Direct finiteness of $\mathrm{CV}_p(G)$ and $\mathrm{PF}_p(G)$

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Directly finite algebras

Definition

A unital algebra A is directly finite (or Dedekind finite) if every $a, b \in A$ satisfying $ab = 1_A$ also satisfy $ba = 1_A$.

Examples

- A commutative and unital
- A finite-dimensional and unital
  (consider left reg rep of A on itself)

If A is directly finite, and B is a subalgebra of A containing 1_A, then B is directly finite.

Examples from functional analysis

- $\mathbb{C}I_X + \mathcal{K}(X)$ is directly finite for every Banach space X.
- If X is any ℓ^p or L^p, then $\mathcal{B}(X)$ is **not** directly finite.

Examples from functional analysis

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Theorem (Kaplansky; Montgomery (1969))

For any discrete group G, the group von Neumann algebra $\mathrm{VN}(G)$ is directly finite.

In particular, $\ell^1(G)$ is directly finite. I don't know any proof which avoids C^*/Hilbertian techniques.

Can apply this to some questions of the form "is every point in the spectrum of a convolution operator an approximate eigenvalue"?

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When G is discrete, $\mathrm{VN}(G)$ has a faithful tracial state

$$T \mapsto \langle T\delta_e \mid \delta_e \rangle$$

and so this theorem indeed generalizes the observation of Kaplansky.

Warning. $\mathrm{VN}(\mathrm{SL}(2,\mathbb{R}))$ is **not** directly finite.

$\mathrm{CV}_p(G)$ for general G

Fix $1 < p < \infty$. For G a locally compact group, write λ_p and ρ_p for the left and right regular representations $G \to \mathcal{B}(L^p(G))$.

Definition

$\mathrm{CV}_p(G) := \{T \in \mathcal{B}(L^p(G)) \colon T\rho_p(x) = \rho_p(x)T$ *for all* $x \in G.\}$

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Some non-obvious results

- $\mathrm{CV}_2(G) = \mathrm{VN}(G)$
  (by VON NEUMANN's double commutant theorem).
- When G is amenable, $\mathrm{CV}_p(G) \subseteq \mathrm{CV}_2(G)$
  [HERZ, 1973]
- For $p \neq 2$, $\mathrm{CV}_p(\mathrm{SL}(2,\mathbb{R})) \not\subseteq \mathrm{CV}_2(\mathrm{SL}(2,\mathbb{R}))$
  [special case of LOHOUE, 1980]

$\mathrm{CV}_p(G)$ FOR DISCRETE G

We would like to embed $\mathrm{CV}_p(G)$ as a unital subalgebra of some directly finite algebra.

Question.

If G is discrete, is $\mathrm{CV}_p(G)$ always contained in $\mathrm{CV}_2(G)$?

Theorem (C., perhaps folklore?)

Let G be discrete. If $T \in \mathrm{CV}_p(G)$ then T is a densely-defined, closed operator on $\ell^2(G)$ that is affiliated to $\mathrm{VN}(G)$. Consequently, $\mathrm{CV}_p(G)$ is directly finite.

In an algebra A we may define the quasi-product
$x \bullet y = x + y - xy$.

Note that if A is a unital, directly finite algebra, and $x, y \in A$ satisfy $x \bullet y = 0$, then $y \bullet x = 0$.

Non-unital algebras

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Definition

Let A be an algebra (with or without identity). We say A is DF if every $x, y \in A$ satisfying $x \bullet y = 0$ also satisfy $y \bullet x = 0$.

General properties

- If A is unital and DF, it is directly finite.
- If A is DF, so is the forced unitization $A \oplus \mathbb{C}1$.
- Any subalgebra of a DF algebra is also DF.

Lemma

Let J be a dense left ideal in an algebra A. If J is DF then so is A.

Theorem (C.)

If G is unimodular, then $\mathrm{C}_r^*(G)$ is DF.

Proof.

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Proof. Since G is unimodular $\mathrm{C}^*_r(G)$ has a densely-defined and faithful trace ϕ.

There is a dense $*$-ideal $J \subset \mathrm{C}^*_r(G)$ on which ϕ is finite-valued.

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There is a dense $*$-ideal $J \subset \mathrm{C}_r^*(G)$ on which ϕ is finite-valued. We can show that J is DF.

Hence A is DF by the earlier lemma. $\square$

$\mathrm{PF}_p(G)$ for G unimodular

Fix $1 < p < \infty$. Recall: $\lambda_p : G \to \mathcal{B}(L^p(G))$ is the left reg rep, by integration we get an injective HM $\lambda_p : C_c(G) \to \mathcal{B}(L^p(G))$.

Definition

$\mathrm{PF}_p(G) := \overline{\lambda_p(C_c(G))}^{\|\cdot\|}$.

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Definition

$\mathrm{PF}_p(G) := \overline{\lambda_p(C_c(G))}^{\|\cdot\|}$.

Of course $\mathrm{PF}_2(G)$ is usually known as $\mathrm{C}^*_r(G)$.

If G is amenable, we know [Herz, 1971] $\mathrm{PF}_p(G) \subseteq \mathrm{PF}_2(G)$.

Corollary

If G is amenable and unimodular then $\mathrm{PF}_p(G)$ is DF.

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Let G be a semisimple Lie group with finite centre. Then $\mathrm{PF}_p(G)$ is DF.

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Proof. Let $J_p(G) = \mathrm{PF}_p(G) \cap L^p(G)$. This is a dense left ideal in $\mathrm{PF}_p(G)$.

By [Cowling, 1973] G has the Kunze–Stein property, that is $L^p(G) \subseteq \mathrm{VN}(G)$. Hence $J_p(G) \subseteq \mathrm{C}^*_r(G)$.

Therefore $J_p(G)$ is DF. Hence $\mathrm{PF}_p(G)$ is DF. □

$L^1(G)$ need not be DF

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Let G be the affine group of either $\mathbb{R}$ or $\mathbb{C}$. Then $L^1(G)$ is not DF.

Ideas in the proof

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Let G be the affine group of either $\mathbb{R}$ or $\mathbb{C}$. Then $L^1(G)$ is not DF.

Ideas in the proof

First show $\mathrm{C}^*_r(G)$ is not DF. This follows from old, explicit calculations of faithful representations in which elements of $\mathbb{C}1 + \mathrm{C}^*_r(G)$ are represented by non-trivial Fredholm operators. [DIEP (1974) for real case; ROSENBERG (1976) for complex case.]

Then transfer the one-sided invertibility to $\mathbb{C}1 + L^1(G)$ using the fact that $L^1(G)$ is a Hermitian Banach $*$-algebra [LEPTIN, 1977]

Quo vadis?

Question.

For which groups G are all $\mathrm{CV}_p(G)$ directly finite?

Question.

Is there a unimodular G and some $p \in (1, \infty)$ such that $\mathrm{PF}_p(G)$ is not DF?

Question.

For which solvable G is $\mathrm{C}^*_r(G)$ DF? What about $L^1(G)$?