Abstract. We construct the cokernel of a categorical crossed module and we establish the universal property of this categorical group. Then we specialize the notion of categorical crossed module to get a convenient notion of normal sub-categorical group. We also prove a suitable 2-dimensional version of the kernel-cokernel lemma for a diagram of categorical crossed modules. Finally, we use this lemma to construct a six-term 2-exact sequence connecting the low-dimensional cohomology categorical groups. These invariants are defined using the categorical crossed module of inner derivations studied in part I [6].

Keywords: categorical crossed module, quotient, cohomology categorical group

MSC 2000 : 18D10, 18G50, 20J05, 20L05.

1 Introduction

A way to think about a crossed module of groups $\delta : H \rightarrow G$ is as a morphism $\delta$ of groups which believes that the codomain $G$ is abelian. In fact, the image of $\delta$ is a normal subgroup of $G$, so that the cokernel $G/\text{Im}(\delta)$ of $\delta$ can be constructed as in the abelian case. This is relevant in non-abelian cohomology of groups, where the first cohomology group of a group $G$ with coefficients in the crossed module $\delta$ is defined as the cokernel of the crossed module given by inner derivations [13, 10].
The present paper is the second of two companion papers devoted to *categorical crossed modules*. In the first paper (which will be quoted just as “Part I”), we introduce the notion of crossed module of categorical groups, unifying and generalizing some previous definitions given by Breen [1] and Carrasco-Martínez [4].

The aim of this paper is to show that the previous intuition on crossed modules of groups can be exploited also at the level of categorical groups. In fact, in Section 2 we associate to a categorical crossed module $T : \mathbb{H} \to \mathbb{G}$ a new categorical group $\mathbb{G}/\langle \mathbb{H}, T \rangle$, which we call the *quotient categorical group*, and we justify our terminology establishing its universal property. As in the case of groups, the notion of categorical crossed module subsumes the notion of normal sub-categorical group. To test our definition, we show that, in the 2-category of categorical groups, normal sub-categorical groups correspond to kernels and quotients correspond to essentially surjective morphisms. Also, for a fixed categorical group $\mathbb{G}$, normal sub-categorical groups of $\mathbb{G}$ and quotients of $\mathbb{G}$ correspond each other.

Next, we turn to cohomology. In Section 3, we establish a higher dimensional version of the kernel-cokernel lemma: We associate a six-term 2-exact sequence of categorical groups to a convenient diagram of categorical crossed modules. In the last Section, we apply the previous results to define the low-dimensional cohomology categorical groups of a categorical group $\mathbb{G}$ with coefficients in a categorical crossed module $T : \mathbb{H} \to \mathbb{G}$, as the kernel and the quotient of the categorical crossed module of inner derivations $\overline{T} : \mathbb{H} \to \text{Der}^*(\mathbb{G}, \mathbb{H})$

studied in Part I. Further, we apply the kernel-cokernel lemma to get a six-term 2-exact sequence for the cohomology categorical groups from a short exact sequence of categorical $\mathbb{G}$-crossed modules. This sequence generalizes and unifies several similar exact sequences connecting cohomology sets, groups, groupoids and categorical groups (c.f. [2, 7, 9, 10, 11, 13, 15, 16, 18]).

Throughout the paper, we follow the terminology and the notations fixed in Part I. We also refer to Part I for basic facts on, and examples of categorical groups and categorical crossed modules.
2 The quotient categorical group.

2.1 The construction

Consider a morphism of categorical groups \( T = (T, \mu) : \mathbb{H} \to \mathbb{G} \). The quotient pointed groupoid \( \mathbb{G}/\langle \mathbb{H}, T \rangle \) is defined in the following way:

- Objects: those of \( \mathbb{G} \);
- Premorphisms: pairs \((A, f) : X \to Y\), with \( A \in \mathbb{H} \) and \( f : X \to T(A) \otimes Y \);
- Morphisms: classes of premorphisms \([A, f] : X \to Y\), where two pairs \((A, f)\) and \((A', f')\) are equivalent if there is \( a : A \to A' \) in \( \mathbb{H} \) such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{f} & T(A) \otimes Y \\
\downarrow{f'} & & \downarrow{T(a) \otimes 1} \\
T(A') \otimes Y & \xrightarrow{T(a) \otimes 1} & T(A') \otimes Y
\end{array}
\]

- Composition: given two morphisms \([A, f] : X \to Y\) \([B, g] : Y \to Z\) we define their composition by \([A \otimes B, ?] : X \to Z\), with arrow-part

\[
? : X \xrightarrow{T(A) \otimes Y} T(A) \otimes T(B) \otimes Z \xrightarrow{\text{can}} T(A \otimes B) \otimes Z
\]

- Identity: the identity \([I, ?] : X \to X\) has arrow-part \(X \simeq T(I) \otimes X\).

Let us just point out that \( \mathbb{G}/\langle \mathbb{H}, T \rangle \) is a groupoid (pointed by \( I \)), indeed the inverse of \([A, f] : X \to Y\) is \([A^*, ?] : Y \to X\) with arrow-part

\[
Y \xrightarrow{\text{can}} T(A^*) \otimes T(A) \xrightarrow{f} T(A^*) \otimes X
\]

Clearly, \( \mathbb{G}/\langle \mathbb{H}, T \rangle \) is the classifying groupoid of a bigroupoid having as 2-cells arrows \( a : A \to A' \) compatible with \( f \) and \( f' \) as before.

Let us now consider a categorical \( \mathbb{G} \)-crossed module \( \langle \mathbb{H}, T : \mathbb{H} \to \mathbb{G}, \nu, \chi \rangle \) as defined in part I [6] (just recall that, if \( X \in \mathbb{G} \) and \( A, B \in \mathbb{H} \), we write \( X^A \))
for $X$ acting on $A$, $\nu_{X,A} : T(XA) \otimes X \to X \otimes T(A)$ is the precrossed structure and $\chi_{A,B} : TABA \to A \otimes B$ is the crossed structure. Then we can define a tensor product on $G/\langle H, T \rangle$ in the following way: given two morphisms

$$[A, f] : X \to Y \quad [B, g] : H \to K$$

we define their tensor product by

$$[A \otimes YB, ?] : X \otimes H \to Y \otimes K$$

with arrow-part

$$X \otimes H \xrightarrow{f \otimes g} T(A) \otimes Y \otimes T(B) \otimes K \xrightarrow{1 \otimes \nu^{-1} \otimes 1} T(A) \otimes T(YB) \otimes Y \otimes K \xrightarrow{\text{can}} T(A \otimes YB) \otimes Y \otimes K$$

Let us just point out that the natural isomorphism $\chi$, and its compatibility with $\nu$, is needed to prove that

$$\otimes : G/\langle H, T \rangle \times G/\langle H, T \rangle \to G/\langle H, T \rangle$$

is a functor.

To complete the monoidal structure of $G/\langle H, T \rangle$, we use the essentially surjective functor

$$P_T : G \to G/\langle H, T \rangle$$

$$P_T(f : X \to Y) = [I, ?] : X \to Y \quad \text{with arrow-part} \quad X \xrightarrow{f} Y \simeq T(I) \otimes Y .$$

Now, as unit and associativity constraints in $G/\langle H, T \rangle$, we take the constraints in $G$. It is long but essentially straightforward to check that $G/\langle H, T \rangle$ is a categorical group and $P_T$ is a monoidal functor. Moreover, there is a 2-cell in $CG$

$$\pi_T : H \to G/\langle H, T \rangle$$

(where 0 is the morphism sending each arrow into the identity of the unit object) defined by $(\pi_T)_A = [A,?] : T(A) \to I$, with arrow-part $T(A) \simeq T(A) \otimes I$. 

4
Remark 2.1 Observe that if a categorical $G$-precrossed module $\langle H, T, \nu \rangle$ has two different crossed structures $\chi$ and $\chi'$, the quotient categorical groups we obtain using $\chi$ and $\chi'$ are equal. This is why we consider the crossed structure as a property.

Example 2.2 i) Let $\delta : H \to G$ be a crossed module of groups. As in Example 4.4 i) of [6] we can look at it as a categorical crossed module. Its quotient $G/[0]/\langle H[0], \delta \rangle$ is the strict (but not discrete) categorical group $G(\delta)$ corresponding to $\delta$ in the biequivalence between crossed modules of groups and categorical groups (see [19, 3]). Note that $\pi_0(G(\delta)) = \text{Coker}(\delta)$ and $\pi_1(G(\delta)) = \text{Ker}(\delta)$.

ii) If $(d : H \to G, \chi)$ is a categorical $G$-crossed module as in [4], then $G/\langle H, d \rangle = G/\langle \pi_0(H)[0], \pi_0(d) \rangle = G(\pi_0(d))$.

iii) If $T : A \to B$ is a morphism of symmetric categorical groups, then $B/\langle A, T \rangle$ is the cokernel of $T$ studied in [21].

iv) Let $H$ be a categorical group and consider the “inner automorphism” categorical crossed module $i : H \to Eq(H)$ as in Example 4.4 vii) of [6]. Explicitly, its quotient

$Out(H) = Eq(H)/\langle H, i \rangle$

is the classifying category of the following 2-category:

- 0-cells are monoidal autoequivalences $F : H \to H$,
- a 1-cell $(H, \alpha) : F \to G$ is given by an object $H \in H$ and a monoidal natural transformation $\alpha : F \Rightarrow i(H) G$,
- a 2-cell $f : (H_1, \alpha_1) \Rightarrow (H_2, \alpha_2)$ is an arrow $f : H_1 \to H_2$ such that, for all $X \in H$,

\[
\begin{array}{ccc}
H_1 \otimes G(X) \otimes H_1^* & \xrightarrow{f \otimes 1 \otimes (f^{-1})^*} & H_2 \otimes G(X) \otimes H_2^* \\
\downarrow^{(\alpha_1)_X} & & \downarrow^{(\alpha_2)_X} \\
F(X) & & F(X)
\end{array}
\]

commutes.

Since the natural family $\alpha_X : F(X) \to H \otimes G(X) \otimes H^*$ corresponds to a natural family $\overline{\alpha}_X : F(X) \otimes H \to H \otimes G(X)$, we recover the 2-category used in [17] to classify extensions of categorical groups. It is also clear that our $Out(H)$ is equivalent to the categorical group (also called $Out(H)$) used in [5] to study obstruction theory for extensions of categorical groups.
2.2 The universal property

The previous construction

\[
\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\pi_T} & \mathbb{G}/\langle \mathbb{H}, \mathbb{T} \rangle \\
\mathbb{T} & \xrightarrow{\rightarrow} & \mathbb{G} \\
\end{array}
\]

is universal with respect to the diagrams in \( \mathcal{C}G \)

\[
\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\delta} & \mathbb{F} \\
\mathbb{T} & \xrightarrow{\rightarrow} & \mathbb{G} \\
\end{array}
\]

satisfying the following condition: For any \( X \in \mathbb{G} \) and \( A \in \mathbb{H} \) the diagram

\[
\begin{array}{ccc}
G(X) \otimes I & \xleftarrow{1 \otimes \delta_A} & G(X) \otimes GT(A) \\
\downarrow & & \downarrow \text{can} \\
I \otimes G(X) & & G(X \otimes T(A)) \\
\delta_{XA} \otimes 1 & & G(\nu_{X,A}) \\
GT(\mathbb{X}A) \otimes G(X) & \xleftarrow{\text{can}} & G(T(\mathbb{X}A) \otimes X) \\
\end{array}
\]

is commutative. In fact, the word *universal* means here two different things.

1) \( \mathbb{G}/\langle \mathbb{H}, \mathbb{T} \rangle \) is a standard homotopy cokernel: for each triple

\[
\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\delta} & \mathbb{F} \\
\mathbb{T} & \xrightarrow{\rightarrow} & \mathbb{G} \\
\end{array}
\]

in \( \mathcal{C}G \), satisfying condition (1) there is a unique morphism

\[
G' : \mathbb{G}/\langle \mathbb{H}, \mathbb{T} \rangle \rightarrow \mathbb{F}
\]

in \( \mathcal{C}G \) such that \( G'P_T = G \) and \( G'\pi_T = \delta \).
2) \( G / \langle \mathbb{H}, T \rangle \) is a bilimit: for each triple

\[
\begin{array}{c}
G \\
\downarrow \delta \\
\mathbb{H} \\
\downarrow 0 \\
F
\end{array}
\]

in \( \mathcal{C} \mathcal{G} \), satisfying condition (1) there are \( G' \) and \( \delta' \) in \( \mathcal{C} \mathcal{G} \)

\[
\begin{array}{c}
G/\langle \mathbb{H}, T \rangle \ \\
\downarrow P_T \\
G \\
\downarrow \delta' \\
F
\end{array}
\]

making commutative the following diagram

\[
\begin{array}{c}
G' P_T T \\
\downarrow G' \pi_T \\
G' 0 \\
\downarrow \delta \\
0
\end{array}
\]

Moreover, if \( G'', \delta'' \) are in \( \mathcal{C} \mathcal{G} \)

\[
\begin{array}{c}
G/\langle \mathbb{H}, T \rangle \ \\
\downarrow P_T \\
G \\
\downarrow \delta'' \\
F
\end{array}
\]

and make commutative the following diagram

\[
\begin{array}{c}
G'' P_T T \\
\downarrow G'' \pi_T \\
G'' 0 \\
\downarrow \delta \\
0
\end{array}
\]

then there is a unique \( \lambda : G' \Rightarrow G'' \) in \( \mathcal{C} \mathcal{G} \) such that the following diagram commutes

\[
\begin{array}{c}
G' P_T \\
\downarrow \delta' \\
G \\
\downarrow \delta''
\end{array}
\]
Observe that the first universal property characterizes $G/\langle H, T \rangle$ up to isomorphism, whereas the second one characterizes it up to equivalence.

The proof of the uniqueness, in both the universal properties, is based on the following lemma.

**Lemma 2.3** Let $[A, f]: X \rightarrow Y$ be an arrow in $G/\langle H, T \rangle$. The following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{[A, f]} & Y \\
\downarrow P_T(f) & & \downarrow \text{can} \\
T(A) \otimes Y & \xrightarrow{(\pi_T)_A \otimes 1} & I \otimes Y \\
\end{array}
$$

Now, as far as the first universal property is concerned, define

$$G': G/\langle H, T \rangle \rightarrow F$$

$G'[A, f]: G(X) \xrightarrow{G(f)} G(T(A) \otimes Y) \simeq G(T(A)) \otimes G(Y) \xrightarrow{\delta \otimes 1} I \otimes G(Y) \simeq G(Y)$

and use condition (1) to check that the monoidal structure of $G'$, which is that of $G$, is natural with respect to the arrows of $G/\langle H, T \rangle$. As far as the second universal property is concerned, just take $\delta'$ to be the identity and define $\lambda$ via the formula

$$\lambda_X = (\delta''_X)^{-1}: G'(X) = G(X) \rightarrow G''(X)$$

**Remark 2.4** The fact that to make the functor

$$G': G/\langle H, T \rangle \rightarrow F$$

monoidal we need a condition (1) relating $\delta$ only to $\nu$ (and not to $\chi$) is not a surprise. The fact that $G'$ is monoidal or not depends only on the definition of the tensor product

$$\otimes: G/\langle H, T \rangle \times G/\langle H, T \rangle \rightarrow G/\langle H, T \rangle$$

and not on its functoriality, and the definition of the tensor in $G/\langle H, T \rangle$ only uses $\nu$ (whereas $\chi$ is needed to make this tensor a functor).
Example 2.5 Consider the categorical $G$-crossed module structure associated to a morphism $T : H \to G$ of braided categorical groups, as in Example 4.4 in [6], and the corresponding quotient categorical group. Consider also $F, G$ and $\delta$ in $CG$ as in the following diagram

$$
\begin{array}{c}
H \\ \downarrow^T \\
G \\ \downarrow^\delta \\
F
\end{array}
\quad
\begin{array}{c}
\delta \phi \\
0
\end{array}
\quad
\begin{array}{c}
P \\
\downarrow^\delta \\
G/\langle H, T \rangle
\end{array}

$$

Condition (1) is satisfied if $F$ is braided and $G$ is compatible with the braiding. (Recall that, as pointed out in [21], $G/\langle H, T \rangle$ in general is not braided. Indeed, to prove that the braiding of $G$ is natural in $G/\langle H, T \rangle$, one needs that the braiding is a symmetry.)

In Example 4.4 of [6] we saw that the kernel of a morphism of categorical groups is a categorical crossed module. In the following proposition we consider the kernel of the ”projection” $P_T : G \to G/\langle H, T \rangle$:

Proposition 2.6 Consider a categorical $G$-crossed module $T : H \to G$ and the factorization $T'$ of $T$ through the kernel of $P_T$

$$
\begin{array}{c}
H \\
\downarrow^T \\
G \\
\downarrow^\delta \\
F \\
\downarrow^KerP_T
\end{array}
\quad
\begin{array}{c}
P_T \\
\downarrow^{\pi_T} \\
G/\langle H, T \rangle
\end{array}
\quad
\begin{array}{c}
\delta \phi \\
0
\end{array}

$$

The functor $T'$ is a morphism of categorical $G$-crossed modules. Moreover, it is full and essentially surjective on objects.

The previous proposition means that the sequence

$$
\begin{array}{c}
\delta \phi \\
0 \\
H \\
\downarrow^{\pi_T} \\
G/\langle H, T \rangle
\end{array}
\quad
\begin{array}{c}
P_T \\
\downarrow^T
\end{array}
\quad
\begin{array}{c}
G \\
\downarrow^\delta \\
G/\langle H, T \rangle
\end{array}

$$

is 2-exact (see Definition 3.4 below). More important, it means that $T'$ is an equivalence if and only if $T$ is faithful. Therefore, we can give the following definition.
Definition 2.7 A normal sub-categorical group of a categorical group $G$ is a categorical $G$-crossed module $T : H \to G$ with $T$ faithful.

Proposition 2.8 Consider a morphism $L : G \to K$ in $C_G$ together with its kernel

$$KerL \xrightarrow{\epsilon_L} G \xrightarrow{L} K,$$

$$\epsilon_L : L e_L \Rightarrow 0 .$$

Consider also the normal sub-categorical group $\langle KerL, e_L \rangle$ of $G$, the corresponding quotient categorical group and the factorization $L'$ of $L$ through the quotient

$$\xymatrix{KerL \ar[r]^-{\epsilon_L} & G \ar[d]_{P_{\epsilon_L}} \ar[r]^-{L} & K \\ G/\langle KerL, e_L \rangle \ar@{_{(}->}[ur]^{L'} & }$$

The functor $L' : G/\langle KerL, e_L \rangle \to K$ is full and faithful.

In the previous proposition, the factorization $L'$ exists because the condition (1) in the universal property of the quotient is verified when $\delta = \epsilon_L$.

The previous proposition means that $L : G \to K$ is essentially surjective on objects if and only if $L' : G/\langle KerL, e_L \rangle \to K$ is an equivalence. In other words, quotients in the the 2-category $C_G$ are, up to equivalence, precisely the essentially surjective morphisms.

Remark 2.9 Let us recall that the image of a precrossed module of groups is a normal subgroup of the codomain. The situation for categorical groups is similar. If, in Proposition 2.8, $G$ is a $K$-categorical group and $L$ is equipped with the structure of categorical $K$-precrossed module, then $L'$ inherits such structure and $P_{\epsilon_L}$ is a morphism of categorical $K$-precrossed modules. In fact, following Remark 4.3 in [6], $L'$ is a categorical $K$-crossed module. (In other words the full image of a categorical precrossed module and the not full image of a categorical crossed module are normal sub-categorical groups.) Let us just describe the action

$$K \times G/\langle KerL, e_L \rangle \to G/\langle KerL, e_L \rangle .$$

On objects, it is given by the action of $K$ over $G$. Now consider an arrow $[(N, \epsilon_N), f] : G_1 \to G_2$ in the quotient and an object $K \in K$. We need an arrow $K_{G_1} \to K_{G_2}$. Since $L'$ is full and faithful, it suffices to define an
arrow $L'(K^G_1) \longrightarrow L'(K^G_2)$ in $\mathbb{K}$. This is given by the following composition:

$\begin{align*}
L(K^G_1) & \xrightarrow{\varphi_{K,G_1}} K \otimes L(G_1) \otimes K^* \xrightarrow{1 \otimes L(f) \otimes 1} K \otimes L(N \otimes G_2) \otimes K^* \\
\downarrow \downarrow & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quar
3 The kernel-cokernel lemma

The aim of this section is to obtain an analogous to the classical kernel-cokernel lemma (see [14]). For it, we first extend the definitions given in [6], Section 3, considering categorical precrossed modules based on different categorical groups:

**Definition 3.1** The 2-category of categorical precrossed modules

$\text{PreCross}$

has as objects the categorical precrossed modules. Given two categorical precrossed modules $⟨H, T : H \to G, \nu⟩$ and $⟨H', T' : H' \to G', \nu'⟩$, a morphism between them consists of a 4-tuple $(F, G, \eta, \alpha)$, as in the following diagram

$$
\begin{array}{ccc}
G \times H & \xrightarrow{ac} & H \\
\downarrow{G \times F} & & \downarrow{F} \\
G' \times H' & \xrightarrow{ac} & H'
\end{array}
\begin{array}{ccc}
& T & \xrightarrow{\alpha} & G \\
\eta & \downarrow & \alpha & \downarrow \\
& T' & \xrightarrow{\nu} & G'
\end{array}
$$

where

- $F : H \to H'$, $G : G \to G'$ and $\alpha : GT \Rightarrow T'F$ are in $CG$.
- $(F, \eta) : H \to H'$ is a morphism in $G − CG$, considering $H'$ a $G$-categorical group via $G : G \to G'$.

Moreover, we ask for the following compatibility condition:

For any $X \in G$ and $A \in H$, the following diagram has to be commutative

$$
\begin{array}{ccc}
G(T(XA) \otimes X) & \xrightarrow{G(\nu_{X,A})} & G(X \otimes T(A)) \\
\downarrow{\text{can}} & & \downarrow{\text{can}} \\
GT(XA) \otimes G(X) & \xrightarrow{\alpha_{X,A} \otimes 1} & G(X) \otimes GT(A) \\
\downarrow{T'F(XA) \otimes G(X)} & & \downarrow{1 \otimes \alpha_A} \\
T'(G(XF(A)) \otimes G(X) & \xrightarrow{\nu'_{G(X),F(A)}} & G(X) \otimes T'F(A)
\end{array}
$$

(2)
Given two parallel morphisms \((F, G, \eta, \alpha)\) and \((F', G', \eta', \alpha')\), a 2-cell is a pair
\[
(\beta, \lambda) : (F, G, \eta, \alpha) \Rightarrow (F', G', \eta', \alpha')
\]
where \(\beta\) and \(\lambda\) are 2-cells in \(C_G\) as in the following diagrams

Moreover we ask for the following compatibility conditions: For any \(X \in G, A \in H\), the following diagrams are commutative

\[
\begin{align*}
F(XA) \xrightarrow{\eta_{X,A}} & G(X)F(A) \\
\beta_{XA} \downarrow & \downarrow \lambda_{X,F(A)} \\
F'(XA) \xrightarrow{\eta'_{X,A}} & G'(X)F'(A)
\end{align*}
\]

\[
\begin{align*}
GT(A) \xrightarrow{\alpha_A} & T'F(A) \\
\lambda_{T(A)} \downarrow & \downarrow T'(\beta_A) \\
G'T(A) \xrightarrow{\alpha'_A} & T'F'(A)
\end{align*}
\]

**Remark 3.2** i) In the previous definition, consider the natural isomorphisms \(\varphi = \varphi_{X,A} : T(XA) \to X \otimes T(A) \otimes X^*\) in \(G\), and \(\varphi' = \varphi'_{X',A'} : T'(X'A') \to X' \otimes T'(A') \otimes X'^*\) in \(G'\), obtained from \(\nu\) and \(\nu'\) respectively, such that \((T, \varphi) : H \to G\) is a morphism in \(G - C_G\) and \((T', \varphi') : H' \to G'\) is a morphism in \(G' - C_G\) (see [6]). Then the compatibility condition (2) means that
\[
\alpha : (GT, can\varphi) \Rightarrow (T'F, \varphi'T'\eta) : H \to G'
\]
is a 2-cell in \(G - C_G\), where the \(G\)-action over \(G'\) is that obtained via \(G : G \to G'\).
ii) The compatibility condition (3) means that $\beta : (\mathbf{F}, \bar{\eta}) \Rightarrow (\mathbf{F}', \eta') : \mathbb{H} \rightarrow \mathbb{H}'$ is a 2-cell in $\mathbb{G} - \mathbb{C}G$, considering in $\mathbb{H}'$ the action given via $\mathbb{G}'$ and where $\bar{\eta}_{X,A} : F(\chi A) \rightarrow G'(\chi)F(A)$ is given by the composition

$$F(\chi A) \xrightarrow{\eta_{X,A}} G(\chi F(A)) \xrightarrow{\lambda_{X,F(A)}} G'(\chi)F(A).$$

Finally the compatibility condition (4) means that $(\beta, \lambda)$ is a 2-cell in the 2-category of morphisms $\mathbb{C}G^{-}$.

iii) If, in the previous definition, we take $\mathbb{G}, \mathbb{G}'$ and $\lambda$ to be identities, we get the 2-category $\mathbb{G} - \mathbb{C}G/\mathbb{G}$ of categorical $\mathbb{G}$-precrossed modules defined in [6].

iv) Given two categorical precrossed modules there is a “zero-morphism” between them

\[
\begin{array}{ccc}
\mathbb{G} \times \mathbb{H} & \xrightarrow{ac} & \mathbb{H} & \xrightarrow{T} & \mathbb{G} \\
\downarrow{0 \times 0} & & \downarrow{0} & & \downarrow{0} \\
\mathbb{G}' \times \mathbb{H}' & \xrightarrow{ac} & \mathbb{H}' & \xrightarrow{T'} & \mathbb{G}'
\end{array}
\]

The consideration of the 2-category $\text{PreCross}$ is justified by the following proposition

**Proposition 3.3** Consider two categorical precrossed modules $\langle \mathbb{H}, T : \mathbb{H} \rightarrow \mathbb{G}, \nu \rangle$ and $\langle \mathbb{H}', T' : \mathbb{H}' \rightarrow \mathbb{G}', \nu' \rangle$, and a morphism between them

\[
\begin{array}{ccc}
\mathbb{G} \times \mathbb{H} & \xrightarrow{ac} & \mathbb{H} & \xrightarrow{T} & \mathbb{G} \\
\downarrow{G \times F} & & \downarrow{F} & & \downarrow{G} \\
\mathbb{G}' \times \mathbb{H}' & \xrightarrow{ac} & \mathbb{H}' & \xrightarrow{T'} & \mathbb{G}'
\end{array}
\]

Assume that the categorical precrossed modules are in fact crossed modules. Then the triple $(\mathbf{F}, \alpha, \mathbf{G})$ extends to a morphism between the quotient categorical groups

$$\hat{\mathbf{G}} : \mathbf{G}/\langle \mathbf{H}, \mathbf{T} \rangle \rightarrow \mathbf{G}'/\langle \mathbf{H}', \mathbf{T}' \rangle$$

(that is, there is a monoidal functor $\hat{\mathbf{G}}$ and a monoidal natural transformation $g : \hat{\mathbf{G}}P_T \Rightarrow P_T \mathbf{G}$ compatible with $\alpha, \pi_T$ and $\pi_{T'}$).
Proof: Consider the natural transformation

\[ P T \xrightarrow{\alpha} P T' \xrightarrow{\pi T' F} 0 F \xrightarrow{} 0 \]

Using the compatibility condition on \((\eta, \alpha)\), one can check that this natural transformation satisfies condition (1) in the universal property of \(G/\langle H, T \rangle\).

Here is the picture resulting from the previous proposition

\[
\begin{array}{c}
\xymatrix{
H \ar[r]^F & H' \\
G \ar[r] & G' \\
G/\langle H, T \rangle \ar[r]^-{\pi T} & G'/\langle H', T' \rangle
}
\end{array}
\]

We now recall two definitions needed for establishing the kernel-cokernel sequence (c.f. [12, 17]). Consider the following diagram in \(CG\)

\[
\begin{array}{c}
\xymatrix{
A \ar[r]^-{0} & \mathbb{C} \\
\mathbb{B} \ar[u]^-{G} \ar[r]_-{\beta G} & \mathbb{B}' \ar[u]^-{G'}
}
\end{array}
\]

**Definition 3.4** We say that the triple \((G, \beta, G')\) as in the previous diagram is 2-exact if the factorization of \(G\) through the kernel of \(G'\) is a full and essentially surjective functor.

We say that the triple \((G, \beta, G')\) as in the previous diagram is an extension

- if it is 2-exact, \(G\) is faithful and \(G'\) is essentially surjective;

- or, equivalently, if the factorization of \(G\) through the kernel of \(G'\) is an equivalence and, moreover, \(G'\) is essentially surjective.

Consider three categorical crossed modules \(\langle \mathbb{H}, T : \mathbb{H} \to \mathbb{G}, \nu, \chi \rangle, \langle \mathbb{H}', T' : \mathbb{H}' \to \mathbb{G}', \nu', \chi' \rangle\) and \(\langle \mathbb{H}'', T'' : \mathbb{H}'' \to \mathbb{G}'', \nu'', \chi'' \rangle\) and two morphisms of categorical precrossed modules.
Consider also a 2-cell \((\beta, \lambda)\) from the composite morphism to the zero-morphism as in the following diagram

Using the universal property of the kernel and Proposition 3.3, we get the following diagram in \(CG\)

Using the previous notation, we get the following facts.
Lemma 3.5

1. If the triple \((F, \beta, F')\) is 2-exact and the morphism \(G\) is faithful, then the triple \((\hat{F}, \hat{\beta}, \hat{F}')\) is 2-exact;

2. If the triple \((G, \lambda, G')\) is 2-exact and the morphism \(F'\) is essentially surjective, then the triple \((\hat{G}, \hat{\lambda}, \hat{G}')\) is 2-exact.

Proposition 3.6 If the triples \((F, \beta, F')\) and \((G, \lambda, G')\) are extensions, then there are a morphism \(D\) and two 2-cells \(\Sigma, \Psi\) in \(\mathcal{C}G\) such that the following sequence is 2-exact in \(\text{Ker} \mathcal{T}', \text{Ker} \mathcal{T}''\), \(H/\langle H, T \rangle\) and \(H'/\langle H', T' \rangle\)

\[
\begin{array}{ccc}
\text{Ker} \mathcal{T} & \xrightarrow{\hat{F}} & \text{Ker} \mathcal{T}' \xrightarrow{\Sigma_H} \mathcal{G}/\langle H, T \rangle \xrightarrow{\hat{\lambda}_H} \mathcal{G}'/\langle H', T' \rangle \xrightarrow{\hat{G}'} \mathcal{G}''/\langle H'', T'' \rangle \\
& \xleftarrow{\hat{\beta}_\Psi} & \text{Ker} \mathcal{T}' \xrightarrow{\lambda_\Psi} \mathcal{G}/\langle H, T \rangle \xrightarrow{\hat{\Sigma}_H} \mathcal{G}'/\langle H', T' \rangle \xrightarrow{\hat{\lambda}_H} \mathcal{G}''/\langle H'', T'' \rangle \xrightarrow{\hat{G}'} \mathcal{G}''/\langle H'', T'' \rangle
\end{array}
\]

Moreover, \(\hat{F}\) is faithful and \(\hat{G}'\) is essentially surjective.

Proof: Let us just describe the functor

\[D : \text{Ker} \mathcal{T}' \to \mathcal{G}/\langle H, T \rangle\]

Observe that, since \(F\) is faithful and \((F, \beta, F')\) is 2-exact, \(F : H \to H'\) inherits from the kernel of \(F'\) a structure of categorical \(H'\)-crossed module. Moreover, since \(F'\) is essentially surjective, up to equivalence \(H''\) is the quotient cat-group \(H'/\langle H, F \rangle\) and \(F'\) is the projection \(P_F\). Observe also that, since \(G\) is faithful and \((G, \lambda, G')\) is 2-exact, \(G\) is equivalent to the kernel of \(G'\) and \(G\) is the injection \(e_{G'}\). We use these descriptions of \(H''\) and \(G\) to construct the functor \(D\).

An object in \(\text{Ker} \mathcal{T}'\) is a pair \((B \in H', b : T''(B) \to I)\). We get an object

\[(T'(B), G'(T'(B)) \xrightarrow{\alpha_B'} T''(P_F(B))) = T''(B) \xrightarrow{b} I)\]

in \(\text{Ker} G'\) and we define

\[D(B, b) = P_T(T'(B), b\alpha_B')\]
An arrow in $Ker T''$ is

$$[A, x] : (B_1, b_1 : T''(B_1) \to I) \longrightarrow (B_2, b_2 : T''(B_2) \to I)$$

with $[A, x] : B_1 \to B_2$ a morphism in $\mathbb{H}'/\langle H, F \rangle$ (with representative $x : B_1 \to F(A) \otimes B_2$) such that the following diagram commutes

\[
\begin{array}{ccc}
T''(B_1) & \xrightarrow{T''[A, x]} & T''(B_2) \\
\downarrow b_1 & & \downarrow b_2 \\
I & \xrightarrow{} & I
\end{array}
\]

We define

$$D[A, x] = [A, ?] : (P_T(T'(B_1)), b_1\alpha'_{B_1}) \xrightarrow{\alpha_{B_1}} (P_T(T'(B_2)), b_2\alpha'_{B_2})$$

where the arrow part must be an arrow

$$? : (T'(B_1), b_1\alpha'_{B_1}) \to T(A) \otimes (T'(B_2), b_2\alpha'_{B_2})$$

in $Ker G'$, that is an arrow

$$? : T'(B_1) \to e_{G'}(T(A)) \otimes T'(B_2)$$

in $G'$ making commutative the following diagram

\[
\begin{array}{ccc}
G'(T'(B_1)) & \xrightarrow{G'(\langle\rangle)} & G'(e_{G'}(T(A)) \otimes T'(B_2)) \\
\downarrow T''(\langle\rangle) & & \downarrow \text{can} \\
T''(B_1) & \xrightarrow{T''(\langle\rangle)} & G'(e_{G'}(T(A))) \otimes G'(T'(B_2)) \\
\downarrow b_1 & & \downarrow \epsilon_{G'}(T'(A)) \otimes \alpha_{B_2} \\
I & \xleftarrow{b_2} & I \otimes T''(B_2) \\
\end{array}
\]

For this, we take

$$? : T'(B_1) \xrightarrow{T'(\langle\rangle)} T'(F(A) \otimes B_2) \simeq T'(F(A)) \otimes T'(B_2) \xrightarrow{\alpha^{-1}_{A} \otimes 1} e_{G'}(T(A)) \otimes T'(B_2)$$

and, to check that it is an arrow in the kernel of $G'$, one uses that $(\beta, \lambda)$ is a 2-cell in the 2-category of morphisms $CG^{-}$ (see ii) in Remark 3.2).
In this way, we have defined

\[ D : Ker T'' \rightarrow G/\langle H, T \rangle \]

It is easy to verify that it is well-defined and that it is a functor. To prove that its (obvious) monoidal structure is natural with respect to the arrows of \( Ker T'' \), one uses the compatibility condition between the precrossed structure and the crossed structure of \( T' : H' \rightarrow G' \) (see condition (cr4) of the definition of categorical crossed module in [6]).

For the sake of generality, let us point out that, to establish the results of Section 2 and Section 3 we do not need condition (cr3) in the definition of categorical crossed module.

4 Cohomology with coefficients in categorical crossed modules.

We will define cohomology categorical groups at dimensions 0 and 1. Let us first remember (c.f. [10, 13]) that if \( G \) is a group and \( H \) is a \( G \)-crossed module, then the cohomology groups \( H^0(G, H) \) and \( H^1(G, H) \) are, respectively, the kernel and the cokernel of the group homomorphism \( H \rightarrow Der(G, H) \), given by inner derivations.

Now consider a categorical \( G \)-crossed module \( \langle H, T, \nu, \chi \rangle \), and let \( \langle H, T : H \rightarrow Der^*(G, H), \nu, \chi \rangle \) be the categorical \( Der^*(G, H) \)-crossed module we have obtained in [6], where \( Der^*(G, H) \) is the Whitehead categorical group of derivations and \( \overline{T} : H \rightarrow Der^*(G, H) \) is the homomorphism given by inner derivations (see sections 5 and 6 of [6]). Then taking into account what we have recalled for groups, we give the following definition:

**Definition 4.1** Let \( G \) be a categorical group and \( \langle H, T, \nu, \chi \rangle \) a categorical \( G \)-crossed module. Then zero-th and first cohomology categorical groups of \( G \) with coefficients in \( \langle H, T, \nu, \chi \rangle \), are defined by

\[
\mathcal{H}^0(G, H) = \text{Ker}(\overline{T} : H \rightarrow Der^*(G, H))
\]

\[
\mathcal{H}^1(G, H) = Der^*(G, H)/\langle H, \overline{T} \rangle
\]

where the second one is the quotient categorical group built in Section 1 for the categorical crossed module \( \langle H, \overline{T}, \nu, \chi \rangle \).
Both definitions are functorial. Indeed, the existence of a 2-functor
\[ \mathcal{H}^0(\mathcal{G}, -) : \mathcal{G} - \text{cross} \to \mathcal{C}_G \]
is consequence of the fact that the kernel construction is 2-functorial. For
the 2-functoriality of \( \mathcal{H}^1 \) we first prove the following lemma:

**Lemma 4.2** Let \( (F, \eta, \alpha) : (\mathcal{H}, \mathcal{T}, \nu, \chi) \to (\mathcal{H}', \mathcal{T}', \nu', \chi') \) be a morphism of categorical \( \mathcal{G} \)-crossed modules (see [6]), then it extends to a morphism in \( \text{PreCross} \) between the associated inner derivations categorical crossed modules:

\[ (F, F\ast, \eta\ast, \alpha\ast) : (\mathcal{H}, \mathcal{T}, \nu, \chi) \to (\mathcal{H}', \mathcal{T}', \nu, \chi) \]
as indicated in the following diagram

\[
\begin{array}{ccc}
\text{Der}^*(\mathcal{G}, \mathcal{H}) \times \mathcal{H} & \xrightarrow{\text{ac}} & \mathcal{H} \\
\downarrow F \times F & & \downarrow F \\
\text{Der}^*(\mathcal{G}, \mathcal{H}') \times \mathcal{H}' & \xrightarrow{\text{ac}} & \mathcal{H}'
\end{array}
\]

Furthermore, if \( \gamma : (F, \eta, \alpha) \Rightarrow (F', \eta', \alpha') \) is a 2-cell in \( \mathcal{G} - \text{cross} \), it induces a 2-cell in \( \text{PreCross} \)

\[ (\gamma, \gamma\ast) : (F, F\ast, \eta\ast, \alpha\ast) \Rightarrow (F', F'\ast, \eta'\ast, \alpha'\ast) \]

**Proof:** We first recall that the functor \( F\ast : \text{Der}^*(\mathcal{G}, \mathcal{H}) \to \text{Der}^*(\mathcal{G}, \mathcal{H}') \) sends a derivation \( (D, \beta) \in \text{Der}^*(\mathcal{G}, \mathcal{H}) \) to \( F\ast(D, \beta) = (FD, F\ast \beta) \), where, for any \( X, Y \in \mathcal{G} \), \( (F\ast)_{X,Y} \) is given by the composition

\[
FD(X \otimes Y) \xrightarrow{(\beta X, Y)} F(D(X) \otimes XD(Y)) \simeq FD(X) \otimes F(XD(Y)) \xrightarrow{1 \otimes \eta_{X, D(Y)}} FD(X) \otimes XD(Y)
\]

The natural isomorphism \( \alpha\ast : F, \mathcal{T} \Rightarrow \mathcal{T}', F \) applies any object \( A \in \mathcal{H} \) to the morphism of derivations \( (\alpha)_{A} : (FD_A, \beta_A) \Rightarrow (D_{F(A)}, \beta_{F(A)}) \) which, for any \( X \in \mathcal{G} \), is given by the commutativity of the following diagram

\[
\begin{array}{ccc}
F(A \otimes XA^*) & \xrightarrow{((\alpha)_{A})_X} & F(A) \otimes XD(A) \\
\downarrow \text{can} & & \downarrow \text{can} \\
F(A) \otimes F(XA^*) & \xrightarrow{1 \otimes \eta_{X, A^*}} & F(A) \otimes XD(A)
\end{array}
\]
On the other hand, for any \((D, \beta) \in \text{Der}^*(G, H)\) and \(A \in H\), the natural isomorphism \(((\eta)_*)_A : F'((D,\beta)A) \rightarrow (FD,F,\beta)F(A)\) is the composition

\[
F(DT(A) \otimes A) \simeq FDT(A) \otimes F(A) \rightarrow FDTF(A) \otimes F(A)
\]

The verification that \((F,F,\eta,\alpha)\) is in fact a morphism of categorical precrossed modules is straightforward. We only point out that the commutativity of diagram (2) follows from the compatibility condition between the natural isomorphisms \(\chi, \chi'\) (see [6]).

Finally the 2-cell \(\gamma_* : F_* \Rightarrow F'_* : \text{Der}^*(G, H) \rightarrow \text{Der}^*(G, H')\) is given, for any \((D, \beta) \in \text{Der}^*(G, H)\) and \(X \in G\), by \(((\gamma)_*)_X = \gamma_{D(X)} : FD(X) \rightarrow F'D(X)\).

Then, by Proposition 3.3 we have a 2-functor:

\[
\mathcal{H}^1(G, -) : \mathbb{G} \rightarrow \text{cross} \rightarrow CG
\]

which applies a morphism \((F, \eta, \alpha) : \langle H, T, \nu, \chi \rangle \rightarrow \langle H', T', \nu', \chi' \rangle\) to the morphism of categorical groups

\[
\tilde{F}_* : \mathcal{H}^1(G, H) \rightarrow \mathcal{H}^1(G, H').
\]

**Remark 4.3** The categorical group \(\mathcal{H}^0(G, H)\) is equivalent to the categorical group of \(G\)-invariant objects \(H^G\) constructed in [9, 15] as follows: A \(G\)-invariant object of \(H\) consists of a pair \((A, \varphi_A)\), where \(A \in H\) and \(\varphi_A = \varphi^X : XA \rightarrow A\) for any \(X \in G\), \(\varphi^X\) is a family of natural isomorphisms in \(H\) such that \(\varphi^{X \otimes Y} = \varphi^X \otimes \varphi^Y \otimes \varphi_{X,Y,A}\) for any \(X, Y \in G\). An arrow \(u : (A, \varphi_A) \rightarrow (B, \varphi_B)\) is an arrow \(u : A \rightarrow B\) in \(H\) such that \(u \varphi^X = \varphi^X u\) for any \(X \in G\).

**Example 4.4** i) If \((H, T, \nu, \chi)\) is a discrete categorical \(G\)-crossed module, i.e. it is induced by a crossed module of groups, then \(\pi_i(\mathcal{H}^1(G, H))\), \(i = 0, 1\), are the cohomology groups of \(G\) with coefficients in \(H\) as defined in [13].

ii) Let \(A\) be a \(G\)-module and consider the categorical \(G\)-crossed module \(0 : A \rightarrow G\) (see Example 4.4 iii) in [6]). Then the cohomology categorical group \(\mathcal{H}^0(G, A)\) and \(\mathcal{H}^1(G, A)\) coincide with those defined in [9]. Therefore we have, when \(G = G[0]\) and \(A\) is symmetric,

\[
\pi_1(\mathcal{H}^0(G, A)) = \pi_1(A)^G = H^0_{U\lambda}(G, A)
\]

\[
\pi_0(\mathcal{H}^1(G, A)) = H^2_{U\lambda}(G, A)
\]
and
\[ \pi_0(\mathcal{H}^0(G, A)) = \pi_1(\mathcal{H}^1(G, A)) = H^1_{\text{Ulb}}(G, A) \]
where \( H^i_{\text{Ulb}}(G, A), i = 0, 1, 2 \), are the cohomology groups defined by Ulbrich in [20].

**Definition 4.5** Consider three categorical \( G \)-crossed modules \( \langle H, T : H \to G, \nu, \chi \rangle \), \( \langle H', T' : H' \to G, \nu', \chi' \rangle \) and \( \langle H'', T'' : H'' \to G, \nu'', \chi'' \rangle \) and two composable morphisms of categorical \( G \)-crossed modules \( (F', \eta', \alpha') : \langle H', T' : H', \nu', \chi' \rangle \to \langle H, T, \nu, \chi \rangle \) and \( (F, \eta, \alpha) : \langle H, T, \nu, \chi \rangle \to \langle H'', T'', \nu'', \chi'' \rangle \). Consider also a 2-cell \( \beta \) from the composite morphism to the zero morphism

\[
\begin{array}{ccc}
H' & \xrightarrow{\beta} & H \\
\downarrow{F'} & & \downarrow{F} \\
0 & & H''
\end{array}
\]

This sequence is called a short exact sequence of categorical \( G \)-crossed modules if the triple \( (F', \beta, F) \) is an extension in the sense of Definition 3.4.

Now we obtain the main result of this section.

**Theorem 4.6** Let

\[
\begin{array}{ccc}
H' & \xrightarrow{\beta} & H \\
\downarrow{F'} & & \downarrow{F} \\
0 & & H''
\end{array}
\]

be a short exact sequence of categorical \( G \)-crossed modules. Then there is a natural induced 2-exact sequence of categorical groups

\[
\begin{array}{ccc}
\mathcal{H}^0(G, H') & \xrightarrow{F'} & \mathcal{H}^0(G, H) \\
\downarrow{\Delta} & & \downarrow{\Delta} \\
\mathcal{H}^1(G, H') & \xrightarrow{F_*'} & \mathcal{H}^1(G, H)
\end{array}
\]

Moreover the functor \( F_*' \) is faithful.
Proof: By lemma 4.2 we have a diagram of arrows and 2-cells in PreCross.

\[
\begin{array}{ccc}
\mathbb{H}' & \xrightarrow{F'} & \mathbb{H} \\
\downarrow{\tau} & & \downarrow{\tau} \\
\text{Der}^*(G, \mathbb{H}') & \xrightarrow{F^*} & \text{Der}^*(G, \mathbb{H}) \\
& \xrightarrow{\beta^*} & \downarrow{\alpha} \\
& & \text{Der}^*(G, \mathbb{H}'')
\end{array}
\]

Now by Proposition 5.5 in [6], the triple \((F^*_*, \beta^*_*, F^*_*)\) is 2-exact. Moreover, the functor \(F^*_*\) is obviously faithfull and then the existence of the six-term 2-exact sequence follows from Proposition 3.6.

Acknowledgements. The authors gratefully acknowledge support by DGES of Spain, BFM2001-2866, and FNRS grant 1.5.116.01.

References


**P. Carrasco, A.R. Garzón**  
*Departamento de Álgebra*  
*Facultad de Ciencias*  
*Universidad de Granada*  
*18071, Granada*  
*Spain*  
*email: mcarrasc@ugr.es; agarzon@ugr.es*

**E.M. Vitale**  
*Département de Mathématique*  
*Université catholique de Louvain*  
*1348 Louvain-la-Neuve*  
*Belgium*  
*email: vitale@math.ucl.ac.be*