On $\mathcal{H}^1$ of categorical groups

Antonio R. Garzón and Aurora del Río
Departamento de Álgebra, Universidad de Granada
18071 Granada, Spain
e-mail: agarzon@ugr.es

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Abstract

In this paper we introduce and study the first cohomology categorical group $\mathcal{H}^1(G, A)$ of a categorical group $G$ with coefficients in a braided categorical group $A$ provided of a coherent $G$-action. The fundamental exact sequence connecting $\mathcal{H}^0$ and $\mathcal{H}^1$ in this context is then established.

1 Introduction

The term categorification, appeared for the first time in [7], involves the basic idea that constructions from usual algebra can be translated to the level of categories, and to higher levels, by substituting equations by isomorphisms which have to satisfy suitable coherence conditions. Categorical groups, that is, monoidal groupoids in which every object is invertible with respect to the tensor product, provide a 2-dimensional vision of groups, and this process of categorification has been shown to be useful in different contexts as Algebraic Topology, Ring theory, or Homological Algebra [1, 3, 5, 6, 13, 14, 15, 16, 17].

In this paper we continue the program (see [1, 4, 5, 6, 9, 10, 12, 13, 15, 16]) of developing a Homological Algebra, together with their topological connotations, in the setting of categorical groups. Recall that, for any categorical group $G$ and any $G$-categorical group $H$, the consideration of derivations from $G$ into $H$ led (see [9]) to the construction of a pointed cohomology groupoid $\mathcal{H}^1(G, H)$, and allowed to show the existence of a 2-exact sequence of categorical groups and pointed groupoids, associated to any “short
exact sequence of $G$-categorical groups”, connecting $\mathcal{H}^0(G, \mathbb{H})$ (the categorical group of invariant objects) and $\mathcal{H}^1(G, \mathbb{H})$. When the $G$-categorical group of coefficients is a symmetric categorical group $\mathbb{H} = \mathbb{A}$, then (see [13]) $\mathcal{H}^1(G, \mathbb{A})$ is actually a symmetric categorical group, and the fundamental exact sequence connecting $\mathcal{H}^0$ and $\mathcal{H}^1$ is, in this case, a 2-exact sequence of symmetric categorical groups.

Braided categorical groups provide an interesting intermediate step between categorical groups and symmetric categorical groups. Relevant examples of this kind of categorical groups arise in several contexts (see [14]). The main objective of this paper is to prove that, taking braided categorical groups as coefficients ($G$-modules in our terminology), the quotient groupoid of derivations modulo inner derivations, $\mathcal{H}^1(G, \mathbb{A})$, has a natural monoidal structure such that it becomes a categorical group (the first cohomology categorical group of $G$ with coefficients in $\mathbb{A}$). Moreover, in this case of braided-enriched coefficients, we show that the corresponding fundamental exact sequence becomes, as well, just in a 2-exact sequence of categorical groups. Finally, we offer some examples which make the importance of the braided enrichment clear, since the derived sequences of the 2-exact sequence, connecting the homotopy invariants ($\pi_0$ and $\pi_1$) of the cohomology categorical groups $\mathcal{H}^0$ and $\mathcal{H}^1$, are then group exact sequences.

Throughout the paper, we freely use notions and terminology in relation with categorical groups (see [1, 10, 16] for general background).

2 The categorical group $\mathcal{H}^1(G, \mathbb{A})$

Fix a categorical group $G$ and let $\mathbb{H}$ be a $G$-categorical group. A derivation from $G$ into $\mathbb{H}$, [11, 12], is a functor $D : G \to \mathbb{H}$ together with a family of natural isomorphisms

$$\beta = \beta_{X,Y} : D(X \otimes Y) \to D(X) \otimes^{X} D(Y), \quad X, Y \in G,$$

which satisfy a suitable coherence condition. Derivations from $G$ into $\mathbb{H}$ are the objects of a category (actually a pointed groupoid) $\text{Der}(G, \mathbb{H})$ whose arrows are natural transformations $\epsilon : D \Rightarrow D'$ such that, for any objects $X, Y \in G$, the condition $(\epsilon_X \otimes^{X} \epsilon_Y) \cdot \beta_{X,Y} = \beta'_{X,Y} \cdot \epsilon_{X \otimes Y}$ holds. When $\mathbb{H} = \mathbb{A}$ is a $G$-module, that is, $\mathbb{A}$ is a braided categorical group (with braiding $c$) provided with a coherent $G$-action, then $\text{Der}(G, \mathbb{A})$ becomes a categorical group, which is symmetric if $\mathbb{A}$ is symmetric (see [12]).

Any object $A \in \mathbb{A}$ defines an inner derivation $(D_A, \beta_A) : G \to \mathbb{A}$ given, for any $X \in G$, by $D_A(X) = A \otimes^{X} A^*$ (with $A^*$ an inverse of $A$) and where

$$\left(\beta_A\right)_{X,Y} : A \otimes^{X \otimes Y} A^* \to A \otimes^{X} A^* \otimes^{X} (A \otimes^{Y} A^*)$$
is a composition of canonical isomorphisms (see [13]). Then we can consider the inner derivation homomorphism of categorical groups

\[ T = (T, \mu) : \mathcal{A} \rightarrow \mathcal{D}(\mathcal{G}, \mathcal{A}), \]

which is defined by \( T(A) = (D_A, \beta_A) \) where, for any \( A, B \in \mathcal{A}, \mu_{A,B} : D_{A \otimes B} \Rightarrow D_A \otimes D_B \) is the morphism in \( \mathcal{D}(\mathcal{G}, \mathcal{A}) \) determined by the natural transformation whose component in the object \( X \in \mathcal{G} \) is given by the diagram

\[
\begin{align*}
(A \otimes B) \otimes \chi (A \otimes B)^* & \xrightarrow{\mu_{A,B}^X} A \otimes X A^* \otimes B \otimes X B^* \\
\text{can} & \downarrow \xrightarrow{\chi_{A^* \otimes B}^{-1}} A \otimes B \otimes X B^* \otimes X A^*.
\end{align*}
\]

Note that, given any homomorphism of categorical groups \( T = (T, \mu) : \mathcal{G} \rightarrow \mathcal{G}' \), there is an associated quotient pointed groupoid \( \mathcal{G}' / \langle \mathcal{G}, T \rangle \) whose objects are those of \( \mathcal{G}' \). The arrows from \( A \) to \( B \) are equivalence classes \([X, f] \) of pairs \((X, f)\), with \( X \in \mathcal{G} \) and \( f : A \rightarrow T(X) \otimes B \) a morphism in \( \mathcal{G}' \), where \([X, f] = [Y, g] \) if there exists a morphism \( \alpha : X \rightarrow Y \) in \( \mathcal{G} \) such that \((T(\alpha) \otimes 1) \cdot f = g\). The composition of two morphisms \([X, f] : A \rightarrow B \) and \([Y, g] : B \rightarrow C \) is given by

\[
[X \otimes Y, A \xrightarrow{f} T(X) \otimes B \xrightarrow{1 \otimes g} T(X) \otimes T(Y) \otimes C \xrightarrow{\mu^{-1} \otimes 1} T(X \otimes Y) \otimes C] : A \rightarrow C.
\]

The homomorphism of categorical groups \((1)\) given by inner derivations, allows then to establish the following:

**Definition 2.1** Let \( \mathcal{G} \) be a categorical group and \((\mathcal{A}, c)\) a \( \mathcal{G}\)-module. The quotient groupoid of derivations from \( \mathcal{G} \) into \( \mathcal{A} \) modulo inner derivations, denoted by \( \mathcal{H}^1(\mathcal{G}, \mathcal{A}) \), is defined as the quotient groupoid \( \mathcal{D}(\mathcal{G}, \mathcal{A}) / \langle \mathcal{A}, T \rangle \) associated to the inner derivation homomorphism.

Thus, \( \mathcal{H}^1(\mathcal{G}, \mathcal{A}) \) has as objects derivations from \( \mathcal{G} \) to \( \mathcal{A} \). Given two derivations, \((D, \beta), (D', \beta') : \mathcal{G} \rightarrow \mathcal{A}\), the arrows are equivalence classes of pairs \((A, g^A)\), where \( A \in \mathcal{A} \) and \( g^A : (D, \beta) \Rightarrow T(A) \otimes (D', \beta') \) is a morphism of derivations, with \( T(A) = (D_A, \beta_A) \), that is, \( g^A \) is a natural transformation from \( D \) to \( D_A \otimes D' \) such that, for any \( X, Y \in \mathcal{G} \), the relation \((g^A_X \otimes g^A_Y) \cdot \beta_{X,Y} = (\beta_A \otimes \beta')_{X,Y} \cdot g^A_X \otimes g^A_Y \) holds. Moreover, \([A, g^A] = [A', g'^A] \) if there exists a morphism in \( \mathcal{A} \), \( u : A \rightarrow A' \), such that, for any \( X \in \mathcal{G} \), the following relation holds

\[
(u \otimes x^X(u^*)^{-1} \otimes 1) \cdot g^A_X = g^A_X'.
\]

\[ (2) \]
Note that the groupoid $\mathcal{H}^1(G, A)$ is pointed by the trivial derivation, that is, the pair $(D_0, \beta_0)$ where $D_0 : G \to A$ is the constant functor with value the unit object $I \in A$, and $\beta_0$ is given by canonical isomorphisms. We remark that, if $A$ is symmetric (i.e., the condition $c^2 = 1$ holds), then this stronger condition on the coefficients allowed us to prove in [13] the existence of a categorical group structure in $\mathcal{H}^1(G, A)$, which was defined there as a quotient of symmetric categorical groups. The main result of this paper is the following theorem where we prove that the braided (non-necessarily symmetric) enrichment on the coefficients is enough to define a monoidal structure on the groupoid $\mathcal{H}^1(G, A)$. More precisely, the crucial fact we now observe is that the braiding is enough to prove that $\otimes$ is a functor, so that the symmetry condition, used in the same proof done in [13], can be avoided.

**Theorem 2.2** Let $G$ be a categorical group and $(A, c)$ a $G$-module. Then, there is a tensor product defined in the in the quotient groupoid $\mathcal{H}^1(G, A)$ of derivations modulo inner derivations

$$\otimes : \mathcal{H}^1(G, A) \times \mathcal{H}^1(G, A) \to \mathcal{H}^1(G, A),$$

that turns it into a categorical group.

**Proof:** The tensor product on objects is defined as the tensor product of derivations defined in [12]. As for morphisms in $\mathcal{H}^1(G, A)$, $[A, c] : (D, \beta) \to (F, \alpha)$ and $[A', c'] : (D', \beta') \to (F', \alpha')$, we define

$$[A, c] \otimes [A', c'] = [A \otimes A', c^{A \otimes A'}] : (D \otimes D', \beta \otimes \beta') \to (F \otimes F', \alpha \otimes \alpha')$$

where, for any $X \in G$, $c^{A \otimes A'}$ is the unique morphism making commutative the diagram

\[
\begin{tikzcd}
(D \otimes D')(X) \ar{r}{A \otimes A'} & A \otimes A' \otimes (A \otimes A')^* \otimes (F \otimes F')(X) \\
A^X \otimes A^* \otimes F X \otimes A^* \otimes F'(X) \ar{u}[swap]{c^A \otimes c^{A'}} \ar{d}[swap]{1 \otimes c^{F(X)} \otimes 1} & A^X \otimes A^* \otimes (A^* \otimes A^*)^* \otimes (F \otimes F')(X) \ar{u}[swap]{\text{can}} \ar{d}[swap]{1 \otimes c^{A^* \otimes A'}} \\
A^X \otimes A^* \otimes F X \otimes A^* \otimes F'(X) \ar{u}[swap]{1 \otimes c^{F(X)} \otimes 1} \ar{d}[swap]{c^X} & A^X \otimes A^* \otimes F X \otimes (F \otimes F')(X) \ar{u}[swap]{\text{can}} \ar{d}[swap]{1 \otimes c^{F(X)} \otimes 1}
\end{tikzcd}
\]

It is straightforward to check that this definition does not depend on the choice of the representatives. Next we prove that this tensor is actually a bifunctor. We will avoid all the other details in the proof since they are similar to the symmetric case (see [13]).
We start by checking the equality

\[ ((D, \beta) \xrightarrow{[I, \varrho]} (D, \beta) \xrightarrow{[A, \varrho]} (F, \alpha)) \otimes ((D', \beta') \xrightarrow{[A', \varrho']} (F', \alpha')) = \]

\[ (D \otimes D', \beta \otimes \beta') \xrightarrow{[I \otimes A', \varrho' \otimes A']} (D \otimes F', \beta \otimes \alpha') \xrightarrow{[A, I, \varrho]} (F \otimes F', \alpha \otimes \alpha'). \]

In the following we will omit all canonical isomorphisms.

On the one hand, \([A, \varrho^A] \cdot [I, \varrho^I] = [I \otimes A, \varrho^{I \otimes A}] : (D, \beta) \to (F, \alpha)\) where, for any \(X \in G\), \(\varrho^I_X \cdot \varrho^A = (1 \otimes c_{X,A \otimes X, \otimes 1}) \cdot (1 \otimes \varrho_X^A) \cdot \varrho^I_X\). On the other hand, \([I, \varrho^I] \cdot [A', \varrho''] = [A' \otimes I, \varrho^{A' \otimes I}] : (D', \beta') \to (F', \alpha')\) where, for any \(X \in G\), \(\varrho_{X') = (1 \otimes c_{X', I \otimes 1}) \cdot (1 \otimes \varrho_X^{I \otimes A}) \cdot \varrho_I^A \cdot \varrho_X^A\). Thus, the first member of the equality is

\[ [I \otimes A, \varrho^{I \otimes A}] \otimes [A' \otimes I, \varrho^{A' \otimes I}] = [I \otimes A \otimes A' \otimes I, \varrho^{I \otimes A \otimes A' \otimes I}] : (D \otimes D', \beta \otimes \beta') \to (F \otimes F', \alpha \otimes \alpha') \]

where, for any \(X \in G\),

\[ \varrho^{I \otimes A \otimes A' \otimes I}_X = (1 \otimes c_{X,A \otimes X, \otimes 1}) \cdot (1 \otimes c_{X,A', I \otimes X, \otimes 1}) \cdot (1 \otimes c_{X,A', I \otimes X, \otimes 1}) \cdot (1 \otimes \varrho_X^A \otimes 1 \otimes \varrho_X^A) \cdot \varrho_I^A \cdot \varrho_X^A(1) \cdot \varrho_I^A \cdot \varrho_X^A. \]

In the second member we have \([I \otimes A', \varrho^{A' \otimes I}] : (D \otimes D', \beta \otimes \beta') \to (D \otimes F', \beta \otimes \alpha')\) where, for any \(X \in G\),

\[ \varrho^{A' \otimes I}_X = (1 \otimes c_{X,A' \otimes X, \otimes 1}) \cdot (1 \otimes c_{X,A', I \otimes X, \otimes 1}) \cdot \varrho_{X'}^{A' \otimes A' \otimes A'} \cdot \varrho_{X'}^{A' \otimes A' \otimes A'} \]

and \([A \otimes I, \varrho^{A \otimes I}] : (D \otimes F', \beta \otimes \alpha') \to (F \otimes F', \alpha \otimes \alpha')\) where, for any \(X \in G\),

\[ \varrho^{A \otimes I}_X = (1 \otimes c_{X,A \otimes X, \otimes 1}) \cdot (1 \otimes c_{X,A, I \otimes X, \otimes 1}) \cdot (1 \otimes \varrho_X^A \otimes 1 \otimes \varrho_X^A) \cdot \varrho_{X'}^{A' \otimes A' \otimes A'} \cdot \varrho_{X'}^{A' \otimes A' \otimes A'}. \]

Then, the second member of the equality is

\[ [A \otimes I, \varrho^{A \otimes I}] \cdot [I \otimes A', \varrho^{A' \otimes I}] = [I \otimes A' \otimes A \otimes I, \varrho^{I \otimes A' \otimes A \otimes I}] : (D \otimes D', \beta \otimes \beta') \Rightarrow (F \otimes F', \alpha \otimes \alpha') \]

where, for any \(X \in G\),

\[ \varrho^{I \otimes A' \otimes A \otimes I}_X = (1 \otimes c_{X,A' \otimes X, \otimes 1}) \cdot (1 \otimes c_{X,A', I \otimes X, \otimes 1}) \cdot (1 \otimes c_{X,A', I \otimes X, \otimes 1}) \cdot (1 \otimes \varrho_{X'}^{A' \otimes A' \otimes A'} \cdot \varrho_{X'}^{A' \otimes A' \otimes A'} \cdot \varrho_{X'}^{A' \otimes A' \otimes A'} \cdot \varrho_{X'}^{A' \otimes A' \otimes A'}. \]
Therefore, the required equality holds if, and only if, the equality $[I \otimes A \otimes A' \otimes I, \varrho^ {IA \otimes A' \otimes I}] = [I \otimes A' \otimes A \otimes I, \varrho^ {IA' \otimes A \otimes I}]$ holds, but this is true thanks to the morphism $\Upsilon = 1 \otimes c_{A,A'} \otimes 1 : I \otimes A \otimes A' \otimes I \to I \otimes A' \otimes A \otimes I$. This morphism actually satisfies relation (2) because it holds if, and only if, the following diagram commutes.

This diagram commutes because all quoted regions in it are commutative. In fact, regions (I) are commutative by naturality, and regions (II) and (III) are commutative due to the coherence conditions of the braiding.

The equality already checked, together with the following one, imply that $\otimes$ is a bifunctor

$$
((D, \beta') \stackrel{[A \otimes I, \varrho^ {A}]}{\longrightarrow} (F, \alpha) \stackrel{[I \otimes \varrho^ {I}]}{\longrightarrow} (F, \alpha)) \otimes ((D', \beta') \stackrel{[A' \otimes I, \varrho^ {A'}]}{\longrightarrow} (F', \alpha') \stackrel{[I \otimes \varrho^ {I}]}{\longrightarrow} (F', \alpha')) = (D \otimes D', \beta \otimes \beta') \stackrel{[A \otimes I, \varrho^ {A}]}{\longrightarrow} (F \otimes D', \alpha \otimes \beta') \stackrel{[I \otimes \varrho^ {I}]}{\longrightarrow} (F \otimes F', \alpha \otimes \alpha').
$$

In the first member of this equality we have

$$
[A \otimes I, \varrho^ {A}][I \otimes A', \varrho^ {A'}] = [A \otimes I \otimes I \otimes A', \varrho^ {A \otimes A' \otimes I \otimes I}](D \otimes D', \beta \otimes \beta') \to (F \otimes F', \alpha \otimes \alpha'),
$$

where, for any $X \in \mathbb{G},$

$$
\varrho^ {A \otimes A'} = \varrho^ {A} \otimes \varrho^ {A'} = (1 \otimes c^{-1}_{X,A' \otimes X}, 1) \cdot (1 \otimes c_{X', A' \otimes X, FX} \otimes 1) \cdot (1 \otimes c_{FX, A' \otimes X} \otimes 1).
$$
In the second member we have

\[ [I \otimes A', \overline{q}_{A'}^{\otimes I}] \cdot [A \otimes I, \overline{q}^{A \otimes I}] = [A \otimes I \otimes I \otimes A', \overline{q}^{A \otimes I \otimes I \otimes A'} : (D \otimes D', \beta \otimes \beta') \to (F \otimes F', \alpha \otimes \alpha'), \]

where, for any \( X \in \mathcal{G}, \)

\[ \overline{q}_{X}^{A \otimes I \otimes I \otimes A'} = \overline{q}_{X}^{A \otimes A'} = (1 \otimes c_{X', A' \otimes X}^{-1} \otimes 1), \]

It is clear that \( \overline{q}^{A \otimes A'} = \overline{q}^{A \otimes A'} \) and then, by considering \( u = id : A \otimes I \otimes I \otimes A' \to A \otimes I \otimes I \otimes A' \) relation (2) holds. Therefore,

\[ [A \otimes I \otimes I \otimes A', \overline{q}_{A}^{A \otimes I \otimes I \otimes A'}] = [A \otimes I \otimes I \otimes A', \overline{q}^{A \otimes I \otimes I \otimes A'}] \]

and the required second equality also holds.

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The categorical group \( \mathcal{H}^1(\mathcal{G}, \mathcal{A}) \) is referred as the first cohomology categorical group of \( \mathcal{G} \) with coefficients in \( \mathcal{A} \). We remark that this construction establishes a 2-functor between the 2-categories of \( \mathcal{G} \)-modules and that of categorical groups.

An interesting example of this cohomology categorical group arises in the following crossed module situation providing a definition for the first cohomology categorical group of a crossed module with coefficients in a braided crossed module (equivalently, a reduced 2-crossed module).

Suppose that \( \mathcal{L} = (H \xrightarrow{\delta} G) \) is a crossed module of groups, \( \mathcal{A} = (L, \rho, M, \{\cdot, \cdot\}) \) is a braided crossed module of groups and there is an action of \( \mathcal{L} \) on \( (\mathcal{A}, \{-, -\}) \) (see [12, 13] for details). Then, the associated categorical group \( \mathcal{G}(\mathcal{L}) \) acts on the braided categorical group \( \mathcal{G}(\mathcal{A}) \) and the categorical group \( \mathcal{H}^1(\mathcal{L}, \mathcal{A}) = \mathcal{H}^1(\mathcal{G}(\mathcal{L}), \mathcal{G}(\mathcal{A})) \) is defined (an explicit description of this groupoid can be seen in [13]). In the particular case that \( \mathcal{L} = (0 \xrightarrow{0} G) \) is the crossed module associated to a group \( G \), and the action of \( \mathcal{L} \) on \( \mathcal{A} \) is the trivial one, an object of \( \mathcal{H}^1(\mathcal{L}, \mathcal{A}) \) is precisely a Dedecker 2-cocycle \( (d : G \to M, l : G^2 \to L) \) of \( G \) with coefficients in \( \mathcal{A} \), [8], and an arrow from \( (d, l) \) to \( (d', l') \) is the class of a pair \((m, \tau)\) where \( m \in M \) and \( \tau \) is precisely an equivalence, [8], between the Dedecker 2-cocycles \((d, l)\) and \((d', l')\). The tensor product of objects is given by \((d, l) \otimes (d', l') = (d \otimes d', \overline{l})\), where \((d \otimes d')(x) = d(x)d'(x)\) and \(\overline{l}_{x,y} = l_{x,y}^{-1} d(x)d'(y)^{-1} \{d(y), d'(x)\}\). Thus, \(\pi_0(\mathcal{H}^1(\mathcal{L}, \mathcal{A}))\), which is a group due to the categorical group structure on \(\mathcal{H}^1(\mathcal{L}, \mathcal{A})\), coincides with the 2nd non-abelian cohomology group, \(\mathbb{H}^2(G, \mathcal{A})\), of \( G \) with coefficients in the reduced 2-crossed module \((\mathcal{A}, \{-, -\})\), whose group structure was already observed
in [2, Proposition 2.4]. Note also that \( \pi_1(\mathcal{H}(\mathcal{L}, \mathcal{A})) = Coker(\rho) \oplus \text{Hom}(G, \text{Ker}(\rho)) \).

Recall now that the zero-th cohomology categorical group \( \mathcal{H}^0(G, A) \), of a categorical group \( G \) with coefficients in a \( G \)-module \( A \), was introduced in [13] as the kernel of the inner derivation homomorphism and was identified as the categorical group \( A^G \) of \( G \)-invariant objects. The categorical group of invariant objects \( \mathbb{H}^G \) was also considered in [9] for any \( G \)-categorical group \( H \). In this setting, a fundamental six-term 2-exact sequence of categorical groups and pointed groupoids, connecting cohomology at levels 0 and 1, was shown. The existence of an analogous sequence, in the context of symmetric \( G \)-modules, was also proved in [13]. In this last case, each term in the sequence was actually a symmetric categorical group, but in [13, Remark 6.4] we already observed that, if the coefficients are just \( G \)-modules, then there is a corresponding 2-exact sequence of braided categorical groups and pointed groupoids (the last three terms). This observation together with the above theorem allow now to assert the following:

**Theorem 2.3** Let \( (A', T' \rightarrow A \rightarrow A'') \) be a short exact sequence of \( G \)-modules. Then, there exists an induced 2-exact sequence of categorical groups

\[
\mathcal{H}^0(G, A') \rightarrow \mathcal{H}^0(G, A) \rightarrow \mathcal{H}^0(G, A'') \rightarrow \mathcal{H}^1(G, A') \rightarrow \mathcal{H}^1(G, A) \rightarrow \mathcal{H}^1(G, A'').
\]

This sequence induces the group exact sequence

\[
0 \rightarrow \pi_1(\mathcal{H}^0(G, A')) \rightarrow \pi_1(\mathcal{H}^0(G, A)) \rightarrow \pi_1(\mathcal{H}^0(G, A'')) \rightarrow \pi_1(\mathcal{H}^1(G, A')) \rightarrow \pi_1(\mathcal{H}^1(G, A)) \rightarrow \pi_1(\mathcal{H}^1(G, A'')).
\]

In order to illustrate the sequences in the above theorem, let us consider

\[
\phi = (\phi_1, \phi_0) : \mathcal{A} = (L \rightarrow M, \{-,-\}) \rightarrow \mathcal{B} = (L'' \rightarrow M'', \{-,-\}),
\]

a surjective morphism of reduced 2-crossed modules of groups (i.e., a morphism \( \phi \) with \( \phi_0 : M \rightarrow M'' \) and \( \phi_1 : L \rightarrow L'' \) epimorphisms). Let \( G \) be a group and suppose that the crossed module \( L = (0 \rightarrow G) \) acts on \( (\mathcal{A}, \{-,-\}) \) and \( (\mathcal{B}, \{-,-\}) \) in such a way that \( \phi \) preserves the action (see [13] for details). If \( L' = Ker(\phi_1) \) and \( M' = Ker(\phi_0) \), let
\[ \mathcal{F} = (L' \xrightarrow{\rho'} M', \{-,-\}) \] be the 2-reduced crossed module fiber of \( \phi \) (where \( M' \) acts on \( L' \) by restriction of the action of \( M \) on \( L \) and \( \{-,-\} : M' \times M' \to L' \) is also induced by restriction). Then \( \mathcal{L} \) also acts on \( (\mathcal{F}, \{-,-\}) \) and since \( \mathcal{G}(\mathcal{F}) \) is equivalent to the kernel of the induced homomorphism of \( \mathcal{G}(\mathcal{L}) \)-modules \( \mathcal{G}(A) \to \mathcal{G}(B) \) and \( \mathcal{G}(\phi) \) is clearly essentially surjective, the sequence \( \mathcal{G}(\mathcal{F}) \to \mathcal{G}(A) \to \mathcal{G}(B) \) is a short exact sequence of \( G[0] \)-modules (where \( G[0] = \mathcal{G}(\mathcal{L}) \) is the strict categorical group associated to the group \( G \)). Therefore, according to the above theorem there is a 2-exact sequence of categorical groups

\[
\mathcal{G}(\mathcal{F})^{G[0]} \to \mathcal{G}(A)^{G[0]} \to \mathcal{G}(B)^{G[0]}
\]

Moreover, whenever the action of \( \mathcal{L} \) is trivial, the group exact sequence in the \( \pi \)'s that this 2-exact sequence induces is the following, according to the known identification of each one of its terms,

\[
0 \to \text{Ker}\rho' \to \text{Ker}\rho \to \text{Ker}\rho''
\]

\[
\text{Coker}\rho' \oplus \text{Hom}(G, \text{Ker}\rho') \to \text{Coker}\rho \oplus \text{Hom}(G, \text{Ker}\rho) \to \text{Coker}\rho'' \oplus \text{Hom}(G, \text{Ker}\rho'')
\]

\[
H^2(G, \mathcal{F}) \to H^2(G, A) \to H^2(G, B).
\]

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