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# On algebraic K-theory categorical groups

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**Abstract** Homotopy categorical groups of any pointed space are defined via the fundamental groupoid of iterated loop spaces. This notion allows, paralleling the group case, to introduce the notion of K-categorical groups $\mathbb{K}_i R$ of any ring $R$. We also show the existence of a fundamental categorical crossed module associated to any fibre homotopy sequence and then, $\mathbb{K}_1 R$ and $\mathbb{K}_2 R$ are characterized, respectively, as the homotopy cokernel and kernel of the fundamental categorical crossed module associated to te fibre homotopy sequence $F(R) \xrightarrow{d_R} BGL(R) \xrightarrow{q_R} BGL(R)^+$. As consequence, the 3th level of the Postnikov tower of the $K$-theory spectrum of $R$ is classified by this categorical crossed module.

**Keywords** categorical group, categorical crossed module, algebraic K-theory

**Mathematics Subject Classification (2000)** 18D10, 19D99, 20L05.

## 1 Introduction

For any ring $R$, if $GL_n(R)$ is the general linear group of invertible matrices $n \times n$ with entries in $R$, there is a sequence $GL_1(R) \subset GL_2(R) \subset GL_3(R) \subset \cdots$. If $GL(R)$ is the direc limit and $E(R)$ is the subgroup of $GL(R)$ generated by the elementary matrices, then $E(R) = [GL(R), GL(R)]$, the derived subgroup, and the abelian quotient group, $GL(R)/E(R)$, is the so called Whitehead group of $R$ denoted by $K_1 R$.

The Steinberg groups $St_n(R)$ were introduced by Milnor as those groups given by generators and relations encapsulating the key rules of the elementary matrices and the canonical homomorphism $\Phi_n : St_n(R) \to E_n(R) \subset GL_n(R)$ induces a homomorphism in the corresponding direct limits

$$St(R) \xrightarrow{\Phi} GL(R) \ ,$$

Partially supported by DGI of Spain and FEDER (Project: MTM2007-65431); Consejería de Innovacion de J. de Andalucía (P06-FQM-1889); MEC de España, 'Ingenio Mathematica(i-Math)´ No.CSD2006-00032 (consolider-Ingenio 2010).

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whose image is actually $E(R)$, and whose kernel gives the definition of the 2-th group of algebraic $K$-theory, that is, $K_2(R) = Ker(\Phi)$.

Higher $K$-groups of a ring $R$, $K_iR$, $i \geq 1$, were defined by Quillen as the composition of covariants functors

$$K_i : R \mapsto GLR \mapsto BGLR \mapsto BGLR^+ \mapsto \pi_i BGLR^+$$

where $BGL(R)$ is the classifying space of the group $GL(R)$ and $BGL(R)^+$, its Quillen's plus-construction satisfies that $\pi_1 BGL(R)^+ \cong \frac{\pi_1 BGL(R)}{E(R)} = \frac{GL(R)}{E(R)} = K_1R$. Moreover, $\pi_2 BGL(R)^+ \cong K_2R$. In this way, the Quillen K-groups $K_1$ and $K_2$ are recognized, respectively, as the cokernel and the kernel of the group homomorphism $St(R) \xrightarrow{\Phi} GL(R)$ which is a crossed module of groups. Actually, $\Phi$ is an instance of crossed module arising from a general process that associates such kind of structure to any map of pointed spaces. For any fibration $p : (X, x_0) \to (B, b_0)$ with fibre $F = p^{-1}(b_0)$, the morphism $\pi_1(F, x_0) \xrightarrow{i} \pi_1(X, x_0)$, induced by the inclusion map $i : (F, x_0) \hookrightarrow (X, x_0)$, is a crossed module of groups, called the 'fundamental crossed module' of the fibration $p$. This structure also can be associated to any map by using the standard procedure in homotopy theory of factoring any map of pointed spaces $f : (X, x_0) \to (Y, y_0)$ through a homotopy equivalence $(X, x_0) \to (\overline{X}, *)$ followed by a fibration $\overline{f} : (\overline{X}, *) \to (Y, y_0)$. In this way we have a functor $f \mapsto \overline{f}$ from maps to fibrations and this functor allows then to define the fundamental categorical crossed module of any map between pointed spaces $f : (X, x_0) \to (Y, y_0)$. In fact, if $Kf$ is the homotopy kernel of $f$, that is, the fibre of the fibration $\overline{f}$, then $\pi_1(Kf, *) \xrightarrow{\pi_1(kf)} \pi_1(X, x_0)$ is called the fundamental crossed module of the fibre homotopy sequence $Kf \xrightarrow{kf} X \xrightarrow{f} Y$.

A basic structure for the Algebraic K-theory is the fibre homotopy sequence

$$F(R) \to BGL(R) \to BGL(R)^+ ,$$

whose associated fundamental crossed module $\pi_1 F(R) \longrightarrow \pi_1 BGL(R)$ is equivalent to the above quoted $St(R) \xrightarrow{\Phi} GL(R)$ and, therefore, its cokernel and kernel are respectively the K-groups $K_1$ and $K_2$.

The main object of our work is to show that all these results are the trace in the group setting of another 2-dimensional ones which live and can be formulated at the level of categorical groups. The key to carry out this process is supported by:

a) Suitable notions of homotopy categorical groups $\wp_n(X, *)$, $n \geq 2$, associated to any pointed space and, also, the existence of 2-exact sequences associated both to any pair of pointed spaces and to any fibration. This is achieved in §3.

b) A suitable notion of crossed module in the 2-category of categorical groups and the existence of such a structure associated to any fibration of pointed spaces, that is, of a ´fundamental categorical crossed module of a fibration‘. To this end is devoted Section §4.

The point a) will allow us to define, in §4, notions of $K$-theory categorical groups of a ring $R$, $\mathbb{K}_iR = \wp_{i+1}BGLR^+$, $i \geq 1$, whereas point b) will allow us to identify, in this §4, the $K$-categorical groups $\mathbb{K}_iR$, $i = 1, 2$, respectively as the homotopy cokernel and the homotopy kernel of the fundamental categorical crossed module associated to the fibre homotopy sequence $F(R) \to BGL(R) \to BGL(R)^+$ . The homotopy groups $\pi_i$, $i = 1, 2, 3$, of the homotopy 3-type represented by this categorical crossed module are then shown to be $K_iR$, $i = 1, 2, 3$, respectively.

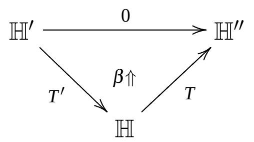
Previously, in §2, we give a minimum number of preliminary notions and results that can be completed, for general background, in the bibliography.

## 2 Preliminaries

Throughout the paper we freely use notions and terminology related to categorical groups (see [3, 10, 15, 16] and the references therein for general background).

If $\mathbb{H}$ is a categorical group, $\pi_0\mathbb{H}$ stands for the group (abelian if $\mathbb{H}$ is braided) of isomorphism classes of objects and $\pi_1\mathbb{H}$ for the abelian group of automorphisms of the unit object.

Given a diagram in the 2-category $\mathscr{C}G$ of categorical groups as the following,




recall (see [14]) that the triple $(T', \beta, T)$ is said to be 2-exact if the factorization of $T'$ through the homotopy kernel of $T$ is a full and essentially surjective functor. If $(T', \beta, T)$ is 2-exact, then $\pi_i(\mathbb{H}' \xrightarrow{T'} \mathbb{H} \xrightarrow{T} \mathbb{H}'')$, $i = 0, 1$, is an exact sequence of groups.

A categorical crossed module $\langle \mathbb{H}, \mathbb{G}, T, \nu, \chi \rangle$, consists of a morphism of categorical groups $T = (T, \mu) : \mathbb{H} \to \mathbb{G}$ together with an action of $\mathbb{G}$ on $\mathbb{H}$, that is, a functor $\mathbb{G} \times \mathbb{H} \longrightarrow \mathbb{H}$, $(X, A) \mapsto {}^{X}A$ and coherent natural isomorphisms

$$\psi_{X,A,B} : {}^{X}(A \otimes B) \to {}^{X}A \otimes {}^{X}B, \quad \Phi_{X,Y,A} : {}^{(X \otimes Y)}A \to {}^{X}({}^{Y}A),$$

and two families of natural isomorphisms in $\mathbb{G}$ and $\mathbb{H}$, respectively

$$\nu = \big(\nu_{X,A} : T({}^{X}A) \otimes X \longrightarrow X \otimes T(A)\big)_{(X,A) \in \mathbb{G} \times \mathbb{H}}$$

$$\chi = \big(\chi_{A,B} : {}^{TA}B \otimes A \longrightarrow A \otimes B\big)_{(A,B) \in \mathbb{H}}$$

such that the coherence conditions of [8] hold.

In [8] it was observed that if $\langle \mathbb{H}, \mathbb{G}, T, \nu, \chi \rangle$ is a categorical crossed module, then $KerT$, the homotopy kernel of $T$, is a braided categorical group and, also, that it is possible to construct a new categorical group, $CokerT$, called the homotopy cokernel of the categorical crossed module. Moreover (see [6, Lemma 2.5]), $\pi_0 KerT \cong \pi_1 CokerT$, and then the homotopy groups of the categorical crossed module $\langle \mathbb{H}, \mathbb{G}, T, \nu, \chi \rangle$ were defined by:

$$\Pi_i \langle \mathbb{H}, \mathbb{G}, T, \nu, \chi \rangle = \begin{cases} \pi_0 CokerT & for \;\; i = 1 \\ \pi_0 KerT \cong \pi_1 CokerT & for \;\; i = 2 \\ \pi_1 KerT & for \; i = 3 \,. \end{cases}$$

## 3 The 2-exact homotopy sequence of a fibration

Below we will denote by $\wp_1(Y)$ the fundamental groupoid of any topological space $Y$. If $(X, x_0)$ is a pointed topological space with base point $x_0 \in X$, then $\wp_2(X, x_0) = \wp_1(\Omega(X, x_0))$, the fundamental groupoid of the loop space $\Omega(X, x_0)$, is enriched with a natural categorical group structure and we shall refer to it as the *fundamental categorical group* of $(X, x_0)$ (see [9, Example 2.4] where it is explicitly described or [11] where alternative arguments, based on an internal notion of categorical group defined in a groupoid-enriched category, are used). These afore-mentioned arguments also allow us to ensure that, if we define for all $n \geq 2$,

$$\wp_n(X,x_0) = \wp_1(\Omega^{n-1}(X,x_0))$$

then $\wp_3(X,x_0)$ is a braided categorical group and $\wp_n(X,x_0)$, $n \geq 4$, are symmetric categorical groups. These constructions $\wp_n$, $n \geq 2$, actually define functors from the category of pointed topological spaces to the category of (braided or symmetric) categorical groups, with $\pi_0\wp_n(X,x_0) \cong \pi_{n-1}(X,x_0)$ and $\pi_1\wp_n(X,x_0) \cong \pi_n(X,x_0)$. These homotopy categorical groups $\wp_n(X,x_0)$ can be identified, through the pointed total singular complex of a pointed space, with the homotopy groupoids defined in [5] for any pointed Kan complex. Remark (see [11]) that there is a categorical group action of of $\rho_2(X,x_0)$ on $\rho_n(X,x_0)$ (for $n = 2$ it is given by conjugation)

The homotopy kernel of a pointed map $f : (X,x_0) \to (Y,y_0)$ is a pair $(Kf,kf)$ where $Kf$ is the subspace of the product space $X \times Y^I$ consisting of the pairs $(x,\omega : y_0 \to f(x))$ (i.e., $\omega$ is a path in $Y$ starting at the base point $y_0$ and ending in $f(x)$). $Kf$ is pointed by the pair $(x_0,\omega_{y_0})$ (where $\omega_{y_0}$ is the constant loop in $Y$ at $y_0$) and the map $kf : (Kf,(x_0,\omega_{y_0})) \to (X,x_0)$ is the projection. In particular, considering for any pointed topological pair $(X,A,x_0)$ the inclusion $i : (A,x_0) \hookrightarrow (X,x_0)$, the homotopy kernel is given by the subspace $Ki = \{(a,\omega) \in A \times X^I \,/\, \omega(0) = x_0,\, \omega(1) = a\}$ (i.e. the set of paths of $X$ starting at the base point $x_0$ and ending in $A$) and the map $ki : (Ki,(x_0,\omega_{x_0})) \to (A,x_0)$ is given by $ki(a,\omega) = a$.

For any pointed topological pair $(X,A,x_0)$ we define:

$$\wp_2(X,A,x_0) = \wp_1(Ki)$$

and, for $n \geq 3$,

$$\wp_n(X,A,x_0) = \wp_1(\Omega^{n-2}(Ki,(x_0,\omega_{x_0}))) \ .$$

Thus, $\wp_2(X,A,x_0)$ is a groupoid, $\wp_3(X,A,x_0)$ is a categorical group, $\wp_4(X,A,x_0)$ is a braided categorical group and $\wp_n(X,A,x_0)$, $n \geq 5$, is a symmetric categorical group. We refer to these categorical groups as the *relative homotopy categorical groups* of the pair $(X,A,x_0)$. Note that, for $n \geq 3$, $\pi_0\wp_n(X,A,x_0) \cong \pi_{n-2}(Ki,(x_0,\omega_{x_0})) = \pi_{n-1}(X,A,x_0)$ and $\pi_1\wp_n(X,A,x_0) \cong \pi_{n-1}(Ki,(x_0,\omega_{x_0})) = \pi_n(X,A,x_0)$.

For any pointed map $f : (X,x_0) \to (Y,y_0)$ it is well-known that the map $q : \Omega(Y,y_0) \to (Kkf,((x_0,\omega_{y_0}),\omega_{x_0}))$, given by $q(\omega) = ((x_0,\omega),\omega_{x_0})$ is a homotopy equivalence. Then the sequence of iterated homotopy kernels

$$\cdots Kkkf \longrightarrow Kkf \xrightarrow{kkf} Kf \xrightarrow{kf} X \xrightarrow{f} Y$$

is homotopy equivalent to the sequence

$$\cdots \Omega Kf \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow Kf \xrightarrow{kf} X \xrightarrow{f} Y$$

and therefore we have:

**Proposition 1** [12, Corollary 4] *For any pointed map $f : (X,x_0) \to (Y,y_0)$, there exists a long 2-exact sequence of categorical groups and pointed groupoids (in the last three terms)*

$$\ldots \to \wp_n(Kf,(x_0,\omega_{y_0})) \to \wp_n(X,x_0) \to \wp_n(Y,y_0) \to \wp_{n-1}(Kf,(x_0,\omega_{y_0})) \to \ldots$$

$$\ldots \to \wp_2(Kf,(x_0,\omega_{y_0})) \to \wp_2(X,x_0) \to \wp_2(Y,y_0) \to \wp_1(Kf) \to \wp_1(X) \to \wp_1(Y).$$

Particularly, considering, for any pointed topological pair $(X,A,x_0)$, the inclusion $i:(A,x_0)\hookrightarrow(X,x_0)$ we have:

**Corollary 1** (The 2-exact sequence of a pair of spaces) *For any pointed topological pair $(X,A,x_0)$ there exist a long 2-exact sequence of categorical groups and pointed groupoids (the last three terms)*

$$\ldots\to\wp_{n+1}(X,A,x_0)\to\wp_n(A,x_0)\to\wp_n(X,x_0)\to\wp_n(X,A,x_0)\to\ldots$$
$$\ldots\to\wp_3(X,A,x_0)\to\wp_2(A,x_0)\to\wp_2(X,x_0)\to\wp_2(X,A,x_0)\to\wp_1(A)\to\wp_1(X).$$

*that is called the* 2-exact homotopy sequence *of the pair* $(X,A,x_0)$.

Note that, by taking $\pi_0$ in the above 2-exact sequences, we obtain, respectively. the well-known exact sequences of groups (and pointed sets in the last three terms)

$$\ldots\to\pi_n(Kf,(x_0,\omega_{y_0}))\to\pi_n(X,x_0)\to\pi_n(Y,y_0)\to\ldots$$
$$\ldots\to\pi_1(Kf,(x_0,\omega_{y_0}))\to\pi_1(X,x_0)\to\pi_1(Y,y_0)\to\pi_0(Kf)\to\pi_0(X)\to\pi_0(Y).$$

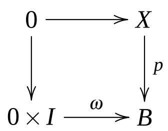
and

$$\pi_{n+1}(X,A,x_0)\to\pi_n(A,x_0)\to\pi_n(X,x_0)\to\pi_n(X,A,x_0)\to\ldots$$
$$\ldots\to\pi_2(X,A,x_0)\to\pi_1(A,x_0)\to\pi_1(X,x_0)\to\pi_1(X,A,x_0)\to\pi_0(A)\to\pi_0(X)\,.$$

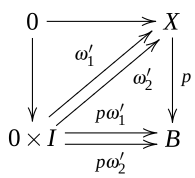
**Theorem 1** *Let $p:X\to B$ a fibration and suppose $b_0\in B'\subset B$. Let $X'=p^{-1}(B')$ and let $x_0\in p^{-1}(b_0)$. Then, $p$ induces a functor $p:\wp_n(X,X',x_0)\longrightarrow\wp_n(B,B',b_0)$ which is a full and essentially surjective functor, for $n=2$, and a monoidal equivalence for all $n\geq 3$.*

*Proof* We begin with $n=2$. Recall that $\wp_2(X,X',x_0)=\wp_1((Ki_X,(x_0,\omega_{x_0}))$ where $i_X:(X',x_0)\hookrightarrow(X,x_0)$ and $Ki_X=\{(x',\omega)\in X'\times X^I\,|\,\omega(o)=x_0\,,\,\omega(1)=x'\}$. Analogously, $\wp_2(B,B',b_0)=\wp_1((Ki_B,(b_0,\omega_{b_0}))$.

Given $\omega\in Ki_B$ we have a commutative diagram



and then we can find a lift $\omega':I\to X$ with $\omega'(0)=x_0$ and $p\omega'=\omega$. Since $p\omega'(1)=\omega(1)\in B'$, we see that $\omega'(1)\in X'$ and therefore $\omega'\in Ki_X$ and $p:\wp_2(X,X',x_0)\longrightarrow\wp_2(B,B',b_0)$ is surjective on objects. Now let us see that $p$ is full, that is, for any $\omega'_1,\omega'_2\in Ki_X$, the induced map $Hom_{\wp_1(Ki_X)}(\omega'_1,\omega'_2)\longrightarrow Hom_{\wp_1(Ki_B)}(p\omega'_1,p\omega'_2)$ is surjective. A morphism from $p\omega'_1$ to $p\omega'_2$ is a path class $[H]$ where $H:I\to Ki_B$ has $p\omega'_1$ as origin and $p\omega'_2$ as final or, equivalently, $H:I\times I\to B$ is a homotopy, rel 0, from $p\omega'_1$ to $p\omega'_2$, that is, $H(s,0)=p\omega'_1$, $H(s,1)=p\omega'_2$ and $H(0,t)=p\omega'_1(0)=b_0=p\omega'_2(0)$. For any such a morphism $[H]$ consider the diagram



where $\omega_1'(0) = x_0 = \omega_2'(0)$ and $p(x_0) = b_0 = H(0,t)$. Then, it is well known that there is a lifting $H' : 0 \times I \times I \longrightarrow X$ of $H$, which is a homotopy rel 0 in $X$ from $\omega_1'$ to $\omega_2'$ whose class is a morphism in $\wp_1(Ki_X)$ from $\omega_1'$ to $\omega_2'$ such that $p([H']) = [H]$.

For $n \geq 3$, since both $\wp_n(X,X',x_0)$ and $\wp_n(B,B',b_0)$ are categorical groups and $p : \pi_q(X,X',x_0) \to \pi_q(B,B',b_0)$ is a bijection for every $q \geq 1$ (see [18]), we have that

$$\pi_0\wp_n(X,X',x_0)) \cong \pi_{n-1}(X,X',x_0) \cong \pi_{n-1}(B,B',b_0) \cong \pi_0\wp_n(B,B',b_0))$$

and

$$\pi_1\wp_n(X,X',x_0)) \cong \pi_n(X,X',x_0) \cong \pi_n(B,B',b_0) \cong \pi_1\wp_n(B,B',b_0))$$

and we can use [16, Proposition 9] to conclude that $p : \wp_n(X,X',x_0) \longrightarrow \wp_n(B,B',b_0)$ is a monoidal equivalence for $n \geq 3$. □

**Corollary 2** *Let $p : (X,x_0) \to (B,b_0)$ be a fibration with fibre $F = p^{-1}(b_0)$. Then, the induced functor $p : \wp_n(X,F,x_0) \to \wp_n(B,b_0)$, is a full and essentially surjective functor, for $n = 2$, and a monoidal equivalence for $n \geq 3$.*

*Proof* This follows from the above theorem on taking $B' = \{b_0\}$ and using the canonical identification $\wp_n(B,\{b_0\},b_0) = \wp_n(B,b_0)$. □

Combining the 2-exact sequence of the pair $(X,F,x_0)$ with the equivalences of this corollary we get:

**Corollary 3** (The 2-exact homotopy sequence of a fibration)

*Let $p : (X,x_0) \to (B,b_0)$ be a fibration with fibre $F = p^{-1}(b_0)$. Then, there exists a long 2-exact sequence*

$$\ldots \to \wp_{n+1}(B,b_0) \xrightarrow{\partial} \wp_n(F,x_0) \xrightarrow{i} \wp_n(X,x_0) \xrightarrow{p} \wp_n(B,b_0) \xrightarrow{\partial} \ldots$$

$$\ldots \to \wp_3(B,b_0) \xrightarrow{\partial} \wp_2(F,x_0) \xrightarrow{i} \wp_2(X,x_0) \to \wp_2(X,F,x_0) \to \wp_1(F,x_0) \to \wp_1(X,x_0)$$

*that is called the* 2-exact homotopy sequence *of the fibration p.*

We remark that $\pi_0\wp_2(X,F,x_0)) \cong \pi_1(X,F,x_0) \cong \pi_1(B,b_0) \cong \pi_0\wp_2(B,b_0))$ and then, applying $\pi_0$ to sequence (2), we obtain the well-known group exact sequence of the fibration $p$:

$$\ldots \to \pi_{n+1}(B,b_0) \xrightarrow{\partial} \pi_n(F,x_0) \xrightarrow{i} \pi_n(X,x_0) \xrightarrow{p} \pi_n(B,b_0) \xrightarrow{\partial} \ldots$$

$$\ldots \to \pi_2(B,b_0) \xrightarrow{\partial} \pi_1(F,x_0) \xrightarrow{i} \pi_1(X,x_0) \to \pi_1(B,b_0) \to \pi_0(F,x_0) \to \pi_0(X,x_0).$$

**Corollary 4** *If $p : (X,x_0) \to (B,b_0)$ is a fibration with fibre $F$ and $X$ is contractible, then the functor*

$$\partial : \wp_{n+1}(B,b_0) \to \wp_n(F,x_0)$$

*is an equivalence for $n \geq 2$.*

## 4 The fundamental categorical crossed module of a fibration

Let us recall that, for any fibration $p:(X,x_0)\to(B,b_0)$ with fibre $F=p^{-1}(b_0)$ and inclusion map $i:(F,x_0)\hookrightarrow(X,x_0)$, the induced map $\pi_1(F,x_0)\stackrel{i}{\longrightarrow}\pi_1(X,x_0)$ is, as firstly Quillen had observed, a crossed module called the 'fundamental crossed module' of the fibration $p$ (see [4] for an explicit proof of this fact). Next, we are going to show, in a direct way, that this result is a trace in the group setting of another one which lives at the higher level of categorical groups.

**Theorem 2** *Let $p:(X,x_0)\to(B,b_0)$ be a fibration with fibre $F=p^{-1}(b_0)$ and consider the induced categorical group homomorphism $\wp_2(F,x_0)\stackrel{i}{\longrightarrow}\wp_2(X,x_0)$ given in the 2-exact homotopy sequence of the fibration p. Then the homotopy categorical group $\wp_2(F,x_0)$ is a $\wp_2(X,x_0)$-categorical group and, for any $\omega\in\wp_2(X,x_0)$ and $\alpha,\alpha'\in\wp_2(F,x_0)$, there are natural isomorphisms*

$$\nu=\nu_{\omega,\alpha}:i({}^{\omega}\alpha)\otimes\omega\to\omega\otimes\alpha\quad,\quad\chi=\chi_{\alpha,\alpha'}:{}^{i(\alpha)}\alpha'\otimes\alpha\to\alpha\otimes\alpha'$$

*such that $\langle\wp_2(F,x_0),\wp_2(X,x_0),i,\nu,\chi\rangle$ is a categorical crossed module that we will call the "fundamental categorical crossed module" of the fibration p.*

*Proof* To define a categorical group action of $\wp_2(X,x_0)$ on $\wp_2(F,x_0)$

$$\wp_2(X,x_0)\times\wp_2(F,x_0)\stackrel{ac}{\longrightarrow}\wp_2(F,x_0)$$

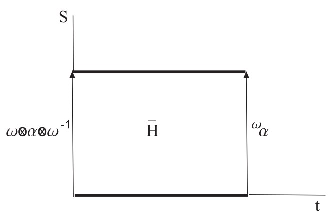
we consider the continuous map

$$\Omega(X,x_0)\times\Omega(F,x_0)\longrightarrow\Omega(F,x_0)\ ,\ \ (\omega,\alpha)\mapsto{}^{\omega}\alpha \tag{1}$$

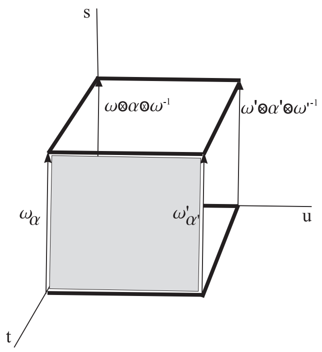
where ${}^{\omega}\alpha$ is defined as follows. Let $\omega\otimes\alpha\otimes\omega^{-1}\in\Omega(X,x_0)$ and consider the projection $p(\omega\otimes\alpha\otimes\omega^{-1})\in\Omega(B,b_0)$ which is homotopic, since $p(\alpha)=b_0$, to the constant loop in $B$ at $b_0$ through a homotopy of loops $H:I\times I\to B$. Then, $H_0(s)=H(s,0)=p(\omega\otimes\alpha\otimes\omega^{-1})(s)$ and $H_1(s)=H(s,1)=b_0$. Since $p$ is a fibration, using the homotopy lifting property in the diagram

$$\begin{array}{ccc} I & \xrightarrow{\omega\otimes\alpha\otimes\omega^{-1}} & X \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ I\times I & \xrightarrow{H} & B \end{array}$$

we get a homotopy of loops $\overline{H}_{\alpha,\omega}=\overline{H}:I\times I\to X$ such that $\overline{H}_0(s)=\overline{H}(s,0)=\omega\otimes\alpha\otimes\omega^{-1}$ and $p\overline{H}=H$. Then, $p\overline{H}_1(s)=p\overline{H}(s,1)=H(s,1)=b_0$ and therefore $Im\overline{H}_1\subseteq F$, that is, $\overline{H}_1\in\Omega(F,b_0)$. We define ${}^{\omega}\alpha=\overline{H}_1$ that is represented in the figure below



Since the fundamental groupoid functor $\wp_1$ preserves products, map (1) induces the functor $ac$ that is given, on objects, by $ac(\omega,\alpha) = {}^{\omega}\alpha$ and, on arrows $(\omega,\alpha) \xrightarrow{([h],[\overline{h}])} (\omega',\alpha')$ , by ${}^{[h]}[\overline{h}] = [{}^{h}\overline{h}] : {}^{\omega}\alpha \to {}^{\omega'}\alpha'$ where, if $J(s,t,u) = \overline{H}_{\overline{h}_u,h_u}(s,t)$ with $\overline{h}_u(s) = \overline{h}(s,u)$ and $h_u(s) = h(s,u)$, then $({}^{h}\overline{h})(s,u) = (\overline{H}_{\overline{h}_u,h_u})_1(s) = \overline{H}_{\overline{h}_u,h_u}(s,1)$, that is, the front face in the cube $J(s,t,u)$ represented below




For any $\omega,\omega' \in \wp_2(X,x_0)$ and $\alpha \in \wp_2(F,x_0)$, next we define a natural isomorphism

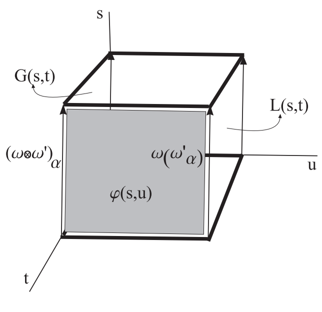
$$\Phi = \Phi_{\omega,\omega',\alpha} : {}^{\omega\otimes\omega'}\alpha \longrightarrow {}^{\omega}({}^{\omega'}\alpha)\,.$$

For this, let us consider the map $G : I\times I \to X$ given by $G(s,t) = \overline{H}_{\alpha,\omega\otimes\omega'}(s,t)$, which is a homotopy of loops from $(\omega\otimes\omega')\alpha\otimes(\omega\otimes\omega')^{-1}$ to ${}^{\omega\otimes\omega'}\alpha$, and also the map $L : I\times I \to X$ given as follows:

$$L(s,t) = \begin{cases} {}^{\omega}\overline{H}_{\alpha,\omega'}(s,2t) & 0 \le t \le \frac{1}{2} \\ \overline{H}_{{}^{\omega'}\alpha,\omega}(s,2t-1) & \frac{1}{2} \le t \le 1\,, \end{cases}$$

where ${}^{\omega}\overline{H}$ denotes the action of $\wp_2(X,x_0)$ on itself by conjugation. Note that ${}^{\omega}\overline{H}_{\alpha,\omega'}$ is a homotopy of loops from $\omega\otimes\omega'\otimes\alpha\otimes\omega'^{-1}\otimes\omega^{-1}$ to $\omega\otimes{}^{\omega'}\alpha\otimes\omega^{-1}$ whereas $\overline{H}_{{}^{\omega'}\alpha,\omega}$ is a homotopy of loops from $\omega\otimes{}^{\omega'}\alpha\otimes\omega^{-1}$ to ${}^{\omega}({}^{\omega'}\alpha)$. It is clear that $pG$ is homotopic to $pL$ through a homotopy $T : I\times I\times I \to B$, with $T(s,1,u) = b_0$, and also, $G(s,0)$ is homotopic to $L(s,0)$ trough the canonical homotopy of loops from $(\omega\otimes\omega')\alpha\otimes(\omega\otimes\omega')^{-1}$ to $\omega\otimes\omega'\otimes\alpha\otimes\omega'^{-1}\otimes\omega^{-1}$. Then, by [17, Theorem10, Chap.2], there is a lifting $J : I\times I\times I \to X$ which is a homotopy from $G$ to $L$ and is an extension of the homotopy $K : G(s,0) \simeq L(s,0)$, that is $J(s,t,0) = G(s,t)$, $J(s,t,1) = L(s,t)$ and $J(s,0,u) = K(s,u)$.

Let $\varphi(s,u) = J(s,1,u)$, that is, the front face of the cube $J(s,t,u)$ represented below



Then, $\varphi = \varphi_{\omega,\omega',\alpha} : I\times I \to F$ is a homotopy of loops from $\varphi(s,0) = J(s,1,0) = G(s,1) = {}^{\omega\otimes\omega'}\alpha$ to $\varphi(s,1) = J(s,1,1) = L(s,1) = {}^{\omega}({}^{\omega'}\alpha)$ and we define the required morphism $\Phi_{\omega,\omega',\alpha}$ as the morphism in $\wp_2(F,x_0)$ given by the class of the homotopy $\varphi$, that is $\Phi_{\omega,\omega',\alpha} = [\varphi]$.

For any $\omega \in \Omega(X,*)$ and $\alpha,\alpha' \in \Omega(F,*)$ the natural isomorphism

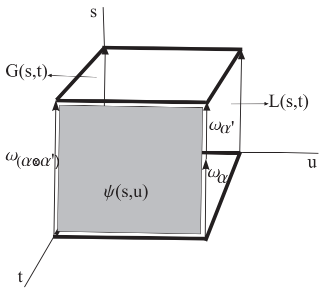
$$\Psi = \Psi_{\omega,\alpha,\alpha'} : {}^{\omega}(\alpha\otimes\alpha') \longrightarrow {}^{\omega}\alpha \otimes {}^{\omega}\alpha'$$

is defined as follows. Consider the map $G : I\times I \to X$ given by $G(s,t) = \overline{H}_{\alpha\otimes\alpha',\omega}(s,t)$, which is a homotopy of loops from $\omega\otimes\alpha\otimes\alpha'\otimes\omega^{-1}$ to ${}^{\omega}(\alpha\otimes\alpha')$, and consider also the map $L : I\times I \to X$, given by

$$L(s,t) = (\overline{H}_{\alpha,\omega}\otimes\overline{H}_{\alpha',\omega})(s,t) = \begin{cases} \overline{H}_{\alpha,\omega}(2s,t) & 0\le s\le \frac{1}{2} \\ \overline{H}_{\alpha',\omega}(2s-1,t) & \frac{1}{2}\le s\le 1 \end{cases},$$

which is a homotopy from $(\omega\otimes\alpha\otimes\omega^{-1})\otimes(\omega\otimes\alpha'\otimes\omega^{-1})$ to ${}^{\omega}\alpha\otimes{}^{\omega}\alpha'$. Then, it is clear that $pG$ is homotopic to $pL$ through a homotopy $T : I\times I\times I \to B$, with $T(s,1,u) = b_0$, and also, $G(s,0)$ is homotopic to $L(s,0)$ trough the canonical homotopy of loops from $\omega\otimes\alpha\otimes\alpha'\otimes\omega^{-1}$ to $\omega\otimes\alpha\otimes\omega`{-1}\otimes\omega\otimes\alpha'\otimes\omega^{-1}$. Then, by [17, Theorem10, Chap.2], there is a lifting $J : I\times I\times I \to X$ which is a homotopy from $G$ to $L$ and is an extension of the homotopy $K : G(s,0) \simeq L(s,0)$, that is $J(s,t,0) = G(s,t)$, $J(s,t,1) = L(s,t)$ and $J(s,0,u) = K(s,u)$.

Let $\psi(s,u) = J(s,1,u)$, that is, the front face of the cube $J(s,t,u)$ represented below




Then, $\psi = \psi_{\omega,\alpha,\alpha'} : I\times I \to F$ is a homotopy of loops from $\psi(s,0) = J(s,1,0) = G(s,1) = {}^{\omega}(\alpha\otimes\alpha')$ to $\psi(s,1) = J(s,1,1) = L(s,1) = {}^{\omega}\alpha\otimes{}^{\omega}\alpha'$ and we define the required morphism $\Psi_{\omega,\alpha,\alpha'}$ as the morphism in $\wp_2(F,x_0)$ given by the class of the homotopy $\psi$, that is $\Psi_{\omega,\alpha,\alpha'} = [\psi]$.

The required coherence conditions for $\Phi$ and $\Psi$ (see [10]) follow by applying suitably [17, Theorem10, Chap.2]. For instance, the coherence condition for $\Phi$ says that, for any loops $\omega,\omega',\omega'' \in \Omega(X,x_0)$ and $\alpha \in \Omega(F,x_0)$, the following equality

$$[{}^{\omega}\varphi_{\omega',\omega'',\alpha}][\varphi_{\omega,\omega'\otimes\omega'',\alpha}][{}^{a_{\omega,\omega',\omega''}}\alpha] = [\varphi_{\omega,\omega',{}^{\omega''}\alpha}][\varphi_{\omega\otimes\omega',\omega'',\alpha}]$$

or, equivalently, the equality

$$[{}^{\omega}\varphi_{\omega',\omega'',\alpha}\cdot\varphi_{\omega,\omega'\otimes\omega'',\alpha}\cdot{}^{a_{\omega,\omega',\omega''}}\alpha\cdot\varphi^{-1}_{\omega\otimes\omega',\omega'',\alpha}\cdot\varphi^{-1}_{\omega,\omega',{}^{\omega''}\alpha}] = [id_{\omega(\omega'(\omega''\alpha))}]$$

must hold. Now, the morphism $M = {}^{\omega}\varphi_{\omega',\omega'',\alpha}\cdot\varphi_{\omega,\omega'\otimes\omega'',\alpha}\cdot{}^{a_{\omega,\omega',\omega''}}\alpha\cdot\varphi^{-1}_{\omega\otimes\omega',\omega'',\alpha}\cdot\varphi^{-1}_{\omega,\omega',{}^{\omega''}\alpha}$ is $M = F_0(s,1,u)$, that is, the front face of a cube $F_0$ that is obtained by gluing the five cubes defining, in their front faces, each one of the morphisms that are composed in the definition of $M$. The back face of $F_0$, $F_0(s,0,u)$, represents the composition of analogue morphisms but for the action by conjugation of $\wp_2(X,x_0)$ on itself. Then, this face is homotopic, through a cube $G$, to the face constant at the loop $\omega\otimes\omega'\otimes\omega''\otimes\alpha\otimes\omega''^{-1}\otimes\omega'^{-1}\otimes\omega^{-1}$. Let us consider now the cube $F_1$ that is constant at the left (or right) face of $F_0$. The front face of $F_1$, $F(s,1,u)$, is constant at the loop ${}^{\omega}({}^{\omega'}({}^{\omega''}\alpha))$ and its back face, which is constant at the loop $\omega\otimes\omega'\otimes\omega''\otimes\alpha\otimes\omega''^{-1}\otimes\omega'^{-1}\otimes\omega^{-1}$, is homotopic through $G$ to the back face of $F_0$. It is clear that $pF_0$ is homotopic to $pF_1$ so that there exists a homotopy $H: I^4\to B$ from $pF_0$ to $pF_1$. Moreover $H(s,1,u,v)$ is constant at $b_0$ and $H(s,0,u,v)=pG(s,0,u,v)$ and therefore, using [17, Theorem10, Chap.2], there is a lifting $J: I^4\to X$ of $H$ which is a homotopy from $F_0$ to $F_1$ and is an extension of $G$. Then, since $J(s,1,t,0)=F_0(s,1,t)=M$ and $J(s,1,t,1)=F_1(s,1,t)=id_{\omega({\omega'}({\omega''}\alpha))}$, $J(s,1,t,v)$ is a cube giving the required homotopy between $M$ and $id_{\omega({\omega'}({\omega''}\alpha))}$ .

Next, for any $\omega\in\wp_2(X,x_0)$ and $\alpha\in\wp_2(F,x_0)$, we define the natural isomorphism in $\wp_2(X,x_0)$

$$\nu=\nu_{\omega,\alpha}: {}^{\omega}\alpha\otimes\omega\longrightarrow\omega\otimes\alpha\,.$$

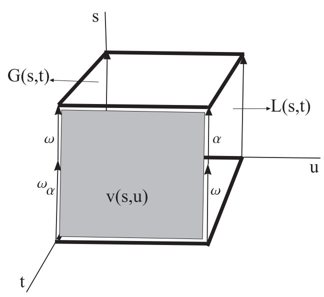
To do that, let us consider the map $G: I\times I\to X$ given by

$$G(s,t)=(\overline{H}_{\alpha,\omega}\otimes\omega)(s,t)=\begin{cases}\overline{H}_{\alpha,\omega}(2s,t) & 0\le s\le\frac{1}{2}\\ \omega(2s-1) & \frac{1}{2}\le s\le 1\,,\end{cases}$$

which is a homotopy of loops from $\omega\otimes\alpha\otimes\omega^{-1}\otimes\omega$ to ${}^{\omega}\alpha\otimes\omega$, and the map $L: I\times I\to X$ given by

$$L(s,t)=(\omega\otimes\alpha)(s,t)=\begin{cases}\omega(2s) & 0\le s\le\frac{1}{2}\\ \alpha(2s-1) & \frac{1}{2}\le s\le 1\,,\end{cases}$$

which is the constant homotopy at $\omega\otimes\alpha$. As above, [17, Theorem10, Chap.2] assures the existence of a homotopy $J: I\times I\times I\longrightarrow X$ such that $J(s,t,0)=G(s,t)$, $J(s,t,1)=L(s,t)$ and $J(s,0,u)$ is the canonical homotopy from $\omega\otimes\alpha\otimes\omega^{-1}\otimes\omega$ to ${}^{\omega}\alpha\otimes\omega$. Let $v(s,u)=J(s,1,u): I\times I\to X$ the front face of the cube




Then, $v$ is a homotopy of loops from $v(s,0)=J(s,1,0)=G(s,1)={}^{\omega}\alpha\otimes\omega$ to $v(s,1)=J(s,1,1)=L(s,1)=\omega\otimes\alpha$ and we define $\nu=\nu_{\omega,\alpha}=[v]$.

Finally, for $\alpha,\alpha'\in\wp_2(F,x_0)$, the definition of the natural isomorphism

$$\chi=\chi_{\alpha,\alpha'}: {}^{\alpha}\alpha'\otimes\alpha\longrightarrow\alpha\otimes\alpha'$$

in $\wp_2(F,x_0)$ is the same done for $\nu$ whenever $\omega\in\Omega(F,x_0)$, that is, $\chi_{\alpha,\alpha'}=\nu_{\alpha,\alpha'}$.

All the required coherence conditions that $\nu$ and $\chi$ must satisfy (see [8]) follow again as straightforward instances of [17, Theorem10, Chap.2]. □

Note that, as we already commented, the projection of this result by $\pi_0$, $\pi_0(\wp_2(F,x_0) \xrightarrow{i} \wp_2(X,x_0))$, gives the fundamental crossed module $\pi_1(F,*) \xrightarrow{i} \pi_1(X,*)$ of the fibration $p$.

The standard procedure in homotopy theory of factoring any map of pointed spaces $f: (X,x_0) \to (Y,y_0)$ through a homotopy equivalence $(X,x_0) \to (\overline{X},\overline{x_0})$ followed by a fibration $\overline{f}: (\overline{X},\overline{x_0}) \to (Y,y_0)$, (where $\overline{X} = \{(x,\omega) \in X \times Y^I \mid \omega(1) = f(x)\}$), gives a functor $f \mapsto \overline{f}$ from maps to fibrations. This functor allows then to define the fundamental crossed module of any map between pointed spaces $f: (X,x_0) \to (Y,y_0)$. In fact, if $((Kf,(x_0,\omega_{y_0})),kf)$ is the homotopy kernel of $f$, that is, the fibre of the fibration $\overline{f}$, then there is a long exact sequence

$$\cdots \to \pi_2(Y,y_0) \xrightarrow{\partial} \pi_1(Kf,(x_0,\omega_{y_0})) \xrightarrow{\pi_1(kf)} \pi_1(X,x_0) \to \pi_1(Y,y_0) \to \cdots$$

and $\pi_1(Kf,(x_0,\omega_{y_0})) \xrightarrow{\pi_1(kf)} \pi_1(X,x_0)$ is called the fundamental crossed module of the fibre homotopy sequence $(Kf,(x_0,\omega_{y_0})) \xrightarrow{kf} (X,x_0) \xrightarrow{f} (Y,y_0)$ .

In the same way, according to Corollary 3, there is also the 2-exact sequence

$$\begin{array}{l} \cdots \longrightarrow \wp_3(Y,y_0) \xrightarrow{\partial} \wp_2(Kf,(x_0,\omega_{y_0})) \xrightarrow{\wp_2(kf)} \wp_2(X,x_0) \longrightarrow \wp_2(\overline{X},K_f,(x_0,\omega_{y_0})) \\ \qquad \longrightarrow \wp_1(K_f,(x_0,\omega_{y_0})) \longrightarrow \wp_1(X,x_0) \end{array} \tag{2}$$

and we make the following:

**Definition 1** The fundamental categorical crossed module of a fibre homotopy sequence $(Kf,(x_0,\omega_{y_0})) \xrightarrow{kf} (X,x_0) \xrightarrow{f} (Y,y_0)$ is defined as the categorical crossed module

$$\wp_2(Kf,(x_0,\omega_{y_0})) \xrightarrow{\wp_2(kf)} \wp_2(X,x_0)$$

obtained from the fibration $\overline{f}: (\overline{X},\overline{x_0}) \to (Y,y_0)$ according to Theorem 2.

*Remark 1* In the particular case in which we consider a pair of pointed topological spaces $(X,A,x_0)$, associated to the inclusion $i: (A,x_0) \hookrightarrow (X,x_0)$ there is the fibration $\overline{A} \to X$ where $\overline{A}$ is the space of paths in $X$ ending at some point of $A$ and the maps send each path to its starting point. The fibre of this fibration is given by the subspace $Ki = \{(a,\omega) \in A \times X^I \,/\, \omega(0) = x_0,\ \omega(1) = a\}$ (i.e. the set of paths of $X$ starting at the base point $x_0$ and ending in $A$) whose homotopy categorical groups are, by definition, those of the pair $(X,A,x_0)$, i.e., $\wp_n(X,A,x_0) = \wp_1(\Omega^{n-2}(Ki,(x_0,\omega_{x_0}))$ , $n \geq 3$. In this way, just we obtain the homotopy categorical crossed module

$$\partial: \wp_3(X,A,x_0) \longrightarrow \wp_2(A,x_0)$$

introduced in [6], so that, as in the group case, the fundamental categorical crossed module of a pair of spaces can be deduced from the fundamental categorical crossed module of a fibration.

## 5 *K*-theory categorical groups

In what follows [1,2] are appropriate references for background. Let us recall that, for any ring $R$, if $GL_n(R)$ is the general linear group of invertible matrices $n\times n$ with entries in $R$, there is a sequence

$$GL_1(R)\subset GL_2(R)\subset GL_3(R)\subset\cdots$$

whose direct limit is denoted $GL(R)$. The subgroup $E(R)$ of $GL(R)$ generated by the elementary matrices is just the derived subgroup $[GL(R),GL(R)]$ and the quotient group, which is an abelian group, is the so called Whitehead group of $R$ denoted by $K_1R$. In this way, $K_1$ is a covariant functor from the category of rings to the category of abelian groups.

The Steinberg groups $St_n(R)$ were introduced by Milnor as those groups given by generators $x_{ij}^{\lambda}$ and relations encapsulating the key rules of the elementary matrices $e_{ij}^{\lambda}$. The canonical homomorphism $\Phi_n : St_n(R)\to E_n(R)\subset GL_n(R)$, $x_{ij}^{\lambda}\mapsto e_{ij}^{\lambda}$, induce a homomorphism in the corresponding direct limits

$$St(R)\xrightarrow{\Phi} GL(R)\ ,$$

whose image is actually $E(R)$, and whose kernel gives the definition of the 2-th group of algebraic $K$-theory, that is, $K_2(R)=Ker(\Phi)$. Again, this construction is evidently functorial.

By considering $BGL(R)$, the classifying space of the group $GL(R)$ then, by Quillen plus-construction, there exists a space $BGL(R)^+$ satisfying that $\pi_1BGL(R)^+\cong\frac{\pi_1BGL(R)}{E(R)}=\frac{GL(R)}{E(R)}=K_1R$. Moreover, $\pi_2BGL(R)^+\cong K_2R$.

Higher $K$-groups of a ring $R$, $K_iR$, $i\geq 1$, were defined by Quillen as the composition of covariants functors (see [2])

$$K_i : R\mapsto GLR\mapsto BGLR\mapsto BGLR^+\mapsto \pi_iBGLR^+\,.$$

Now, it suggests the following:

**Definition 2** For any ring $R$ we define $K$-categorical groups $\mathbb{K}_iR$, $i\geq 1$, as the composition of covariants functors

$$\mathbb{K}_i : R\mapsto GLR\mapsto BGLR\mapsto BGLR^+\mapsto \wp_{i+1}BGLR^+\,.$$

We then have that $\mathbb{K}_2R$ is a braided categorical group and, for $i\geq 3$, $\mathbb{K}_iR$ is a symmetric categorical group.

Let us observe that, for any $i\geq 1$,

$$\pi_0\mathbb{K}_iR=\pi_0\wp_{i+1}BGLR^+=\pi_iBGLR^+=K_iR$$

and

$$\pi_1\mathbb{K}_iR=\pi_1\wp_{i+1}BGLR^+=\pi_{i+1}BGLR^+=K_{i+1}R\,.$$

Thus, according to the classification of braided categorical groups given in [13, Theorem 3.3], $K$-categorical groups $\mathbb{K}_iR$, $i\geq 2$, are completely determined, up to isomorphism, by the $K$-groups $K_iR$ and $K_{i+1}R$ and the quadratic map $K_iR\to K_{i+1}R$ assigning to each object $[A]\in K_iR$ its signature.

Above we observed that the Quillen's $K$-groups $K_1R$ and $K_2R$ can be recognized, respectively, as the cokernel and the kernel of the group homomorphism $St(R)\xrightarrow{\Phi} GL(R)$. This morphism is a crossed module and it is, actually, an instance of crossed module arising from the general process that associates, via the fundamental crossed module of any fibre homotopy sequence, such kind of structures to any map of pointed spaces. In fact, a basic structure for the algebraic $K$-theory is the homotopy fibration

$$F(R) \xrightarrow{d_R} BGL(R) \xrightarrow{q_R} BGL(R)^+ . \tag{3}$$

and the main point to emphasize here is that its associated crossed module

$$\pi_1 F(R) \xrightarrow{\pi_1(d_R)} \pi_1 BGL(R)$$

is equivalent, since $\pi_1 F(R) \cong St(R)$, to

$$St(R) \xrightarrow{\Phi} GL(R)$$

and therefore its cokernel and kernel are, respectively, $K_1R$ and $K_2R$.

According to Definition 2, associated to the homotopy fibration (3) there is also the categorical crossed module

$$\wp_2 F(R) \xrightarrow{\wp_2 d_R} \wp_2 BGL(R)$$

and, paralleling the group case, we have:

**Theorem 3** *For any ring R, the K-categorical groups $\mathbb{K}_1 R$ and $\mathbb{K}_2 R$ are, respectively, up to monoidal equivalence, the homotopy cokernel and kernel of the categorical crossed module $\wp_2 d_R$, that is:*

$$\mathbb{K}_1 R \simeq Coker\, \wp_2 d_R \;\;,\;\; \mathbb{K}_2 R \simeq Ker\, \wp_2 d_R \,.$$

*Proof* Consider the 2-exact sequence (2) associated to the homotopy fibration (3)

$$\ldots \to \wp_3(BGL(R)^+) \to \wp_2(F(R)) \xrightarrow{\wp_2 d_R} \wp_2(BGL(R)) \to \wp_2(\overline{BGL(R)}, F(R)) \to \cdots$$

and the induced functor (see Corollary 2) $q_R : \wp_2(\overline{BGL(R)}, F(R)) \longrightarrow \wp_2(BGL(R)^+)$. Then, deduced from the universal property of the homotopy cokernel, there exists a morphism $Coker\wp_2 d_R \to \wp_2 BGL(R)^+$ that induces isomorphisms in the homotopy groups $\pi_i$, $i = 0,1$. In fact, since $\wp_2 BGL(R)$ is the categorical group associated to the group $GL(R)$, then (see [8, Example 3.2 ii)]) $Coker\wp_2 d_R$ is the strict categorical group associated to the crossed module $St(R) \xrightarrow{\Phi} GL(R)$ . Thus $\pi_0 Coker\wp_2 d_R \cong K_1 R \cong \pi_0 \mathbb{K}_1(R)$ and $\pi_1 Coker\wp_2 d_R \cong K_2 R \cong \pi_1 \mathbb{K}_2(R)$ and therefore, according to [16, Proposition 9], $Coker\wp_2 d_R \to \mathbb{K}_1 R$ is a monoidal equivalence.

On the other hand, deduced from the universal property of the kernel and the above 2-exact sequence, there is a canonical homomorphism $\wp_3 BGL(R)^+ \to Ker\wp_2 d_R$ which is also a monoidal equivalence. In fact, from the group exact sequence

$$0 \longrightarrow \pi_1 Ker\wp_2 d_R \longrightarrow \pi_1 \wp_2 F(R) \longrightarrow \pi_1 \wp_2 BGL(R)$$

and since $\pi_1 \wp_2 BGL(R) = \pi_2 BGL(R) = 0$, we deduce that $\pi_1 Ker\wp_2 d_R \cong \pi_1 \wp_2 F(R) = \pi_2 F(R)$. Also, from the group exact sequence deduced from the homotopy fibration (3) we have

$$\cdots \pi_3 BGL(R) \longrightarrow \pi_3 BGL(R)^+ \longrightarrow \pi_2 F(R) \longrightarrow \pi_2 BGL(R)$$

and since $\pi_3 BGL(R) = \pi_2 BGL(R) = 0$ we obtain $\pi_1 Ker\wp_2 d_R \cong \pi_3 BGL(R)^+ = \pi_1 \mathbb{K}_2(R)$. Then, by [16, Proposition 9], we conclude that $\mathbb{K}_2 R \to Ker\wp_2 d_R$ is a monoidal equivalence. $\square$

Categorical crossed modules are algebraic models for homotopy 3-types (see [6,7]) and the homotopy groups of the classifying space of any categorical crossed module $\langle \mathbb{H}, \mathbb{G}, T, \nu, \chi \rangle$ are given by $\pi_0 CokerT$, $\pi_0 KerT \cong \pi_1 CokerT$ and $\pi_1 KerT$, that is, they are the so-called homotopy groups of the categorical crossed module. The following corollary reveals that the 3-type associated to $\wp_2 d_R$ is controlled by the $K$-groups $K_i R$, $i = 1,2,3$, or, in other words, that the 3th level of the Postnikov tower of the $K$-theory spectrum of $R$ is classified by the categorical crossed module $\wp_2 d_R$.

**Corollary 5** *For any ring R, the categorical crossed module $\wp_2 d_R$ induces isomorphisms,*

$$\pi_i \wp_2 d_R \cong K_i R, \quad i = 1,2,3\,.$$

*Proof*

$$\pi_0 Coker \wp_2 d_R \cong K_1 R\,, \quad \pi_1 Coker \wp_2 d_R \cong K_2 R\, (\cong \pi_0 Ker \wp_2 d_R) \ \text{ and } \ \pi_1 Ker \wp_2 d_R \cong K_3 R\,.\square$$

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