New results on semigroups of analytic functions

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The basic definitions

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The papers and their authors

**BCDMPS**

*Semigroups of composition operators and integral operators in spaces of analytic functions*


**BCDMS**

*Semigroups of composition operators in BMOA and the extension of a theorem of Sarason*


*Authors:*

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Semigroups of analytic functions

A (one-parameter) semigroup of analytic functions is any continuous homomorphism \( \Phi : (\mathbb{R}^+, +) \rightarrow \{ f \in H^\infty(\mathbb{D}) : \|f\|_\infty \leq 1 \} \), that is

\[
t \mapsto \Phi(t) = \varphi_t
\]

from the additive semigroup of nonnegative real numbers into the composition semigroup of all analytic functions which map \( \mathbb{D} \) into \( \mathbb{D} \).
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from the additive semigroup of nonnegative real numbers into the composition semigroup of all analytic functions which map \( \mathbb{D} \) into \( \mathbb{D} \). \( \Phi = (\varphi_t) \) consists of \( \varphi_t \in \mathcal{H}(\mathbb{D}) \) with \( \varphi_t(\mathbb{D}) \subseteq \mathbb{D} \) and satisfying

1. \( \varphi_0 \) is the identity in \( \mathbb{D} \),
2. \( \varphi_{t+s} = \varphi_t \circ \varphi_s \), for all \( t, s \geq 0 \),
3. \( \varphi_t(z) \to \varphi_0(z) = z \), as \( t \to 0 \), \( z \in \mathbb{D} \).
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from the additive semigroup of nonnegative real numbers into the composition semigroup of all analytic functions which map $\mathbb{D}$ into $\mathbb{D}$. $\Phi = (\varphi_t)$ consists of $\varphi_t \in \mathcal{H}(\mathbb{D})$ with $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ and satisfying

1. $\varphi_0$ is the identity in $\mathbb{D}$,
2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
3. $\varphi_t(z) \rightarrow \varphi_0(z) = z$, as $t \rightarrow 0$, $z \in \mathbb{D}$.

**Examples:**

- $\varphi_t(z) = e^{-t}z$ (Dilation semigroup)
- $\varphi_t(z) = e^{it}z$ (Rotation semigroup)
- $\varphi_t(z) = e^{-t}z + (1 - e^{-t})$
Generators of analytic semigroups

(E. Berkson, H. Porta (1978))
The infinitesimal generator of \((\varphi_t)\) is the function

\[
G(z) := \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t} = \frac{\partial \varphi_t}{\partial t}(z)\big|_{t=0}, \ z \in \mathbb{D}.
\]
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\[
G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z) \frac{\partial \varphi_t(z)}{\partial z}, \quad z \in \mathbb{D}, \quad t \geq 0. \tag{1}
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G has a unique representation

\[
G(z) = (bz - 1)(z - b)P(z), \quad z \in \mathbb{D}
\]

where \(b \in \overline{\mathbb{D}}\) (called the Denjoy-Wolff point of the semigroup) and \(P \in \mathcal{H}(\mathbb{D})\) with \(\text{Re} P(z) \geq 0\) for all \(z \in \mathbb{D}\).
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- \(G(z) = -z\) for the dilation semigroup \((b = 0, \; P(z) = 1)\)
- \(G(z) = iz\) for the rotation semigroup \((b = 0, \; P(z) = -i)\)
- \(G(z) = -(z - 1)\) for \(\varphi_t(z) = e^{-t}z + 1 - e^{-t} \quad (b = 1, \; P(z) = \frac{1}{1-z})\)
Each semigroup of analytic functions gives rise to a semigroup \((C_t)\) consisting of composition operators on \(H(\mathbb{D})\) via composition

\[ C_t(f) := f \circ \varphi_t, \quad f \in H(\mathbb{D}). \]
Semigroups of operators

Each semigroup of analytic functions gives rise to a semigroup \((C_t)\) consisting of composition operators on \(\mathcal{H}(D)\) via composition

\[
C_t(f) := f \circ \varphi_t, \quad f \in \mathcal{H}(D).
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Given a Banach space \(X \subset \mathcal{H}(D)\) and a semigroup \((\varphi_t)\), we say that \((\varphi_t)\) generates a semigroup of operators on \(X\) if \((C_t)\) is a \(C_0\)-semigroup of bounded operators in \(X\), i.e.
Semigroups of operators

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- \(C_t(f) \in X\) for all \(t \geq 0\) and for every \(f \in X\)
- \(\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0\).
Semigroups of operators

Each semigroup of analytic functions gives rise to a semigroup \((C_t)\) consisting of composition operators on \(H(D)\) via composition

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- \(C_t(f) \in X\) for all \(t \geq 0\) and for every \(f \in X\)
- \(\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0\).

Given a semigroup \((\varphi_t)\) and a Banach space \(X\) contained in \(H(D)\) we denote by \([\varphi_t, X]\) the maximal closed linear subspace of \(X\) such that \((\varphi_t)\) generates a semigroups of operators on it.
Previous results on semigroups of analytic functions

Theorem

1. Every semigroup of analytic functions generates a semigroup of operators on the Hardy spaces $H^p$ ($1 \leq p < \infty$), the Bergman spaces $A^p$ ($1 \leq p < \infty$) and the Dirichlet space, i.e. $[\phi_t, X] = X$ in these cases.
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2. *No non-trivial semigroup generates a semigroup of operators in the space* $H^\infty$ *of bounded analytic functions*, i.e. $[\varphi_t, H^\infty] = H^\infty$ implies $\Phi = 0$. 

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A theorem with proof

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Previous results on semigroups of analytic functions

Theorem

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2. No non-trivial semigroup generates a semigroup of operators in the space $H^\infty$ of bounded analytic functions, i.e. $[\varphi_t, H^\infty] = H^\infty$ implies $\Phi = 0$.

3. There are plenty of semigroups (but not all) which generate semigroups of operators in the disk algebra.
The case $X = BMOA$

**Definition**

An analytic function $f$ is said to belong to $BMOA$ if

$$
\|f\|_*^2 = \sup \frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1 - |z|^2) dA(z) < \infty
$$

where the sup is taken over all arcs $I \subset \partial \mathbb{D}$, $R(I)$ is the Carleson rectangle determined by $I$, $|I|$ denotes the normalized length of $I$ and $dA(z)$ the normalized Lebesgue measure on $\partial \mathbb{D}$. 
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$VMOA$ is the subspace of functions satisfying

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1 - |z|^2) dA(z) = 0$$

It is known that $VMOA$ is the closure of the polynomials in $BMOA$ and that $(VMOA)^{**} = BMOA$. 
The problem for $BMOA$

Here it is our starting motivation:

**Theorem A.** (Sarason) *Suppose $f \in BMOA$, then the following are equivalent:*

1. $f \in VMOA$.
2. $\lim_{t \to 0^+} \| f(e^{it} \cdot) - f \|_* = 0$.
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*Note that* \( \lim_{t \to 0^+} \| f(e^{it} \cdot) - f \|_* = 0 \) *means* \( f \in [e^{it}z, BMO] \).
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Note that $\lim_{t \to 0^+} \| f(e^{it} \cdot) - f \|_\star = 0$ means $f \in [e^{it}z, \text{BMO}]$.

Note that $\lim_{r \to 1} \| f(r \cdot) - f \|_\star = 0$ can be written $\lim_{t \to 0^+} \| f(e^{-t} \cdot) - f \|_\star = 0$. 

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*Problems:*

1. Describe $(\varphi_t)$ such that $VMOA = [\varphi_t, BMOA]$. 
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*Problems:*
1. *Describe* \( (\phi_t) \) *such that* \( VMOA = [\phi_t, BMOA] \).
2. *Given* \( (\phi_t) \) *calculate* \( [\phi_t, BMOA] \).
The case $X = \text{Bloch}$

**Definition**

An analytic function $f$ is said to belong to Bloch if

$$\|f\|_{\text{Bloch}} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) < \infty,$$
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$\text{bloch}$ is the subspace of functions such that

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$$

It is known that $bloch$ is the closure of polynomials in the Bloch space and $(bloch)^{**} = \text{Bloch}$
The problem for *Bloch*

We also start with the well known result

**Theorem B.** (Anderson-Clunie-Pommerenke) Suppose $f \in Bloch$. Then

$$ f \in bloch \iff \lim_{r \to 1} \| f(r\cdot) - f \|_{Bloch} = 0. $$
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\]

**Problems:**

3.- Does it hold that \( \text{bloch} = [e^{it}z, Bloch] \)?

4.- Describe \( (\varphi_t) \) such that \( \text{bloch} = [\varphi_t, Bloch] \).
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We also start with the well known result

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**Problems:**

3.- **Does it hold that** $bloch = [e^{it}z, Bloch]$?
4.- **Describe** $(\varphi_t)$ **such that** $bloch = [\varphi_t, Bloch]$.
5.- **Given** $(\varphi_t)$ **calculate** $[\varphi_t, Bloch]$. 

Oscar Blasco

New results on semigroups of analytic functions
In general

\[ VMOA \subset [\varphi_t, BMOA] \subset BMOA. \]
A basic calculation

In general

$$VMOA \subsetneq [\varphi_t, BMOA] \subsetneq BMOA.$$ 

Let $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$. Then

$$f(z) = \log\left(\frac{1}{1-z}\right) \in [\varphi_t, BMOA] \setminus VMOA.$$
A basic calculation

In general

\[ VMOA \subsetneq [\varphi_t, BMOA] \subsetneq BMOA. \]

Let \( \varphi_t(z) = e^{-t}z + 1 - e^{-t} \). Then
\[ f(z) = \log\left(\frac{1}{1-z}\right) \in [\varphi_t, BMOA] \setminus VMOA. \] Indeed
\[
    f(\varphi_t(z)) = \log\left(\frac{1}{1 - \varphi_t(z)}\right) = tf(z)
\]
and therefore
\[
    \lim_{t \to 0} \| f \circ \varphi_t - f \|_* = 0.
\]
Results on **BMOA**

**Theorem**

*Every semigroup \((\varphi_t)\) generates a semigroup of operators on \(VMOA\), i.e. \(VMOA = [\varphi_t, VMOA]\).*
Every semigroup \((\varphi_t)\) generates a semigroup of operators on \(VMOA\), i.e. \(VMOA = [\varphi_t, VMOA]\).

Let \(G\) be the infinitesimal generator of \((\varphi_t)\). Then,

\[
[\varphi_t, BMOA] = \{ f \in BMOA : Gf' \in BMOA \}.
\]
More results on BMOA

Theorem

Let \( (\varphi_t) \) be a semigroup with infinitesimal generator \( G \). Assume that for some \( 0 < \alpha < 1 \),

\[
\frac{(1 - |z|)^\alpha}{G(z)} = O(1) \quad \text{as} \quad |z| \to 1.
\] (2)

Then \( VMOA = [\varphi_t, BMOA] \).
More results on $BMOA$

**Theorem**

Let $(\varphi_t)$ be a semigroup with infinitesimal generator $G$. Assume that for some $0 < \alpha < 1$,

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\frac{(1 - |z|)^\alpha}{G(z)} = O(1) \text{ as } |z| \to 1.
$$

(2)

Then $VMOA = [\varphi_t, BMOA]$.

**Corollary**

Suppose $(\varphi_t(z))$ is a semigroup whose generator $G$ satisfies the condition (2). Then for a function $f \in BMOA$ the following are equivalent

1. $f \in VMOA$.
2. $\lim_{t \to 0^+} \|f \circ \varphi_t - f\|_* = 0$. 

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Results on Bloch

Theorem

Any semigroup of analytic functions $(\varphi_t)$ generates a $C_0$-semigroup in Bloch, i.e. $[\varphi_t, \text{bloch}] = \text{bloch}$. 
Results on *Bloch*

**Theorem**

*Any semigroup of analytic functions* $(\varphi_t)$ *generates a* $C_0$-*semigroup in Bloch*, *i.e.* $[\varphi_t,\text{bloch}] = \text{bloch}$.

**Theorem**

*There are not non-trivial semigroups of analytic functions* $(\varphi_t)$ *generating a* $C_0$-*semigroup in Bloch*, *i.e.* if $[\varphi_t,\text{Bloch}] = \text{Bloch}$ then $\varphi_t(z) = 0$. 
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**Results on Bloch**

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**Theorem**

Let \( G \) be the infinitesimal generator of \( (\varphi_t) \). Then,

\[
[\varphi_t, \text{Bloch}] = \{ f \in \text{Bloch} : Gf' \in \text{Bloch} \}.
\]
Main results on *Bloch* and *BMOA*

Suppose now that $X$ is either *VMOA* or *bloch* so that the second dual $X^{**}$ is *BMOA* or *Bloch* respectively. Let $(\varphi_t)$ be a semigroup on $\mathbb{D}$ and let $(C_t)$ be the induced semigroup of composition operators on $X^{**}$ and denote $S_t = C_t|_X$.

Theorem

Let $(\varphi_t)$ be a semigroup and $X$ be one of the spaces *VMOA* or *bloch*. Denote by $\Gamma$ the generator of the induced composition semigroup $(S_t)$ on $X$ and let $\lambda \in \rho(\Gamma)$. Then

1. $[\varphi_t, \text{BMOA}] = \text{VMOA}$ if and only if $R(\lambda, \Gamma) = (\lambda I - \Gamma)^{-1}$ is weakly compact on $\text{VMOA}$.

2. $[\varphi_t, \text{Bloch}] = \text{Bloch}$ if and only if $R(\lambda, \Gamma)$ is weakly compact on $\text{Bloch}$. 

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1. $[\varphi_t, BMOA] = VMOA$ if and only if $R(\lambda, \Gamma) = (\lambda I - \Gamma)^{-1}$ is weakly compact on $VMOA$.

2. $[\varphi_t, Bloch] = bloch$ if and only if $R(\lambda, \Gamma)$ is weakly compact on bloch.
A theorem and its proof

Theorem

Let $G$ be the infinitesimal generator of $(\varphi_t)$. Then,

$$\{ f \in BMOA : Gf' \in BMOA \} \subset [\varphi_t, BMOA].$$

Proof:

Let $f \in BMOA$ such that $m := Gf' \in BMOA$. First of all, one shows that

$$(f \circ \varphi_t)'(z) - f'(z) = \int_0^t (m \circ \varphi_s)'(z) \, ds; \quad \text{for } t \geq 0, \ z \in \mathbb{D}.$$
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$$(f \circ \varphi_t)'(z) - f'(z) = \int_0^t (m \circ \varphi_s)'(z)ds; \quad \text{for } t \geq 0, \ z \in \mathbb{D}.$$

Next let $I$ be an interval in $\partial \mathbb{D}$ and $R(I)$ the corresponding Carleson rectangle.
For $0 \leq t \leq 1$ we have

$$
\int_{R(I)} |(f \circ \varphi_t)'(z) - f'(z)|^2 (1 - |z|^2) dA(z)
$$

$$
= \int_{R(I)} \left( \int_0^t |(m \circ \varphi_s)'(z)|^2 ds \right)^2 (1 - |z|^2) dA(z)
$$

$$
\leq \int_{R(I)} t \left( \int_0^1 |(m \circ \varphi_s)'(z)|^2 ds \right) (1 - |z|^2) dA(z)
$$

where we have applied Cauchy-Schwarz in the inside integral.
Hence

\[
\|f \circ \varphi_t - f\|_\star = \sup_{I \subseteq \partial \mathbb{D}} \left( \frac{1}{|I|} \int_{R(I)} \left| (f \circ \varphi_t)'(z) - f'(z) \right|^2 (1 - |z|^2) dA(z) \right)^{\frac{1}{2}}
\]

\[
\leq \sup_{I \subseteq \partial \mathbb{D}} \left( \frac{1}{|I|} \int_{R(I)} t \left( \int_0^1 \left| (m \circ \varphi_s)'(z) \right|^2 ds \right) (1 - |z|^2) dA(z) \right)^{\frac{1}{2}}
\]

\[
\leq \sup_{I \subseteq \partial \mathbb{D}} \left( t \int_0^1 \left( \frac{1}{|I|} \int_{R(I)} \left| (m \circ \varphi_s)'(z) \right|^2 (1 - |z|^2) dA(z) \right) ds \right)^{\frac{1}{2}}
\]

\[
\leq \left( t \int_0^1 \|m \circ \varphi_s\|_\star^2 ds \right)^{\frac{1}{2}}
\]

\[
\leq \sqrt{t} \sup_{s \in [0,1]} \|m \circ \varphi_s\|_\star
\]

\[
\leq \sqrt{t} C \|m\|_\star \sup_{s \in [0,1]} (1 - \log(1 - \varphi_s(0))) \leq C' \sqrt{t},
\]

where we have used \( \|m \circ \psi\|_\star \leq C \|m\|_\star \log\left( \frac{e}{1 - \psi(0)} \right) \) for any \( \psi : \mathbb{D} \to \mathbb{D} \) analytic.

Therefore \( f \in [\varphi_t, BMOA] \).