New results on semigroups of analytic functions

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The papers and their authors

- BCDMPS
  Semigroups of composition operators and integral operators in spaces of analytic functions
  Ann. Acad. Scient. Fennicae Math. **38** (2013), 1-23.
- BCDMS
  Semigroups of composition operators in BMOA and the extension of a theorem of Sarason
  Int. Eq. Oper. Theory **61** (2008), 45-62.

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Semigroups of analytic functions

A (one-parameter) semigroup of analytic functions is any continuous homomorphism $\Phi : (\mathbb{R}^+, +) \to \{f \in H^\infty(\mathbb{D}) : \|f\|_\infty \leq 1\}$, that is

$$t \mapsto \Phi(t) = \varphi_t$$

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from the additive semigroup of nonnegative real numbers into the composition semigroup of all analytic functions which map $\mathbb{D}$ into $\mathbb{D}$.
$\Phi = (\varphi_t)$ consists of $\varphi_t \in \mathscr{H}(\mathbb{D})$ with $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ and satisfying

1. φ_0 is the identity in $\mathbb{D}$,
2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
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Examples:

- $\phi_t(z) = e^{-t}z$ (Dilation semigroup)
- $\phi_t(z) = e^{it}z$ (Rotation semigroup)
- $\phi_t(z) = e^{-t}z + (1 - e^{-t})$

Generators of analytic semigroups

(E. Berkson, H. Porta (1978))
The infinitesimal generator of (φ_t) is the function

$$G(z) := \lim_{t\to 0^+} \frac{\varphi_t(z) - z}{t} = \frac{\partial \varphi_t}{\partial t}(z)|_{t=0}, \; z \in \mathbb{D}.$$

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G has a unique representation

$$G(z) = (\overline{b}z - 1)(z - b)P(z), \ z \in \mathbb{D}$$

where $b \in \overline{\mathbb{D}}$ (called the Denjoy-Wolff point of the semigroup) and $P \in \mathscr{H}(\mathbb{D})$ with $\operatorname{Re} P(z) \geq 0$ for all $z \in \mathbb{D}$.

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- $G(z) = -z$ for the dilation semigroup $(b = 0, \; P(z) = 1)$
- $G(z) = iz$ for the rotation semigroup $(b = 0, \; P(z) = -i)$
- $G(z) = -(z-1)$ for $\phi_t(z) = e^{-t}z + 1 - e^{-t}$ $(b = 1, \; P(z) = \frac{1}{1-z})$

Semigroups of operators

Each semigroup of analytic functions gives rise to a semigroup (C_t) consisting of composition operators on $\mathscr{H}(\mathbb{D})$ via composition

$$C_t(f) := f \circ \varphi_t, \qquad f \in \mathscr{H}(\mathbb{D}).$$

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- $\text{lím}_{t\to 0^+} \|C_t(f) - f\|_X = 0.$

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Given a semigroup (φ_t) and a Banach space X contained in $\mathscr{H}(\mathbb{D})$ we denote by $[\varphi_t, X]$ the maximal closed linear subspace of X such that (φ_t) generates a semigroups of operators on it.

Previous results on semigroups of analytic functions

Theorem

1. *Every semigroup of analytic functions generates a semigroup of operators on the Hardy spaces* H^p $(1 \leq p < \infty)$, *the Bergman spaces* A^p $(1 \leq p < \infty)$ *and the Dirichlet space, i.e.* $[\varphi_t, X] = X$ *in these cases.*

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2. *No non-trivial semigroup generates a semigroup of operators in the space* H^∞ *of bounded analytic functions, i.e.* $[\varphi_t, H^\infty] = H^\infty$ *implies* $\Phi = 0$.
3. *There are plenty of semigroups (but not all) which generate semigroups of operators in the disk algebra.*

The case $X = BMOA$

Definition

An analytic function f is said to belong to BMOA if

$$\|f\|_*^2 = \sup_I \frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1-|z|^2) dA(z) < \infty$$

where the sup is taken over all arcs $I \subset \partial\mathbb{D}$, $R(I)$ is the Carleson rectangle determined by I, $|I|$ denotes the normalized length of I and $dA(z)$ the normalized Lebesgue measure on $\partial\mathbb{D}$.

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VMOA is the subspace of functions satisfying

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1 - |z|^2) dA(z) = 0$$

It is known that VMOA is the closure of the polynomials in BMOA and that $(VMOA)^{**} = BMOA$.

The problem for *BMOA*

Here it is our starting motivation:

Theorem A. (Sarason) *Suppose* $f \in BMOA$*, then the following are equivalent:*

1. $f \in VMOA$.
2. $\lim_{t\to 0^+} \|f(e^{it}\cdot) - f\|_\star = 0$.
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1.- Describe (φ_t) such that $VMOA = [\varphi_t, BMOA]$.
2.- Given (φ_t) calculate $[\varphi_t, BMOA]$.

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The problem for *Bloch*

We also start with the well known result

Theorem B. (Anderson-Clunie-Pommerenke) Suppose $f \in Bloch$. Then

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3.- Does it hold that $bloch = [e^{it}z, Bloch]$**?**

4.- Describe (φ_t) **such that** $bloch = [\varphi_t, Bloch]$**.**

5.- Given (φ_t) **calculate** $[\varphi_t, Bloch]$**.**

A basic calculation

In general

$$VMOA \subsetneq [\varphi_t, BMOA] \subsetneq BMOA.$$

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Let $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$. Then
$f(z) = \log(\frac{1}{1-z}) \in [\varphi_t, BMOA] \setminus VMOA$.

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In general

$$VMOA \subsetneq [\varphi_t, BMOA] \subsetneq BMOA.$$

Let $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$. Then $f(z) = \log(\frac{1}{1-z}) \in [\varphi_t, BMOA] \setminus VMOA$. Indeed

$$f(\varphi_t(z)) = \log(\frac{1}{1-\varphi_t(z)}) = tf(z)$$

and therefore

$$\lim_{t\to 0} \|f \circ \varphi_t - f\|_* = 0.$$

Results on *BMOA*

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Let G be the infinitesimal generator of (φ_t). Then,

$$[\varphi_t, BMOA] = \overline{\{f \in BMOA : Gf' \in BMOA\}}.$$

More results on *BMOA*

Theorem

Let (φ_t) *be a semigroup with infinitesimal generator* G*. Assume that for some* $0 < \alpha < 1$,

$$\frac{(1-|z|)^\alpha}{G(z)} = O(1) \ \text{as}\ |z| \to 1. \tag{2}$$

Then $VMOA = [\varphi_t, BMOA]$.

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Then $VMOA = [\varphi_t, BMOA]$.

Corollary

Suppose $(\varphi_t(z))$ is a semigroup whose generator G satisfies the condition (2). Then for a function $f \in BMOA$ the following are equivalent

1. $f \in VMOA$.
2. $\lim_{t\to 0^+} ||f \circ \varphi_t - f||_\star = 0$.

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Theorem

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Theorem

There are not non-trivial semigroups of analytic functions (φ_t) generating a C_0-semigroup in Bloch, i.e. if $[\varphi_t, Bloch] = Bloch$ then $\varphi_t(z) = 0$.

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Main results on *Bloch* and *BMOA*

Suppose now that X is either *VMOA* or *bloch* so that the second dual X^{**} is *BMOA* or *Bloch* respectively. Let (φ_t) be a semigroup on $\mathbb{D}$ and let (C_t) be the induced semigroup of composition operators on X^{**} and denote $S_t = C_t|_X$.

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Theorem

Let (φ_t) be a semigroup and X be one of the spaces VMOA or bloch. Denote by Γ the generator of the induced composition semigroup (S_t) on X and let $\lambda \in \rho(\Gamma)$. Then
(1) $[\varphi_t, BMOA] = VMOA$ if and only if $\mathscr{R}(\lambda, \Gamma) = (\lambda I - \Gamma)^{-1}$ is weakly compact on VMOA.

(2) $[\varphi_t, Bloch] = bloch$ if and only if $\mathscr{R}(\lambda, \Gamma)$ is weakly compact on bloch.

A theorem and its proof

Theorem

Let G be the infinitesimal generator of (φ_t). Then,

$$\{f \in BMOA : Gf' \in BMOA\} \subset [\varphi_t, BMOA].$$

Proof:

Let $f \in BMOA$ such that $m := Gf' \in BMOA$. First of all, one shows that

$$(f \circ \varphi_t)'(z) - f'(z) = \int_0^t (m \circ \varphi_s)'(z) ds; \text{ for } t \geq 0, \ z \in \mathbb{D}.$$

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Next let I be an interval in $\partial\mathbb{D}$ and $R(I)$ the corresponding Carleson rectangle.

For $0 \leq t \leq 1$ we have

$$\begin{aligned}
&\int_{R(I)} \left|(f \circ \varphi_t)'(z) - f'(z)\right|^2 (1 - |z|^2) dA(z) \\
&= \int_{R(I)} \left|\int_0^t (m \circ \varphi_s)'(z) ds\right|^2 (1 - |z|^2) dA(z) \\
&\leq \int_{R(I)} t \left(\int_0^1 \left|(m \circ \varphi_s)'(z)\right|^2 ds\right) (1 - |z|^2) dA(z)
\end{aligned}$$

where we have applied Cauchy-Schwarz in the inside integral.

Hence

$$
\begin{aligned}
\|f\circ\varphi_t - f\|_\star &= \sup_{I\subseteq\partial\mathbb{D}}\left(\frac{1}{|I|}\int_{R(I)}\left|(f\circ\varphi_t)'(z)-f'(z)\right|^2(1-|z|^2)dA(z)\right)^{\frac{1}{2}}\\
&\le \sup_{I\subseteq\partial\mathbb{D}}\left(\frac{1}{|I|}\int_{R(I)} t\left(\int_0^1\left|(m\circ\varphi_s)'(z)\right|^2 ds\right)(1-|z|^2)dA(z)\right)^{\frac{1}{2}}\\
&\le \sup_{I\subseteq\partial\mathbb{D}}\left(t\int_0^1\left(\frac{1}{|I|}\int_{R(I)}\left|(m\circ\varphi_s)'(z)\right|^2(1-|z|^2)dA(z)\right)ds\right)^{\frac{1}{2}}\\
&\le \left(t\int_0^1 \|m\circ\varphi_s\|_\star^2\, ds\right)^{\frac{1}{2}}\\
&\le \sqrt{t}\sup_{s\in[0,1]}\|m\circ\varphi_s\|_\star\\
&\le \sqrt{t}C\|m\|_\star \sup_{s\in[0,1]}(1-\log(1-\varphi_s(0)) \le C'\sqrt{t},
\end{aligned}
$$

where we have used $\|m\circ\psi\|_* \le C\|m\|_*\log(\frac{e}{1-\psi(0)})$ for any $\psi:\mathbb{D}\to\mathbb{D}$ analytic.

Therefore $f\in[\varphi_t, BMOA]$.