigroups

A **semigroup** is a non-empty set S with a binary operation such that

$$r(st) = (rs)t \quad (r, s, t \in S)\,.$$

Eg, (1) $(\mathbb{N}, +)$, (2) $\mathbb{S}_n$, which is the free semigroup on n generators, (3) any group, such as $\mathbb{F}_2$, the free group on 2 generators.

An element $p \in S$ is **idempotent** if $p^2 = p$; the set of these is $E(S)$.

For $s \in S$, set $L_s(t) = st$, $R_s(t) = ts$ for $t \in S$.

An element $s \in S$ is **cancellable** if both L_s and R_s are injective, and S is **cancellative** if each $s \in S$ is cancellable. Also **weakly cancellative**.

A subset I is a **left ideal** if $L_s(I) \subset I$ for $s \in S$, etc.

An abelian, cancellative semigroup is embeddable in a group, but this is not true for all cancellable semigroups.

Two semigroup algebras

Let S be a semigroup. Then $(\ell^1(S), \star, \|\cdot\|_1)$ is the **semigroup algebra** of S.

The space $\mathbb{C}S$, the **algebraist's semigroup algebra**, is a dense subalgebra of our Banach algebra $\ell^1(S)$.

Obvious question When are $\ell^1(S)$ and/or $\mathbb{C}S$ semisimple?

For S abelian, $\ell^1(S)$ is semisimple if and only if S is **separating**, in the sense that $s = t$ whenever $s, t \in S$ and $s^2 = t^2 = st$. (Hewitt and Zuckerman, 1956)

Notation: The radicals are $J(S)$ and $J_0(S)$, respectively; the quasi-nilpotents of $\ell^1(S)$ are $\mathcal{Q}(S)$.

Some answers

In the case where G is a group, $\ell^1(G)$ and $\mathbb{C}G$ are semisimple [Rickart 1960]. Further, $\mathcal{Q}(G) = \{0\}$ for each abelian group G.

Easy examples show that there are finite, abelian semigroups S such that $\ell^1(S) = \mathbb{C}S$ is not semisimple. For example, set $S = \{o, s\}$ where

$$o^2 = os = so = s^2 = o\,,$$

so that S is a **zero semigroup**. Set $f = \delta_o - \delta_s$. Then $J(S) = \mathbb{C}f \neq \{0\}$.

Some open questions

I do not know if it is a general truth that the semi-simplicity of one of the algebras $\mathbb{C}S$ and $\ell^1(S)$ follows from the semi-simplicity of the other.

It is not known if either or both are semi-simple whenever S is a cancellative semigroup, or even whenever S is a sub-semigroup of a group. (This is true when S is also abelian or ordered.)

For $S = \mathbb{F}_2$, we have $J(S) = \{0\}$, but $\mathcal{Q}(S)$ is very large. For $S = \mathbb{S}_n$, we have $J(S) = \mathcal{Q}(S) = \{0\}$.

Stone–Čech compactifications

The **Stone–Čech compactification** of a set S is denoted by βS; set $S^* = \beta S \setminus S$, this is the **growth** of S. The space βS is each of the following:

• - characterized by a universal property: βS is a compactification of S such that each bounded function from S to a compact space K has an extension to a continuous map from βS to K;

• - the space of ultrafilters on S;

• - the Stone space of the Boolean algebra $\mathcal{P}(S)$, the power set of S;

• - the character space of the commutative C^*-algebra $\ell^\infty(S)$, so that $\ell^\infty(S) = C(\beta S)$ (and βS is compact).

Suppose that $|S| = \kappa$. Then $|\beta S| = 2^{2^\kappa}$. Topologically βS is a **Stonean space**: it is **extremely disconnected**.

Semigroup compactifications

Let S be a semigroup.

For each $s \in S$, the map $L_s : S \to \beta S$ has an extension to a continuous map $L_s : \beta S \to \beta S$. For each $u \in \beta S$, define $s \,\Box\, u = L_s(u)$.

Next, the map $R_u : s \mapsto s \,\Box\, u, \; S \to \beta S$, has an extension to a continuous map $R_u : \beta S \to \beta S$ for each $u \in \beta S$. Define

$$u \,\Box\, v = R_v(u) \quad (u, v \in \beta S)\,.$$

Then $(\beta S, \Box)$ is a semigroup.

Often the binary operation on $\beta\mathbb{N}$ from the semigroup $(\mathbb{N}, +)$ is denoted by $(\beta\mathbb{N}, +)$. But note that $x + y \neq y + x$, in general.

Starting from a group G, we obtain a semigroup $(\beta G, \Box)$. But it is never a group (for infinite G).

It is easy to stumble across open questions about $(\beta\mathbb{N}, +)$.

Compact, right topological semigroup

Definition A semigroup V with a topology τ is a **compact, right topological semigroup** if (V, τ) is a compact space and the map R_v is continuous with respect to τ for each $v \in V$.

In general, the maps L_v are not continuous.

For example, $V = (\beta S, \Box)$ and $V = (S^*, \Box)$ for weakly cancellative S are compact, right topological semigroups. Here L_v is continuous on $(\beta S, \Box)$ if and only if $v \in S$.

Our semigroups

Let S be a semigroup. Then we are interested in the semigroup algebras $\ell^1(\beta S, \Box)$ and $\ell^1(S^*, \Box)$. Are they semisimple? Is $\ell^1(\mathbb{N}^*, +)$ semisimple?

[The set S^* is an ideal in $(\beta S, \Box)$ iff S is weakly cancellative. In this case,

$$\ell^1(\beta S, \Box) = \ell^1(S) \ltimes \ell^1(S^*, \Box)$$

as a semi-direct product. When $\ell^1(S)$ is semisimple, we have $J(\beta S, \Box) = J(S^*, \Box)$.]

An example

Example For $m, n \in \mathbb{N}$, define

$$m \vee n = \max\{m, n\}\,,$$

and set $S = (\mathbb{N}, \vee)$. Then S is a countable, weakly cancellative, abelian semigroup, and $\ell^1(S)$ is semisimple because S is separating, and so $J(\beta S) = J(S^*)$. Then

$$u \,\Box\, v = v \quad (u, v \in S^*)\,,$$

and so $(S^*, \Box)$ is a right zero semigroup. Thus

$$J(\beta S, \Box) = \left\{ f \in \ell^1(S^*) : \sum_{u \in S^*} f(u) = 0 \right\}\,,$$

and so $\ell^1(S^*)$ is not semisimple. $\Box$

The structure theorem

The study of our semigroups is based on the following **structure theorem**.

Theorem Let V be a compact, right topological semigroup. (Eg, V is $(\beta S, \Box)$ or $(S^*, \Box)$.)

(i) There is a unique minimum ideal $K(V)$ in V. The families of minimal left ideals and of minimal right ideals of V both partition $K(V)$.

(ii) For each minimal right and left ideals R and L in V, there exists an element $p \in E(V) \cap R \cap L$ such that $R \cap L = RL = pVp$ is a group; these groups are maximal in $K(V)$, are pairwise isomorphic, and the family of these groups partitions $K(V)$.

(iii) For each $p, q \in K(V)$, the subset $pK(V)q$ is a subgroup of V, and there exists $r \in E(K(V))$ with $rp = p$ and $qr = q$. $\Box$

[Considerably more is known.]

$K(\beta\mathbb{N})$ is big

It is easy to see that $K(\beta\mathbb{N})$ is equal to $K(\mathbb{N}^*)$.

Theorem (Hindman and Pym) The semigroup $K(\mathbb{N}^*)$ contains many isomorphic copies of $\mathbb{F}_2$ as a subgroup. $\square$

Thus $\ell^1(K(\mathbb{N}^*))$ contains many isometric and (algebra) isomorphic copies of $\ell^1(\mathbb{F}_2)$ as a closed subalgebra, and hence there are many quasi-nilpotents in $\ell^1(K(\mathbb{N}^*))$.

A semigroup R of the form $A \times B$, where

$$(a,b)(c,d) = (a,d) \quad (a,c \in A,\ b,d \in B)$$

is a **rectangular semigroup**. It is easy to see that $\dim J(R) \geq |A|$. But it is a deep result of Yevhen Zelenyuk that $K(\mathbb{N}^*)$ contains a rectangular semigroup $A \times B$ with $|A| = |B| = 2^{\mathfrak{c}}$. Thus there is a 'very large' sub-semigroup R of $K(\mathbb{N}^*)$ with $\ell^1(R)$ far from semisimple.

Second duals of Banach algebras

Let A be a Banach algebra. There are two natural products on A'' of A; they are the **Arens products**, and are denoted by $\Box$ and $\Diamond$, respectively.

We give the definitions. For $a \in A$ and $\lambda \in A'$, define $a \cdot \lambda$ and $\lambda \cdot a$ in A' by

$$\langle b,\, a\cdot\lambda\rangle = \langle ba,\, \lambda\rangle\,, \quad \langle b,\, \lambda\cdot a\rangle = \langle ab,\, \lambda\rangle \quad (b \in A)\,.$$

For each $\lambda \in A'$ and $\mathsf{M} \in A''$, define $\lambda \cdot \mathsf{M} \in A'$ and $\mathsf{M} \cdot \lambda \in A'$ by

$$\langle a,\, \lambda\cdot\mathsf{M}\rangle = \langle \mathsf{M},\, a\cdot\lambda\rangle\,, \quad \langle a,\, \mathsf{M}\cdot\lambda\rangle = \langle \mathsf{M},\, \lambda\cdot a\rangle\,,$$

for $a \in A$.

For $\mathsf{M}, \mathsf{N} \in A''$, define

$$\langle \mathsf{M}\Box\mathsf{N},\, \lambda\rangle = \langle \mathsf{M},\, \mathsf{N}\cdot\lambda\rangle\,, \quad \langle \mathsf{M}\Diamond\mathsf{N},\, \lambda\rangle = \langle \mathsf{N},\, \lambda\cdot\mathsf{M}\rangle\,,$$

for $\lambda \in A'$.

Easier to remember

Take M $= \lim_\alpha a_\alpha$ and N $= \lim_\beta b_\beta$ in A'' (in the weak-$*$ topology). Then

$$\mathsf{M} \,\Box\, \mathsf{N} = \lim_\alpha \lim_\beta a_\alpha b_\beta\,, \qquad \mathsf{M} \diamond \mathsf{N} = \lim_\beta \lim_\alpha a_\alpha b_\beta\,.$$

Arens regularity

Theorem Let A be a Banach algebra. Then $(A'', \Box)$ and $(A'', \diamond\,)$ are Banach algebras containing A as a closed subalgebra. $\Box$

In general, the two products $\Box$ and $\diamond$ on A'' are not the same.

Definition A Banach algebra A is **Arens regular** if $\Box$ and $\diamond$ coincide on A'', and **strongly Arens irregular** if they agree only when one term in the product is in A itself.

These are very contrasting properties.

Second duals of semi-group algebras

Start with a semigroup S and the semigroup algebra $A = (\ell^1(S), \star)$.

Then $A' = \ell^\infty(S) = C(\beta S)$ as a Banach space, and so $A'' = M(\beta S)$.

We can transfer the Arens products $\Box$ and $\Diamond$ to $M(\beta S)$, and so we can define

$$\mu \,\Box\, \nu \text{ and } \mu \diamond \nu \quad \text{for} \quad \mu, \nu \in M(\beta S)\,.$$

In particular, we define $\delta_u \,\Box\, \delta_v$ for $u, v \in \beta S$, and, of course, $\delta_u \,\Box\, \delta_v = \delta_{u \,\Box\, v}$.

It is very rare to have $\mu \,\Box\, \nu = \mu \diamond \nu$.

Fact Let G be a group. Then the algebra $\ell^1(\beta G, \Box)$ is semisimple if and only if $\ell^1(\beta G, \Diamond)$ is semisimple. We do not know if this is always true when we replace G by a (cancellative) semigroup.

Semi-simplicity of $M(\beta S)$

We do have the following. A group G is **amenable** if there is an invariant mean on G: this is a translation-invariant linear functional λ on $\ell^\infty(G)$ with $\|\lambda\| = \langle 1, \lambda\rangle = 1$

Theorem (Granirer) Let S be an infinite, amenable group or $S = (\mathbb{N}, +)$. Then

$$\dim J(M(\beta S, \Box)) \geq 2^{\mathfrak{c}}.$$

Proof There are lots of invariant means, and these can be used to build elements of the radical. $\Box$

Open question Is $M(\beta \mathbb{F}_2, \Box)$ semisimple?

Semi-simplicity of $\ell^1(\beta\mathbb{N}, +)$

We seek a condition for this.

Lemma Let S be a countable semigroup that is a subsemigroup of a group G, and suppose that $(x_n : n \in \mathbb{N})$ is a sequence in $S^* \setminus K(S^*)$. Then there is an infinite subset A of S such that, for each $u \in A^*$:

(i) u is cancellable;

(ii) $u \,\Box\, x_n$ is right cancellable for each $n \in \mathbb{N}$;

(iii) for each $m, n \in \mathbb{N}$, either $x_m \in G x_n$ or $((\beta G) \,\Box\, u \,\Box\, x_m) \cap ((\beta G) \,\Box\, u \,\Box\, x_n) = \emptyset$. $\Box$

A first theorem

We denote the semigroup operation in G^* by '$+$'; for $x \in \beta G$ and $n \in \mathbb{N}$, we write $n * x$ for $x + \cdots + x$, where there are n copies of x.

Theorem Let S be a subsemigroup of an abelian group G. Take $f \in J(S^*)$. Then

$$\text{supp } f \subset K(S^*)\,.$$

Proof Assume that supp $f \not\subset K(S^*)$, and set

$$X = \text{supp } f \setminus K(S^*)\,,$$

so that X is a countable, non-empty set.

By the lemma, there exists cancellable $u \in \beta S$ with $u+x$ right cancellable for each $x \in X$, and such that, for each $x, y \in X$, either $x \in G + y$ or $(\beta G + u + x) \cap (\beta G + u + y) = \emptyset$.

Proof - continued

By replacing each $x \in X$ by $u+x$ and f by $\delta_u \star f$, we may suppose that x is right cancellable for each $x \in X$ and that, for each $x, y \in X$, either $x \in G + y$ or $(\beta G + x) \cap (\beta G + y) = \emptyset$.

We have not changed the value of $\|f\|$ because u is cancellable.

Suppose that $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_m} \in X$ and

$$x_{i_1} + \cdots + x_{i_k} \in G + x_{j_1} + \cdots + x_{j_m} .$$

Then $(\beta G + x_{i_k}) \cap (\beta G + x_{j_m}) \neq \emptyset$, and so $x_{i_k} \in G + x_{j_m}$. Since x_{i_k} and x_{j_m} are right cancellable,

$$x_{i_1} + \cdots + x_{i_{k-1}} \in G + x_{j_1} + \cdots + x_{j_{m-1}} .$$

Continuing, we see that that $k = m$ and that $x_{i_r} \in G + x_{j_r}$ for all $r \in \{1, ..., k\}$.

Proof - continued

Choose $x \in X$, and set $T_n = G + n * x$ for $n \in \mathbb{N}$. Set $h = f \mid T_1$, so that $h \in \ell^1(S^*)$. Since $f(x) \neq 0$, we have $h(x) \neq 0$.

By the above remark, $h^{*n} = f^{*n} \mid T_n$, and so $\|h^{*n}\|_1 \leq \|f^{*n}\|_1$. Consequently $h \in \mathcal{Q}(S^*)$.

Now define $\varphi \in \ell^1(G)$ by

$$\varphi(y) = h(y + x) \quad (y \in G)\,.$$

Then $\|\varphi^{*n}\|_1 = \|h^{*n} \mid T_n\|_1 \leq \|h^{*n}\|_1$, and hence $\varphi \in \mathcal{Q}(G)$. However $\mathcal{Q}(G) = \{0\}$ because G is abelian.

Hence $h(x) = 0$, a contradiction. $\square$

The main theorem

We set $J = J(\beta\mathbb{N}, +)$ and $K = K(\mathbb{N}^*)$.

Theorem Let $f \in \ell^1(\beta\mathbb{N}, +)$. Then $f \in J$ if and only if supp $f \subset K$ and $\delta_p \star f \star \delta_q = 0$ for each $p, q \in K$.

Proof Suppose that $f \in J$. By the previous theorem, supp $f \subset K$. Take $p, q \in K$, and set $G = p + K + q$, a subgroup of K. Then

$$\text{supp}\,(\delta_p \star f \star \delta_q) \subset G\,,$$

and so it is an element of $J(G)$, which is $\{0\}$.

For the converse, it follows that $g \star f$ is nilpotent of degree at most 3 for each $g \in \ell^1(\beta\mathbb{N}, +)^\sharp$, and so $f \in J$. $\square$

Consequence

Corollary The following are equivalent:

(a) $\ell^1(\beta\mathbb{N}, +)$ is semisimple;

(b) $\ell^1(K(\mathbb{N}^*))$ is semisimple;

(c) $\ell^1(G^*)$ is semisimple for some/every infinite, countable, abelian group G. $\square$

Is the condition of the theorem satisfied? Essentially, it says:

Theorem The algebra $\ell^1(\beta\mathbb{N}, +)$ is **not** semisimple if and only if there exist $x, y \in K(\mathbb{N}^*)$ with $x \neq y$ such that $p + x + q = p + y + q$ for each $p, q \in K(\mathbb{N}^*)$. $\square$

Whether or not this holds is a famous open question about $(\beta\mathbb{N}, +)$, raised at least 40 years ago.

Corollary Assume that $\ell^1(\mathbb{N}^*, +)$ is not semisimple. Then $(M(\beta\mathbb{F}_2), \Box)$ is not semisimple. $\square$