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The relationship between the diagonal and the bar constructions on a bisimplicial set

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Abstract

The aim of this paper is to prove that the homotopy type of any bisimplicial set X is modelled by the simplicial set $\overline{W}X$, the bar construction on X. We stress the interest of this result by showing two relevant theorems which now become simple instances of it; namely, the Homotopy colimit theorem of Thomason, for diagrams of small categories, and the generalized Eilenberg–Zilber theorem of Dold–Puppe for bisimplicial Abelian groups. Among other applications, we give an algebraic model for the homotopy theory of (not necessarily path-connected) spaces whose homotopy groups vanish in degree 4 and higher.
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1. Introduction and summary

Let X be a bisimplicial set. When X is regarded as a simplicial object in the simplicial set category and one takes geometric realizations, then one obtains a simplicial topological space whose Segal's realization [23] is homeomorphic to $|\operatorname{diag} X|$, the realization of the simplicial set diagonal of X. Hence the diagonal of a bisimplicial set is a "correct" simplicial set model for its homotopy type. Indeed, in [19] it is shown a closed model structure on the category of bisimplicial sets and it is proved that the diagonal functor induces an equivalence of homotopy categories.

However, there is another canonical way of associating a space to a bisimplicial set through the bar construction $X \mapsto \overline{W}X$. This functor $\overline{W}$, from bisimplicial sets to simplicial sets, may be defined as the functor right adjoint to Illusie's *total Dec* functor [14], and it was first considered by Artin and Mazur in [1] (see Section 2 for details). We must remark that the restriction of $\overline{W}$ to the categories of (nerves of) simplicial groups or simplicial monoids gives our "old" $\overline{W}$ of Eilenberg and Mac Lane [11], Moore [21] and Kan [15]. Also, for (nerves of) simplicial groupoids with discrete simplicial set of objects it agrees with the classifying complex construction $\overline{W}$ of Dwyer and Kan [10].

Since we have two different spaces associated to any bisimplicial set X (namely $|\operatorname{diag} X|$ and $|\overline{W}X|$), it seems natural to compare them. In a paper by Cordier [6] it is asserted that both spaces are indeed homotopically equivalent. However, no proof of this is given or even outlined in Cordier's paper and the only reference there-in is to a "*personal communication*" from Zisman. Furthermore, $\operatorname{diag}(-)$ is a left adjoint functor and good behavior with the left adjoint functor $|-|$ would be expected, but $\overline{W}$ is a right adjoint so it is far from obvious that they have equivalent realizations. The main purpose of this article is to prove the theorem below.

Theorem 1.1. *For any bisimplicial set X there is a natural weak homotopy equivalence*

$$\phi : \operatorname{diag} X \to \overline{W}X.$$

After this introduction and summary, Section 2 of this article contains a minimal amount of needed notation and terminology to prove Theorem 1.1, whose proof is given in Section 3. The remainder of the paper is devoted to examples and applications, and it is organized as follows:

Section 4. In this section, we shall examine the effective power of Theorem 1.1 by showing a relevant theorem which now becomes simple instance of it; that is, the Homotopy colimit theorem by Thomason [25], namely: Let $F : D \to \mathbf{Cat}$ be any diagram of small categories and let NF be the diagram of simplicial sets obtained by composing with the nerve functor N. Then, there is a natural weak homotopy equivalence

$$\operatorname{hocolim} NF \to N \int_D F$$

from the homotopy colimit of NF to the nerve of the Grothendieck construction on F.

Section 5. Here we show how another classic result is quickly deduced from Theorem 1.1: the generalized Eilenberg–Zilber theorem of Dold and Puppe [7], which states that for any bisimplicial Abelian group A there is an inducing homology isomorphism natural chain complex map

$$\mathrm{N}\,\mathrm{diag}\,A \to \mathrm{Tot}\,\mathrm{N}A$$

from the normalized (or Moore) chain complex of the diagonal of A to the total chain complex of the normalized bicomplex of A.

Section 6. As an easy new application of Theorem 1.1, in this section our objective is to provide a description of the homotopy type of a 2-category by showing a pleasing simplicial model for its classifying space. Our conclusion here can be seen as a generalization of a result by Moerdijk and Svensson [20, Section 2, Remark] and it may quite possibly be of interest not only to researchers in higher dimensional category theory, but also to K-theorists, since strict monoidal ("tensor") categories can be identified with 2-categories having a single object.

Section 7. This is a short section in which Theorem 1.1 is applied to prove that the adjoint functors $\mathrm{Dec} \dashv \overline{W} : \mathbf{S}^2 \leftrightarrows \mathbf{S}$ induce an equivalence of categories

$$\mathrm{Ho}\big(\mathbf{S}^2\big) \simeq \mathrm{Ho}(\mathbf{S})$$

between the homotopy category $\mathrm{Ho}(\mathbf{S}^2)$ of bisimplicial sets and the homotopy category $\mathrm{Ho}(\mathbf{S})$ of simplicial sets.

Section 8. Further to all the above quick consequences we dedicate this last section to showing a more substantial new application of Theorem 1.1, by giving an algebraic model for the homotopy theory of (not necessarily path-connected) spaces whose homotopy groups vanish in degree 4 and higher.

There are several notable precedents for our result here. The first is the well-known equivalence between the homotopy category of pointed (path) connected CW-complexes (X, x) such that $\pi_i(X, x) = 0$ for $i \geqslant 2$ (i.e., connected homotopy 1-types) and the category of groups. In [18], Mac Lane and Whitehead proved there is a similar equivalence between the homotopy category of pointed connected CW-complexes with trivial homotopy groups at dimensions $\geqslant 3$ (connected homotopy 2-types) and the homotopy category of crossed modules. These crossed modules were seen to be equivalent to 1-cat-groups (i.e., category objects within the category of groups) and based on this fact, Mac Lane and Whitehead's theorem was extended by Loday [17, Theorem 1.7] to a more general result which, in particular, shows that 2-cat-groups are algebraic models for connected homotopy 3-types, that is, of pointed connected CW-complexes (X, x) such that $\pi_i(X, x) = 0$ for $i \geqslant 4$ (see [5] or [22] for other different proofs of Loday's theorem). Examples such as G-spaces and equivariant homotopy types supply topological contexts where the assumption of path-connectedness is too restrictive. Indeed, one needs to find a sufficiently functorial generalization of the above results without assuming connectedness and choice of a base point. For the case of 1-types, it is well known that such a generalization is obtained by replacing groups by groupoids. For the case of 2-types, Moerdijk and Svensson [20] prove that the Mac Lane–Whitehead equivalence has a satisfactory generalization to not necessarily connected spaces by replacing the notion of crossed modules by that of a 2-groupoid.

Here we deal with non-connected 3-types, and for this case we have chosen to use an algebraic structure that we call 2-*cat-groupoid*. Such a 2-cat-groupoid consists of a groupoid endowed with four endomorphisms satisfying some nice conditions (see Definition 8.2) and it can be thought of as a groupoid enriched with two independent categorical structures. These 2-cat-groupoids relate to Loday's 2-cat-groups as ordinary groupoids relate to groups or as well as 2-groupoids relate to crossed modules: a 2-cat-group is a 2-cat-groupoid with exactly one object. We will show that the Loday equivalence can now be generalized to an equivalence between on the one hand the homotopy category of spaces X such that $\pi_i(X,x)=0$ for $i\geqslant 4$ and for any choice of base point $x\in X$, and, on the other hand, the homotopy category of 2-cat-groupoids. We should remark that this equivalence is completely functorial, explicit and essentially elementary.

2. Some notations and preliminaries

We employ the standard symbolism and nomenclature which can be found in texts on simplicial homotopy theory ([12], for example). For definiteness or emphasis we state the following.

We denote by Δ the category of ordered sets $[n]=\{0,1,\dots,n\}$, $n\geqslant 0$, with order-preserving maps between them. Recall that all maps in Δ are in fact generated by the injections $d^i:[n-1]\to[n]$ (cofaces), $0\leqslant i\leqslant n$, which miss out the ith element and the surjections $s^i:[n+1]\to[n]$ (codegeneracies), $0\leqslant i\leqslant n$, which repeat the ith element.

It is often convenient to see a bisimplicial set $X:\Delta^{op}\times\Delta^{op}\to\mathbf{Set}$ as a (horizontal) simplicial object in the category of (vertical) simplicial sets. If $\alpha:[p]\to[p']$ and $\beta:[q]\to[q']$ are any two maps in Δ, then we will write $\alpha^{*_h}:X_{p',q}\to X_{p,q}$ and $\beta^{*_v}:X_{p,q'}\to X_{p,q}$ by the images $X(\alpha,\mathrm{id})$ and $X(\mathrm{id},\beta)$ respectively. In particular, the horizontal and vertical face and degeneracy maps are $d_i^h=(d^i)^{*_h}$, $d_i^v=(d^i)^{*_v}$, $s_i^h=(s^i)^{*_h}$ and $s_i^v=(s^i)^{*_v}$ respectively.

2.1. *The constructions* diag, Dec *and* $\overline{W}$

By composing with the diagonal functor $\delta:\Delta\to\Delta\times\Delta$, $[n]\mapsto([n],[n])$, we get the diagonal functor, $X\mapsto\operatorname{diag}X=\delta^*(X)$, from bisimplicial sets to simplicial sets. We will use several times the following fact (see [2, XII, lemma in 2.3 and 4.3] for example): if $f:X\to X'$ is a bisimplicial map such that the maps $f_{p,\star}:X_{p,\star}\to X'_{p,\star}$ (respectively $f_{\star,q}:X_{\star,q}\to X'_{\star,q}$) are weak homotopy equivalences for all p (respectively q), then so is the map $\operatorname{diag}f:\operatorname{diag}X\to\operatorname{diag}X'$.

Similarly, composition with the ordinal sum functor $\mathrm{or}:\Delta\times\Delta\to\Delta$, $([p],[q])\mapsto[p+q+1]$, gives Illusie's *total Dec* functor $Y\mapsto\operatorname{Dec}Y=\mathrm{or}^*(Y)$, from simplicial to bisimplicial sets [14, VI, 1.5]. More specifically, for any simplicial set Y, $\operatorname{Dec}Y$ is the bisimplicial set with $\operatorname{Dec}_{p,q}Y=Y_{p+q+1}$, and whose simplicial operators are $d_i^h=d_i:Y_{p+q+1}\to Y_{p+q}$, and analogously $s_i^h=s_i$, whereas $d_j^v=d_{j+p+1}:Y_{p+q+1}\to Y_{p+q}$, and analogously $s_j^v = s_{j+p+1}$. Viewing $\mathrm{Dec}\, Y$ as a simplicial object in the category of simplicial sets, we have the canonical simplicial augmentation

$$d_0 : \mathrm{Dec}\, Y \to Y$$

which induces a weak equivalence $\mathrm{diag}\,\mathrm{Dec}\, Y \to Y$ on the diagonals (regarding Y as a bisimplicial set which is constant in the horizontal direction), since, por any $q \geqslant 0$, the augmented simplicial set $\mathrm{Dec}_{\star,q}\, Y \to Y_q$ has a simplicial contraction given by the degeneracies $s_i : Y_{q+i} \to Y_{q+i+1}$, $i \geqslant 0$.

The functor Dec has a right adjoint $\overline{W}$ [8], whose description is as follows (cf. [1, Section III]). For any bisimplicial set X, the set of p-simplices of $\overline{W}X$ is

$$\overline{W}_p(X) = \left\{ (x_0, x_1, \dots, x_p) \in \prod_{i=0}^{p} X_{i,p-i} \mid d_0^v x_i = d_{i+1}^h x_{i+1},\ \text{for all } 0 \leqslant i < p \right\},$$

and for $0 \leqslant i \leqslant p$ the faces and degeneracies of an element of $\overline{W}_p(X)$ are given by

$$d_i(x_0, \dots, x_p) = \left(d_i^v x_0, d_{i-1}^v x_1, \dots, d_1^v x_{i-1}, d_i^h x_{i+1}, d_i^h x_{i+2}, \dots, d_i^h x_p\right),$$
$$s_i(x_0, \dots, x_n) = \left(s_i^v x_0, s_{i-1}^v x_1, \dots, s_0^v x_i, s_i^h x_i, s_i^h x_{i+1}, \dots, s_i^h x_p\right).$$

We have the natural comparison map of Theorem 1.1

$$\phi : \mathrm{diag}\, X \to \overline{W}X, \tag{1}$$

which takes a p-simplex $x \in \mathrm{diag}\, X$ to

$$\phi(x) = \left(\left(d_1^h\right)^p x, \left(d_2^h\right)^{p-1} d_0^v x, \dots, \left(d_{i+1}^h\right)^{p-i} \left(d_0^v\right)^i x, \dots, \left(d_0^v\right)^p x\right).$$

2.2. *The simplicial category $N(\Delta)$*

In this paper we use the nerve construction on small categories of [2, XI,2.1]; thus, the nerve $N\mathcal{C}$ of a category $\mathcal{C}$ is the simplicial set whose p-simplices are the diagrams in $\mathcal{C}$ of the form

$$c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_p.$$

The i-face (respectively degeneracy) of this simplex is obtained by deleting the object c_i (respectively replacing c_i by $c_i \xleftarrow{\mathrm{id}} c_i$) in the standard way.

The nerve $N\Delta^{op}$ of the category Δ^{op} is the simplicial set of objects of a simplicial category, denoted by $\mathbf{N}\Delta$, that is defined as follows. For each $p \geqslant 0$, $\mathbf{N}_p\Delta$ is the category whose objects are all strings of p composable maps in Δ,

$$\Sigma = [n_0] \xrightarrow{\sigma_1} [n_1] \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_p} [n_p], \tag{2}$$

and whose arrows $f : \Sigma \to \Sigma'$ are those p-tuples $f = (f_k : [n_k] \to [n'_k])_{0 \leqslant k \leqslant p}$ of maps in Δ satisfying the conditions (3) below.

$$\begin{aligned} &f_k \sigma_k = \sigma'_k f_{k-1} && \text{for all } 1 \leqslant k \leqslant p, \\ &f_k(n_k) = n'_k && \text{for all } 0 \leqslant k \leqslant p. \end{aligned} \tag{3}$$

The composition in $\mathbf{N}_p(\Delta)$ is defined in the obvious way. The face and degeneracy functors

$$\mathbf{N}_{p+1}(\Delta) \xleftarrow{s_i} \mathbf{N}_p(\Delta) \xrightarrow{d_i} \mathbf{N}_{p-1}(\Delta)$$

act in the standard way: for Σ as in (2),

$$d_i\,\Sigma = \begin{cases} [n_1] \xrightarrow{\sigma_2} [n_2] \to \cdots \xrightarrow{\sigma_p} [n_p] & i = 0,\\ [n_0] \xrightarrow{\sigma_1} \cdots \to [n_{i-1}] \xrightarrow{\sigma_{i+1}\sigma_i} [n_{i+1}] \to \cdots \xrightarrow{\sigma_p} [n_p] & 0 < i < p,\\ [n_0] \xrightarrow{\sigma_1} [n_1] \to \cdots \xrightarrow{\sigma_{p-1}} [n_{p-1}] & i = p, \end{cases}$$

$$s_i\,\Sigma = [n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_i} [n_i] \xrightarrow{\mathrm{id}} [n_i] \xrightarrow{\sigma_{i+1}} [n_{i+1}] \to \cdots \xrightarrow{\sigma_p} [n_p], \tag{4}$$

and for $f = (f_k) : \Sigma \to \Sigma'$ any arrow in $\mathbf{N}_p\Delta$,

$$d_i\,f = (f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_p) : d_i\,\Sigma \to d_i\,\Sigma',$$
$$s_i\,f = (f_0, \ldots, f_i, f_i, \ldots, f_p) : s_i\,\Sigma \to s_i\,\Sigma'. \tag{5}$$

For any $p \geqslant 0$, we will denote by $\mathbf{d}$ the object of $\mathbf{N}_p\Delta$ (i.e., the p-simplex of $N\Delta^{op}$) bellow:

$$\mathbf{d} = [0] \xrightarrow{d^1} [1] \xrightarrow{d^2} \cdots \xrightarrow{d^p} [p]. \tag{6}$$

Then, for any $\Sigma \in \mathbf{N}_p\Delta$, say (2), we have a canonical morphism in $\mathbf{N}\Delta$,

$$\widetilde{\Sigma} = \bigl(\widetilde{\Sigma}_k\bigr) : \mathbf{d} \to \Sigma, \tag{7}$$

which is given by the "last vertex" maps

$$\widetilde{\Sigma}_k : [k] \to [n_k] \quad 0 \leqslant k \leqslant p, \tag{8}$$

defined by

$$\widetilde{\Sigma}_k(j) = \sigma_k \cdots \sigma_{j+1}(n_j) \quad 0 \leqslant j \leqslant k \ \bigl(\text{so } \widetilde{\Sigma}_k(k) = n_k\bigr).$$

The morphism (7) is natural in Σ, that is, for any given morphism $f : \Sigma \to \Sigma'$ in the category $\mathbf{N}_p\Delta$, the triangle below is commutative:

$$\begin{array}{ccc} & \mathbf{d} & \\ {\scriptstyle\widetilde{\Sigma}}\swarrow & & \searrow{\scriptstyle\widetilde{\Sigma}'} \\ \Sigma & \xrightarrow{\quad f \quad} & \Sigma' \end{array} \tag{9}$$

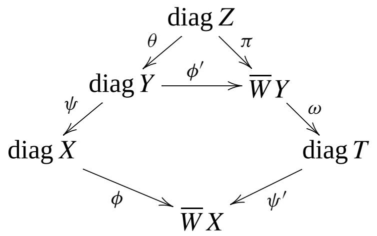
Since the identity functor of $\mathbf{N}_p\Delta$ is connected by a natural transformation to the constant functor $\mathbf{d}$, it follows that this category $\mathbf{N}_p\Delta$ is weakly contractible (in the sense that its classifying space is contractible).

Finally, and in relation with the maps (8), let us point out that the diagrams below are commutative for all $0 \leqslant i \leqslant k < p$ (which is straightforward to verify). The commutativity of these diagrams is needed many times along the paper to check simplicial identities in several constructions:

$$\begin{array}{ccc} [k] & \xrightarrow{d^{k+1}} & [k+1] \\ {\scriptstyle\widetilde{\Sigma}_k}\downarrow & & \downarrow{\scriptstyle\widetilde{\Sigma}_{k+1}} \\ [n_k] & \xrightarrow{\sigma_{k+1}} & [n_{k+1}] \end{array} \qquad \begin{array}{ccccc} [k] & \xrightarrow{\quad d^i \quad} & [k+1] & \xrightarrow{\quad s^i \quad} & [k] \\ & {\scriptstyle(\widetilde{d_i\Sigma})_k}\searrow \ \swarrow{\scriptstyle\widetilde{\Sigma}_{k+1}} & & {\scriptstyle(\widetilde{s_i\Sigma})_{k+1}}\searrow \ \swarrow{\scriptstyle\widetilde{\Sigma}_k} & \\ & [n_{k+1}] & & [n_k] & \end{array} \tag{10}$$

3. Proof of Theorem 1.1

The plan is as follows. Given X any bisimplicial set, we will build a diagram of simplicial maps



(Diagram 1)

in which Y, Z and T are certain bisimplicial sets, and we will prove that the maps ψ, θ, π, ω and ψ' are all weak homotopy equivalences, that the pentagon is commutative (i.e., $\psi'\omega\phi' = \phi\psi$) and also that the triangle on the top is commutative up to homotopy (for this we will show a simplicial homotopy $H:\pi \to \phi'\theta$). Then, as a consequence, we will have that the comparison map ϕ (1) is a weak equivalence as required.

The remainder of this section is devoted to carrying out the above objectives, and so we divide it into nine subsections.

3.1. The simplicial sets in Diagram 1

3.1.1. The bisimplicial set Y and the simplicial set $\overline{W}Y$

The bisimplicial set Y is defined to be the simplicial replacement [2, XII, 5.1] of X (viewing X as a Δ^{op}-diagram of simplicial sets); thus, its set of p,q-simplices is

$$Y_{p,q} = \big\{(\Sigma, x) \mid \Sigma = [n_0]\xrightarrow{\sigma_1}[n_1]\xrightarrow{\sigma_2}\cdots\xrightarrow{\sigma_p}[n_p] \in N_p\Delta^{op},\ x \in X_{n_p,q}\big\}. \tag{11}$$

The vertical face and degeneracy maps are defined by those of X, that is,

$$d_j^v(\Sigma, x) = \big(\Sigma, d_j^v x\big), \qquad s_j^v(\Sigma, x) = \big(\Sigma, s_j^v x\big)$$

and the horizontal face and degeneracy maps by those of $N\Delta^{op}$ (see (4)), that is,

$$d_i^h(\Sigma, x) = (d_i\Sigma, x), \qquad s_i^h(\Sigma, x) = (s_i\Sigma, x)$$

except $d_p^h : Y_{p,q} \to Y_{p-1,q}$ which is defined by

$$d_p^h(\Sigma, x) = \big(d_p\Sigma, \sigma_p^{*_h} x\big).$$

To work easily with the simplicial set $\overline{W}Y$, we will identify any p-simplex of $\overline{W}Y$ with a pair

$$(\Sigma, \mathbf{x}) \mid \Sigma = [n_0]\xrightarrow{\sigma_1}[n_1]\xrightarrow{\sigma_2}\cdots\xrightarrow{\sigma_p}[n_p] \in N_p\Delta^{op},$$

$$\mathbf{x} = (x_0, \ldots, x_p) \in \prod_{k=0}^{p} X_{n_k, p-k}, \tag{12}$$

such that the following equalities hold

$$d_0^v x_k = \sigma_{k+1}^{*_h} x_{k+1} \quad 0 \leqslant k < p;$$

this corresponds to the genuine p-simplex of $\overline{W}Y$ given by the tuple

$$\big(\big([n_0], x_0\big), \big([n_0] \xrightarrow{\sigma_1} [n_1], x_1\big), \dots, \big([n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_p} [n_p], x_p\big)\big) \in \prod_{k=0}^{p} Y_{k,p-k}.$$

After this identification, the face and degeneracy maps of $\overline{W}Y$ are given by

$$d_i(\Sigma, \mathbf{x}) = \begin{cases} \big(d_0\Sigma, (x_1, \dots, x_p)\big) & i = 0, \\ \big(d_i\Sigma, \big(d_i^v x_0, \dots, d_1^v x_{i-1}, x_{i+1}, \dots, x_p\big)\big) & 0 < i < p, \\ \big(d_p\Sigma, \big(d_p^v x_0, \dots, d_1^v x_{p-1}\big)\big) & i = p, \end{cases} \tag{13}$$
$$s_i(\Sigma, \mathbf{x}) = \big(s_i\Sigma, \big(s_i^v x_0, \dots, s_0^v x_i, x_i, \dots, x_p\big)\big).$$

3.1.2. The bisimplicial set Z

The bisimplicial set Z is that obtained by pulling back Illusie's augmented total-Dec of the nerve of Δ^{op} (see Section 2.1), $d_0 : \mathrm{Dec}\, N\Delta^{op} \to N\Delta^{op}$, along the canonical simplicial projection $\overline{W}Y \to N\Delta^{op}$ which applies a simplex $(\Sigma, \mathbf{x})$ of $\overline{W}Y$ (8) to $\Sigma \in N\Delta^{op}$. More precisely, Z is the bisimplicial set such that an element

$$(\Gamma, \alpha, \Sigma, \mathbf{x}) \in Z_{p,q} \tag{14}$$

consists of a p-simplex $\Gamma = [m_0] \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_p} [m_p]$ in $N\Delta^{op}$, a q-simplex $(\Sigma, \mathbf{x}) = ([n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_q} [n_q], (x_0, \dots, x_q))$ of $\overline{W}Y$ and an arrow $\alpha : [m_p] \to [n_0]$ in Δ, and such that its horizontal and vertical face and degeneracies maps are given by

$$d_i^h(\Gamma, \alpha, \Sigma, \mathbf{x}) = \begin{cases} (d_i\Gamma, \alpha, \Sigma, \mathbf{x}) & 0 \leqslant i < p, \\ (d_p\Gamma, \alpha\gamma_p, \Sigma, \mathbf{x}) & i = p, \end{cases}$$
$$s_i^h(\Gamma, \alpha, \Sigma, \mathbf{x}) = (s_i\Gamma, \alpha, \Sigma, \mathbf{x}) \quad 0 \leqslant i \leqslant p,$$
$$d_j^v(\Gamma, \alpha, \Sigma, \mathbf{x}) = \begin{cases} \big(\Gamma, \sigma_1\alpha, d_0(\Sigma, \mathbf{x})\big) & j = 0, \\ \big(\Gamma, \alpha, d_j(\Sigma, \mathbf{x})\big) & 0 < j \leqslant q, \end{cases}$$
$$s_j^v(\Gamma, \alpha, \Sigma, \mathbf{x}) = \big(\Gamma, \alpha, s_j(\Sigma, \mathbf{x})\big) \quad 0 \leqslant j \leqslant q,$$

where $d_i\Gamma$ and $s_i\Gamma$ are as in (4) while $d_j(\Sigma, \mathbf{x})$ and $s_j(\Sigma, \mathbf{x})$ are as in (13).

3.1.3. The bisimplicial set T

In order to define T, let us first observe that $\overline{W}Y$ is the simplicial set of objects of a simplicial category, denoted $\overline{\mathbf{W}}Y$, defined as follows. For each $p \geqslant 0$, category $\overline{\mathbf{W}}_pY$ has $\overline{W}_pY$ as set of objects and its morphisms

$$f : (\Sigma, \mathbf{x}) \to (\Sigma', \mathbf{x}')$$

are those arrows $f : \Sigma \to \Sigma'$ in the category $\mathbf{N}_p\Delta$ (see Section 2.2) such that $f^{*_h}(\mathbf{x}') = \mathbf{x}$ (i.e., $f_k^{*_h}(x_k') = x_k$ for all $0 \leqslant k \leqslant p$). The face and degeneracy functors of $\overline{\mathbf{W}}Y$ are defined by (13) on objects and by (5) on morphisms.

This simplicial category $\overline{\mathbf{W}}Y$ defines, by taking nerve of each category $\overline{\mathbf{W}}_pY$, the bisimplicial set T in Diagram 1. Thus,

$$T = N\overline{\mathbf{W}}Y : \big([p],[q]\big) \mapsto N_q\overline{\mathbf{W}}_pY.$$

More explicitly, an element $\chi \in T_{p,q}$ can be described as a string

$$\chi = \big(\Sigma^0, \mathbf{x}^0\big) \xleftarrow{f^1} \big(\Sigma^1, \mathbf{x}^1\big) \leftarrow \cdots \leftarrow \big(\Sigma^{q-1}, \mathbf{x}^{q-1}\big) \xleftarrow{f^q} \big(\Sigma^q, \mathbf{x}^q\big)$$

of q composable arrows in $\overline{\mathbf{W}}_pY$. The vertical face and degeneracy operators are those of the nerve $N\overline{\mathbf{W}}_pY$, that is, d_i^v acts by deleting $(\Sigma^i, \mathbf{x}^i)$ in the string χ and using composition, if needed, to form the new string $d_i^v\chi$. In the same way, s_i^v duplicates $(\Sigma^i, \mathbf{x}^i)$ at place $i+1$. The horizontal face and degeneracy operators are induced by those of $\overline{\mathbf{W}}Y$, that is,

$$d_j^h(\chi) = d_j\big(\Sigma^0, \mathbf{x}^0\big) \xleftarrow{d_jf^1} d_j\big(\Sigma^1, \mathbf{x}^1\big) \leftarrow \cdots \xleftarrow{d_jf^q} d_j\big(\Sigma^q, \mathbf{x}^q\big),$$
$$s_j^h(\chi) = s_j\big(\Sigma^0, \mathbf{x}^0\big) \xleftarrow{s_jf^1} s_j\big(\Sigma^1, \mathbf{x}^1\big) \leftarrow \cdots \xleftarrow{s_jf^q} s_j\big(\Sigma^q, \mathbf{x}^q\big).$$

3.2. The weak homotopy equivalence ψ in Diagram 1

There is a natural bisimplicial map

$$\Psi : Y \to X \tag{15}$$

sending a p,q-simplex (Σ, x) of Y as in (11) to the simplex $\widetilde{\Sigma}_p^{*_h}(x) \in X_{p,q}$; here $\widetilde{\Sigma}_p : [p] \to [n_p]$ is the pth map in (8). That Ψ is actually a bisimplicial map is an immediate consequence of the commutativity of diagrams (10).

The simplicial map ψ is then the induced by Ψ on the diagonals

$$\psi = \operatorname{diag}\Psi : \operatorname{diag}Y \to \operatorname{diag}X;$$

so that ψ acts on a p-simplex $(\Sigma, x) \in Y_{p,p}$ of $\operatorname{diag}Y$ (see (11)) by

$$\psi(\Sigma, x) = \widetilde{\Sigma}_p^{*_h}(x) \in X_{p,p}. \tag{16}$$

It is already known that ψ is a weak equivalence. Indeed, ψ is the map given by Bousfield and Kan in [2, XII, 3.4] establishing a weak equivalence between the homotopy colimit of X ($= \operatorname{diag}Y$) and the diagonal of X. So we have the following:

Proposition 3.1. *The map $\psi : \operatorname{diag}Y \to \operatorname{diag}X$ is a weak homotopy equivalence.*

3.3. The weak equivalence ω in Diagram 1

In this subsection we prove that the simplicial category $\overline{\mathbf{W}}Y$, defined in 3.1.3, has the same homotopy type as its simplicial set of objects $\overline{W}Y$. Indeed, regarding $\overline{W}Y$ as a bisimplicial set, constant in the vertical direction, we have a bisimplicial inclusion

$$\Omega : \overline{W}Y \to T = N\overline{\mathbf{W}}Y, \tag{17}$$

sending a p,q-simplex $(\Sigma,\mathbf{x}) \in \overline{W}_p X$ (8) to the p,q-simplex

$$\big(s_0^v\big)^q(\Sigma,\mathbf{x}) = \big((\Sigma,\mathbf{x}) \xleftarrow{\mathrm{id}} (\Sigma,\mathbf{x}) \leftarrow \overset{(q}{\cdots} \leftarrow (\Sigma,\mathbf{x}) \xleftarrow{\mathrm{id}} (\Sigma,\mathbf{x})\big) \in T_{p,q} = N_q\overline{\mathbf{W}}_p Y,$$

which is in fact a weak equivalence, as the next proposition proves.

Proposition 3.2. *The induced map* $\omega = \operatorname{diag}\Omega : \overline{W}Y \to \operatorname{diag} T$ *is a weak homotopy equivalence.*

Proof. It suffices to prove that for all $q \geqslant 0$ the simplicial map $\Omega_{\star,q} = (s_0^v)^q : \overline{W}Y \to N_q\overline{\mathbf{W}}Y$ is a weak equivalence. This is obvious for $q = 0$ and for $q \geqslant 1$ follows by an iterative application of the following fact: each map $s_0^v : N_{q-1}\overline{\mathbf{W}}Y \to N_q\overline{\mathbf{W}}Y$ is a simplicial homotopy equivalence, which is proved below.

Since $d_0^v s_0^v = \mathrm{id}$, we only have to show the existence of a simplicial homotopy $H : \mathrm{id} \to s_0^v d_0^v$. To do that, we first construct a $(p+1)$-simplex

$$\big(\Sigma^k(f), \mathbf{x}^k(f)\big) \in \overline{W}_{p+1}Y$$

associated to any arrow $f : (\Sigma,\mathbf{x}) \to (\Sigma',\mathbf{x}')$ in $\overline{W}_p Y$ and any $0 \leqslant k \leqslant p$. This simplex is given by

$$\Sigma^k(f) = [n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_k} [n_k] \xrightarrow{f_k} [n'_k] \xrightarrow{\sigma'_{k+1}} \cdots \xrightarrow{\sigma'_p} [n'_p],$$
$$\mathbf{x}^k(f) = \big(s_k^v x_0, \ldots, s_0^v x_k, x'_k, \ldots, x'_p\big).$$

Next, we consider the arrows in $\overline{\mathbf{W}}_{p+1}Y$

$$h_k(f) : s_k(\Sigma,\mathbf{x}) \to \big(\Sigma^k(f), \mathbf{x}^k(f)\big)$$

defined by $h_k(f) = (\mathrm{id}, \ldots, \mathrm{id}, f_k, \ldots, f_p)$, that is,

$$\begin{array}{ccccccccccc}
s_k\Sigma : [n_0] & \xrightarrow{\sigma_1} & \cdots & \xrightarrow{\sigma_k} & [n_k] & \xrightarrow{\mathrm{id}} & [n_k] & \xrightarrow{\sigma_{k+1}} & \cdots & \xrightarrow{\sigma_p} & [n_p] \\
{\scriptstyle h_k(f)}\big\downarrow{\scriptstyle \mathrm{id}} & & \cdots & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle f_k} & & \cdots & & \big\downarrow{\scriptstyle f_p} \\
\Sigma^k(f) : [n_0] & \xrightarrow{\sigma_1} & \cdots & \xrightarrow{\sigma_k} & [n_k] & \xrightarrow{f_k} & [n'_k] & \xrightarrow{\sigma'_{k+1}} & \cdots & \xrightarrow{\sigma'_p} & [n'_p]
\end{array}$$

And, finally, the announced homotopy $H : \mathrm{id} \to s_0^v d_0^v$ is given by the maps $H_k : N_q\overline{\mathbf{W}}_p Y \to N_q\overline{\mathbf{W}}_{p+1}Y$, $0 \leqslant k \leqslant p$, which send any p-simplex

$$\chi = \big(\Sigma^0, \mathbf{x}^0\big) \xleftarrow{f^1} \big(\Sigma^1, \mathbf{x}^1\big) \xleftarrow{f^2} \cdots \xleftarrow{f^q} \big(\Sigma^q, \mathbf{x}^q\big)$$

in $N_q\overline{\mathbf{W}}Y$ to the $(p+1)$-simplex

$$H_k(\chi) = \big(\Sigma^k\big(f^1\big), \mathbf{x}^k\big(f^1\big)\big) \xleftarrow{h_k(f^1)} s_k\big(\Sigma^1, \mathbf{x}^1\big) \xleftarrow{s_k f^2} \cdots \xleftarrow{s_k f^q} s_k\big(\Sigma^q, \mathbf{x}^q\big).$$

We leave to the reader to check that H is indeed a simplicial homotopy as required. □

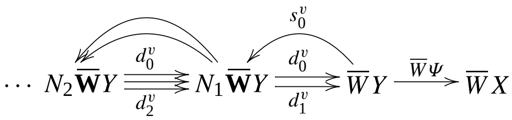
3.4. *The weak equivalence ψ' in Diagram 1*

The bisimplicial map (15), $\Psi : Y \to X$, induces a corresponding simplicial map $\overline{W}\Psi : \overline{W}Y \to \overline{W}X$ that acts on a p-simplex $(\Sigma, \mathbf{x})$ of $\overline{W}Y$ as in (8) by

$$\overline{W}\Psi(\Sigma, \mathbf{x}) = \widetilde{\Sigma}^{*_h}(\mathbf{x}) \in \overline{W}_p X, \tag{18}$$

where for short we are writing $\widetilde{\Sigma}^{*_h}(\mathbf{x})$ by $(\widetilde{\Sigma}_0^{*_h}(x_0), \widetilde{\Sigma}_1^{*_h}(x_1), \ldots, \widetilde{\Sigma}_p^{*_h}(x_p))$. Here the maps $\widetilde{\Sigma}_k : [k] \to [n_k]$ are those defined in (8).

This simplicial map $\overline{W}\Psi$ defines an augmentation on the (vertical) simplicial object of (horizontal) simplicial sets $T = N\overline{\mathbf{W}}Y$,



since, for any morphism $f : (\Sigma, \mathbf{x}) \to (\Sigma', \mathbf{x}')$ in $\overline{\mathbf{W}}Y$ (see 3.1.3), we have

$$\overline{W}\Psi(\Sigma', \mathbf{x}') \overset{(18)}{=} \widetilde{\Sigma'}^{*_h}(\mathbf{x}') \overset{(9)}{=} \widetilde{\Sigma}^{*_h} f^{*_h}(\mathbf{x}') = \widetilde{\Sigma}^{*_h}(\mathbf{x}) = \overline{W}\Psi(\Sigma, \mathbf{x}).$$

Therefore, there is another induced simplicial map

$$\psi' : \operatorname{diag} T \to \overline{W}X \tag{19}$$

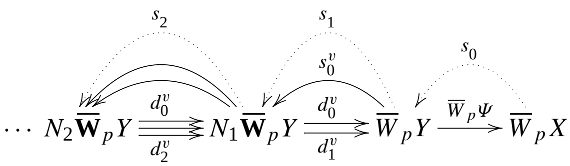
given by

$$\psi'\big(\big(\Sigma^0, \mathbf{x}^0\big) \overset{f^1}{\longleftarrow} \cdots \overset{f^p}{\longleftarrow} \big(\Sigma^p, \mathbf{x}^p\big)\big) = \overline{W}\Psi\big(\Sigma^p, \mathbf{x}^p\big) = \widetilde{\Sigma}^{p*_h}\big(\mathbf{x}^p\big);$$

and this is the map so termed ψ' in Diagram 1.

Proposition 3.3. *The map $\psi' : \operatorname{diag} T \to \overline{W}X$ is a weak homotopy equivalence.*

Proof. The proposition follows from the fact that, for any $p \geqslant 0$, the augmented simplicial set $N\overline{\mathbf{W}}_p Y \overset{\overline{W}_p\Psi}{\longrightarrow} \overline{W}_p X$ has a simplicial contraction



given by the extra degeneracies maps $s_0 : \overline{W}_p X \to \overline{W}_p Y$ and $s_{q+1} : N_q\overline{\mathbf{W}}_p Y \to N_{q+1}\overline{\mathbf{W}}_p Y$, $q \geqslant 0$, defined respectively by

$$s_0(\mathbf{x}) = (\mathbf{d}, \mathbf{x}),$$

$$s_{q+1}\big(\big(\Sigma^0, \mathbf{x}^0\big) \overset{f^1}{\longleftarrow} \cdots \overset{f^q}{\longleftarrow} \big(\Sigma^q, \mathbf{x}^q\big)\big)$$

$$= \big(\big(\Sigma^0, \mathbf{x}^0\big) \overset{f^1}{\longleftarrow} \cdots \overset{f^q}{\longleftarrow} \big(\Sigma^q, \mathbf{x}^q\big) \overset{\widetilde{\Sigma}^q}{\longleftarrow} \big(\mathbf{d}, \widetilde{\Sigma}^{q*_h}\big(\mathbf{x}^q\big)\big)\big),$$

where $\mathbf{d}$ is as in (6), $\widetilde{\Sigma}^q : \mathbf{d} \to \Sigma^q$ as in (7) and $\widetilde{\Sigma}^{q*_h}(\mathbf{x})$ as in (18). □

3.5. The map ϕ' and the commutativity of the pentagon in Diagram 1

The map $\phi' : \operatorname{diag} Y \to \overline{W}Y$ is the corresponding comparison map (1) for the bisimplicial set Y. Thus, by taking into account the identification made by (8) of the simplices of $\overline{W}Y$, ϕ' can be described as the simplicial map that sends a p-simplex $(\Sigma = [n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_p} [n_p], x) \in \operatorname{diag} Y$ (so that $x \in X_{n_p,p}$) to the p-simplex of $\overline{W}Y$

$$\phi'(\Sigma, x) = \big(\Sigma, \big((\sigma_p \cdots \sigma_1)^{*_h} x, d_0^v(\sigma_p \cdots \sigma_2)^{*_h} x, \ldots, \big(d_0^v\big)^{p-1} \sigma_p^{*_h} x, \big(d_0^v\big)^p x\big)\big). \quad (20)$$

Proposition 3.4. *In Diagram* 1 *the equality $\psi'\omega\phi' = \phi\psi$ holds.*

Proof. Let (Σ, x) be any p-simplex of $\operatorname{diag} Y$ as above. Then, we have

$$\begin{aligned}
\psi'\omega\phi'(\Sigma, x) &\overset{(20)}{=} \psi'\omega\big(\Sigma, \big(\big(d_0^v\big)^k (\sigma_p \cdots \sigma_{k+1})^{*_h} x\big)_{0 \leqslant k \leqslant p}\big) \\
&\overset{(17),\ (19),\ (18)}{=} \big(\widetilde{\Sigma}_k^{*_h} \big(d_0^v\big)^k (\sigma_p \cdots \sigma_{k+1})^{*_h} x\big)_{0 \leqslant k \leqslant p} \\
&= \big(\big(d_0^v\big)^k \big(\sigma_p \cdots \sigma_{k+1} \widetilde{\Sigma}_k\big)^{*_h} x\big)_{0 \leqslant k \leqslant p} \\
&\overset{(10)}{=} \big(\big(d_0^v\big)^k \big(\widetilde{\Sigma}_p d^p \cdots d^{k+1}\big)^{*_h} x\big)_{0 \leqslant k \leqslant p} \\
&= \big(\big(d_0^v\big)^k d_{k+1}^h \cdots d_p^h \widetilde{\Sigma}_p^{*_h} x\big)_{0 \leqslant k \leqslant p} \\
&= \big(\big(d_0^v\big)^k \big(d_{k+1}^h\big)^{p-k} \widetilde{\Sigma}_p^{*_h} x\big)_{0 \leqslant k \leqslant p} \\
&\overset{(1)}{=} \phi\big(\widetilde{\Sigma}_p^{*_h} x\big) \overset{(16)}{=} \phi\psi(\Sigma, x),
\end{aligned}$$

as required. □

3.6. The weak equivalence θ in Diagram 1

There is a natural bisimplicial map

$$\Theta : Z \to Y \quad (21)$$

sending a p, q-simplex $(\Gamma, \alpha, \Sigma, \mathbf{x})$ of Z as in (14) to the simplex $(\Gamma, \alpha^{*_h} x_0)$ of Y; recall that $(\Sigma, \mathbf{x}) \in \overline{W}_q Y$, so $x_0 \in X_{n_0,q}$ and therefore $\alpha^{*_h} x_0 \in X_{m_p,q}$ whence $(\Gamma, \alpha^{*_h} x_0) \in Y_{p,q}$.

The simplicial map θ in Diagram 1 is then that induced by Θ on the diagonals, that is,

$$\theta = \operatorname{diag} \Theta : \operatorname{diag} Z \to \operatorname{diag} Y. \quad (22)$$

Proposition 3.5. *The map θ is a weak homotopy equivalence.*

Proof. It follows directly from next Lemma 3.6. □

Lemma 3.6. *For any fixed $p \geqslant 0$, $\Theta : Z_{p,\star} \to Y_{p,\star}$ is a simplicial homotopy equivalence.*

Proof. Let $\mu : Y_{p,\star} \to Z_{p,\star}$ be the simplicial map defined on a q-simplex $(\Sigma, x) = ([n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_p} [n_p], x)$ of $Y_{p,\star}$ by

$$\mu(\Sigma, x) = \big(\Sigma, [n_p] \xrightarrow{\text{id}} [n_p], [n_p] \xrightarrow{\text{id}} \overset{q)}{\cdots} \xrightarrow{\text{id}} [n_p], \big(x, d_0^v x, \ldots, \big(d_0^v\big)^q x\big)\big) \in Z_{p,q}.$$

Then, we have that $\Theta\mu = \text{id}$ and it only remains to show a simplicial homotopy $H : \text{id} \to \mu\Theta$. Indeed, such a simplicial homotopy is given by the maps

$$H^k : Z_{p,q} \to Z_{p,q+1}, \quad 0 \leqslant k \leqslant q$$

defined on a p,q-simplex of Z as in (14), say

$$(\Gamma, \alpha, \Sigma, \mathbf{x}) = \big([m_0] \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_p} [m_p], [m_p] \xrightarrow{\alpha} [n_0], [n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_q} [n_q], (x_0, \ldots, x_q)\big),$$

by

$$H^k(\Gamma, \alpha, \Sigma, \mathbf{x}) = \begin{cases} \big(\Gamma, [m_p] \xrightarrow{\text{id}} [m_p], [m_p] \xrightarrow{\alpha} [n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_q} [n_q], \mathbf{x}^0\big) & k = 0, \\ \big(\Gamma, [m_p] \xrightarrow{\text{id}} [m_p], \\ [m_p] \xrightarrow{\text{id}} \overset{k)}{\cdots} \xrightarrow{\text{id}} [m_p] \xrightarrow{\sigma_k \cdots \sigma_1 \alpha} [n_k] \xrightarrow{\sigma_{k+1}} \cdots \xrightarrow{\sigma_q} [n_q], \mathbf{x}^k\big) & 0 < k \leqslant q, \end{cases}$$

where

$$\mathbf{x}^0 = \big(s_0^v \alpha^{*_h} x_0, x_0, \ldots, x_q\big)$$

and for $0 < k \leqslant q$,

$$\begin{aligned} \mathbf{x}^k &= \big(s_k^v \alpha^{*_h} x_0, s_{k-1}^v (\sigma_1 \alpha)^{*_h} x_1, \ldots, s_0^v (\sigma_k \cdots \sigma_1 \alpha)^{*_h} x_k, x_k, \ldots, x_q\big) \\ &= \big(s_k^v \alpha^{*_h} x_0, s_{k-1}^v d_0^v \alpha^{*_h} x_0, \ldots, s_0^v \big(d_0^v\big)^k \alpha^{*_h} x_0, x_k, \ldots, x_q\big). \end{aligned}$$

It is straightforward to check that H, so defined, verifies the appropriate simplicial identities, and we will therefore leave them to the reader. □

3.7. *The weak equivalence π in Diagram 1*

Regarding the simplicial set $\overline{W}Y$ as a bisimplicial set that is constant in the horizontal direction (so that $(\overline{W}Y)_{p,q} = \overline{W}_q Y$), there is a natural bisimplicial map

$$\Pi : Z \to \overline{W}Y$$

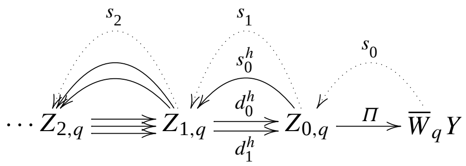
sending a p,q-simplex $(\Gamma, \alpha, \Sigma, \mathbf{x})$ of Z as in (14) to the simplex $(\Sigma, \mathbf{x}) \in \overline{W}_q Y$.

The simplicial map π in Diagram 1 is then that induced by Π on the diagonals, that is,

$$\pi = \operatorname{diag} \Pi : \operatorname{diag} Z \to \overline{W}Y. \tag{23}$$

Proposition 3.7. *The map π is a weak homotopy equivalence.*

Proof. The proposition follows from the fact that, for any $q \geqslant 0$, the augmented simplicial set $Z_{\star,q} \xrightarrow{\Pi} \overline{W}_q Y$ has a simplicial contraction



given by the extra degeneracies maps $s_0 : \overline{W}_q Y \to Z_{0,q}$ and $s_{p+1} : Z_{p,q} \to Z_{p+1,q}$, $p \geqslant 0$, defined respectively by

$$s_0(\Sigma, \mathbf{x}) = \big([n_0], [n_0] \xrightarrow{\mathrm{id}} [n_0], \Sigma, \mathbf{x}\big),$$
$$s_{p+1}(\Gamma, \alpha, \Sigma, \mathbf{x}) = \big([m_0] \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_p} [m_p] \xrightarrow{\alpha} [n_0], [n_0] \xrightarrow{\mathrm{id}} [n_0], \Sigma, \mathbf{x}\big),$$

where we are assuming that $\Gamma = [m_0] \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_p} [m_p]$ and $\Sigma = [n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_q} [n_q]$, as in (14). $\square$

3.8. The homotopy commutativity of the triangle in Diagram 1

Proposition 3.8. *There is a simplicial homotopy $H : \pi \to \phi'\theta$.*

Proof. The simplicial homotopy H is given by the maps

$$H^k : Z_{p,p} \to \overline{W}_{p+1} Y, \quad 0 \leqslant k \leqslant p$$

defined on a p-simplex $(\Gamma, \alpha, \Sigma, \mathbf{x})$ of $\operatorname{diag} Z$ (see (14)), say

$$(\Gamma, \alpha, \Sigma, \mathbf{x}) = \big([m_0] \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_p} [m_p], [m_p] \xrightarrow{\alpha} [n_0],$$
$$[n_0] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_p} [n_p], (x_0, \ldots, x_p)\big),$$

by

$$H^k(\Gamma, \alpha, \Sigma, \mathbf{x}) = \big([m_0] \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_k} [m_k] \xrightarrow{\sigma_k \cdots \sigma_1 \alpha \gamma_p \cdots \gamma_{k+1}} [n_k] \xrightarrow{\sigma_{k+1}} \cdots \xrightarrow{\sigma_p} [n_p],$$
$$\big(x_0^k, \ldots, x_{p+1}^k\big)\big),$$

where, for each $0 \leqslant k \leqslant p$ and $0 \leqslant j \leqslant p+1$,

$$x_j^k = \begin{cases} s_{k-j}^v (\sigma_j \cdots \sigma_1 \alpha \gamma_p \cdots \gamma_{j+1})^{*_h} x_j & 0 \leqslant j \leqslant k, \\ x_{j-1} & k < j \leqslant p+1 \end{cases}$$
$$= \begin{cases} s_{k-j}^v \big(d_0^v\big)^j (\alpha \gamma_p \cdots \gamma_{j+1})^{*_h} x_0 & 0 \leqslant j \leqslant k, \\ x_{j-1} & k < j \leqslant p+1. \end{cases}$$

The verification that H is actually a simplicial homotopy from π to $\phi'\theta$ is quite straightforward and we leave it to the reader. $\square$

3.9. The end of the proof of Theorem 1.1

We now are definitely ready to complete the proof of the result in this paper. Looking at Diagram 1, by Propositions 3.5 and 3.7 we have that both θ and π are weak equivalences and then, by Proposition 3.8, we deduce that ϕ' also is a weak equivalence. Since $\psi'\omega\phi' = \phi\psi$, by Proposition 3.4, and the three maps ψ', ω and ψ are all weak equivalences by Propositions 3.3, 3.2 and 3.1, we conclude that $\phi : \operatorname{diag} X \to \overline{W} X$ is also a weak equivalence as stated in Theorem 1.1.

4. Homotopy colimits of small categories

In this section we see how the Homotopy colimit theorem by Thomason [25, Theorem 1.2] for diagrams of small categories follows from Theorem 1.1 as an instance.

Let $F : D \to \mathbf{Cat}$ be any given diagram of categories. The Grothendieck construction [13] assembles the diagram F into a category, denoted $\int_D F$. The objects of this category are pairs (a, u) where a is an object of D and u is an object of $F(a)$. An arrow $(a, u) \to (b, v)$ in $\int_D F$ is a pair (f, σ) with $f : a \to b$ a morphism in D and $\sigma : {}^f u \to v$ a morphism in $F(b)$, where we are writing ${}^f u$ by $F(f)(u)$. Arrows in $\int_D F$ compose according to the diagram

$$(c, w) \xleftarrow{(g,\tau)} (b, v) \xleftarrow{(f,\sigma)} (a, u), \qquad (a,u) \xrightarrow{(gf,\, \tau\, {}^g\sigma)} (c,w)$$

where we are writing ${}^g\sigma$ by $F(g)(\sigma)$. The nerve $N \int_D F$ of the Grothendieck construction on F is then a simplicial set canonically associated to the D-diagram of categories F.

On the other hand, the simplicial replacement construction [2, XII, 5.1] assembles the D-diagram of simplicial sets NF, obtained by composing F with the nerve functor, into a bisimplicial set, denoted $\coprod_* NF$. The p, q-simplices of this bisimplicial set are pairs of diagrams

$$(A, U) = \left(a_0 \xleftarrow{f_1} \cdots \xleftarrow{f_p} a_p, u_0 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_q} u_q\right)$$

in D and $F(a_p)$ respectively. The vertical face and degeneracy maps are defined on a p, q-simplex (A, U) as above, by those of $NF(a_p)$, that is, $d_i^v(A, U) = (A, d_i U)$, $s_i^v(A, U) = (A, s_i U)$; and similarly the horizontal ones are defined by those of ND, that is, $d_j^h(A, U) = (d_j A, U)$, $s_j^h(A, U) = (s_j A, U)$, except the last horizontal face map which is defined by $d_p^h(A, U) = (d_p A, {}^{f_p} U)$.

The homotopy colimit theorem of Thomason states that both the bisimplicial set $\coprod_* NF$ and the simplicial set $N \int_D F$ have the same homotopy type. More precisely, by definition $\operatorname{hocolim} NF = \operatorname{diag} \coprod_* NF$ and Thomason proved that *there is a weak homotopy equivalence*

$$\eta : \operatorname{hocolim} NF \to N \int_D F$$

given by

$$\eta\big(a_0 \xleftarrow{f_1} a_1 \cdots \xleftarrow{f_p} a_p, u_0 \xleftarrow{\sigma_1} u_1 \cdots \xleftarrow{\sigma_p} u_p\big)$$
$$= \big(a_0, {}^{f_1\cdots f_p}u_0\big) \xleftarrow{(f_1, {}^{f_1\cdots f_p}\sigma_1)} \big(a_1, {}^{f_2\cdots f_p}u_1\big) \xleftarrow{(f_2, {}^{f_2\cdots f_p}\sigma_2)} \cdots$$
$$\big(a_{p-1}, {}^{f_p}u_{p-1}\big) \xleftarrow{(f_p, {}^{f_p}\sigma_p)} (a_p, u_p).$$

Now, the simplicial set $N\int_D F$ can be actually identified with the simplicial set $\overline{W}\coprod_* NF$ (cf. [26, Theorem 3.2.12]) by means of the simplicial isomorphism

$$j\colon N\int_D F \cong \overline{W}\coprod\nolimits_* NF$$

that carries a p-simplex

$$(a_0, u_0) \xleftarrow{(f_1,\sigma_1)} \cdots \xleftarrow{(f_p,\sigma_p)} (a_p, u_p)$$

of the nerve of the Grothendieck construction on F to the p-simplex $(x_0, \ldots, x_p) \in \overline{W}\coprod_* NF$, where, for each $0 \leqslant k \leqslant p$,

$$x_k = \big(\, a_0 \xleftarrow{f_1} \cdots \xleftarrow{f_k} a_k\ ,\ u_k \xleftarrow{\sigma_{k+1}} {}^{f_{k+1}}u_{k+1} \xleftarrow{{}^{f_{k+1}}\sigma_{k+2}} \cdots \xleftarrow{{}^{f_{k+1}\cdots f_{p-1}}\sigma_p} {}^{f_{k+1}\cdots f_p}u_p\,\big).$$

Therefore Thomason's result follows, since the map $\phi : \operatorname{diag}\coprod_* NF \to \overline{W}\coprod_* NF$ (1) is a weak equivalence, by Theorem 1.1, and since the equality $j\eta = \phi$ holds.

5. Bisimplicial Abelian groups

Our purpose in this section is to deduce from Theorem 1.1 a well-known and useful result by Dold and Puppe [7, Theorem 2.9] for bisimplicial Abelian groups.

Recall that the normalized complex $\mathrm{N}A$ of a simplicial Abelian group A has the group

$$\mathrm{N}_m A = \bigcap_{i=0}^{m-1} \operatorname{Ker}(d_i : A_m \to A_{m-1})\ (= A_0 \quad \text{if } m = 0)$$

as m-chains, and has boundary $\partial : \mathrm{N}_m A \to \mathrm{N}_{m-1} A$ the induced homomorphism by $d_m : A_m \to A_{m-1}$ (0, if $m = 0$). It is a fact that $\pi_m(A, 0) = H_m \mathrm{N}A$ for all $m \geqslant 0$.

Hereafter suppose that A is a bisimplicial Abelian group.

The normalized complex $\mathrm{N}\operatorname{diag} A$ of its diagonal simplicial Abelian group is then a chain complex canonically associated to A. On the other hand, the normalized bicomplex $\mathrm{N}A$, of the bisimplicial Abelian group A, is the bicomplex having p, q-chains

$$\mathrm{N}_{p,q} A = \left(\bigcap_{i=0}^{p-1} \operatorname{Ker}\big(d_i^h : A_{p,q} \to A_{p-1,q}\big)\right) \cap \left(\bigcap_{j=0}^{q-1} \operatorname{Ker}\big(d_j^v : A_{p,q} \to A_{p,q-1}\big)\right),$$

horizontal boundary $\partial^h : \mathrm{N}_{p,q}(A) \to \mathrm{N}_{p-1,q}$ given by $\partial^h(x) = d^h_p(x)$ and vertical boundary $\partial^v : \mathrm{N}_{p,q}(A) \to \mathrm{N}_{p,q-1}$ given by $\partial^v(x) = (-1)^p d^v_q(x)$. Then, the total chain complex of this bicomplex, $\mathrm{Tot}\,\mathrm{N}A$, is also a chain complex naturally defined by the bisimplicial Abelian group A.

The generalized Eilenberg–Zilber theorem by Dold and Puppe states that *there exists a quasi-isomorphism of complexes* (*i.e., a chain complex homomorphism inducing isomorphisms on the homology groups*)

$$\mathrm{N}\,\mathrm{diag}\,A \to \mathrm{Tot}\,\mathrm{N}A. \tag{24}$$

Now, the normalized chain complex $\mathrm{N}\overline{W}A$, of the simplicial Abelian group obtained as the bar construction on A, can be identified with the chain complex $\mathrm{Tot}\,\mathrm{N}A$ by means of the chain complex isomorphism

$$\ell\colon\ \mathrm{N}\overline{W}A \cong \mathrm{Tot}\,\mathrm{N}(A) \tag{25}$$

defined as follows. Observe that p-chains of $\mathrm{N}\overline{W}A$ are those p-tuples

$$(x_0, \dots, x_p) \in A_{0,p} \oplus A_{1,p} \oplus \cdots \oplus A_{p,0}$$

satisfying

$$\begin{aligned}
&d^v_0 x_i = d^h_{i+1} x_{i+1} && 0 \leqslant i < p,\\
&d^h_i x_j = 0 && 0 \leqslant i < j \leqslant p,\\
&d^v_i x_j = 0 && 0 \leqslant j < p,\ 0 < i < p-j,
\end{aligned}$$

and whose boundary is

$$\partial(x_0, \dots, x_p) = \big(d^v_p x_0, \dots, d^v_{p-k} x_k, \dots, d^v_1 x_{p-1}\big),$$

whereas a p-chain of $\mathrm{Tot}\,\mathrm{N}A$ is also a p-tuple

$$(y_0, \dots, y_p) \in A_{0,p} \oplus A_{1,p} \oplus \cdots \oplus A_{p,0}$$

but in this case satisfying

$$\begin{aligned}
&d^v_i y_j = 0 && 0 \leqslant j < p,\ 0 \leqslant i < p-j,\\
&d^h_i y_j = 0 && 0 \leqslant i < j \leqslant p,
\end{aligned}$$

and whose boundary is

$$\begin{aligned}
&\partial(y_0, \dots, y_p)\\
&\quad = \big(d^v_p y_0 + d^h_1 y_1, \dots, (-1)^k d^v_{p-k} y_k + d^h_{k+1} y_{k+1}, \dots, (-1)^{p-1} d^v_1 y_{p-1} + d^h_p y_p\big).
\end{aligned}$$

Then, the chain complex isomorphism (25) is given by

$$\ell(x_0, \dots, x_p) = (y_0, \dots, y_p),$$

where, for each $0 \leqslant k \leqslant p$,

$$y_k = (-1)^{(p+1)k}\left(x_k - \sum_{i=0}^{p-k-1} (-1)^i s^v_i d^h_{k+1} x_{k+1}\right).$$

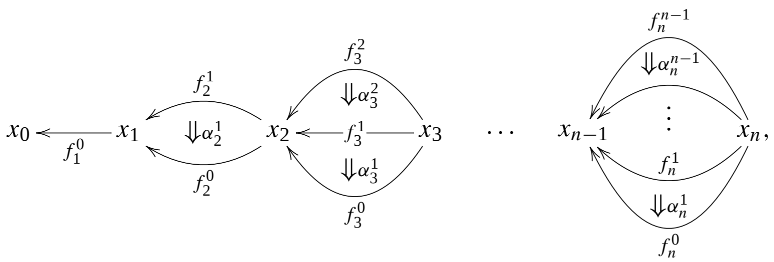
Therefore, since Theorem 1.1 gives the weak homotopy equivalence $\phi : \mathrm{diag}\, A \to \overline{W}A$, we get Dold–Puppe's chain complex quasi-isomorphism (24) by composing the induced quasi-isomorphism $\mathrm{N}(\phi) : \mathrm{N}\,\mathrm{diag}\, A \to \mathrm{N}\overline{W}A$ with the chain complex isomorphism ℓ: $\mathrm{N}\overline{W}A \cong \mathrm{Tot}\,\mathrm{N}A$.

6. Classifying spaces of 2-categories

As a new application of Theorem 1.1, in this section we shall derive the corollary below in which a simplicial model for the classifying space of a 2-category is shown (cf. [3] where a different model, the Duskin–Street's *geometric nerve* [9], is proved).

Recall that a 2-*category* $\mathcal{C}$ is a category enriched in the category of small categories. Thus, $\mathcal{C}$ is a category in which the hom-set between any two objects $x, y \in \mathcal{C}$ is the set of objects of a category $\mathcal{C}(x, y)$, whose arrows are called deformations (2-cells) and are denoted $f : u \Rightarrow v$. Moreover, the composition is a bifunctor $\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$ which is associative and has identities $\mathrm{Id}_x \in \mathcal{C}(x, x)$. This bifunctor produces compositions of arrows and deformations respectively, both denoted here by juxtaposition. On the other hand, composition in each category $\mathcal{C}(x, y)$ is denoted by "$\circ$".

Corollary 6.1. *The classifying space of a* 2-*category* $\mathcal{C}$ *is naturally homotopy equivalent to the geometric realization of the simplicial set* $\overline{W}\mathcal{C}$, *whose* n-*simplices are diagrams* (x, f, α) *in* $\mathcal{C}$ *of the form*



consisting of objects (0-*cells*) x_k, $0 \leqslant k \leqslant n$, *arrows* (1-*cells*) $f_m^k : x_m \to x_{m-1}$, $0 \leqslant k < m \leqslant n$, *and deformations* (2-*cells*) $\alpha_m^k : f_m^k \Rightarrow f_m^{k-1}$, $0 < k < m \leqslant n$. *The simplicial operators act much as for the usual nerve of an ordinary category*: *the face operators* $d_i : \overline{W}_n\mathcal{C} \to \overline{W}_{n-1}\mathcal{C}$, $0 \leqslant i \leqslant n$, *are given by* $d_i(x, f, \alpha) = (y, g, \beta)$, *where*

$$y_k = x_{d^i(k)} \quad 0 \leqslant k \leqslant n-1,$$

$$g_m^k = \begin{cases} f_{d^i(m)}^{d^i(k)} & m \neq i, \\ f_i^k f_{i+1}^k & m = i,\ 0 \leqslant k < m \leqslant n-1, \end{cases}$$

$$\beta_m^k = \begin{cases} \alpha_{d^i(m)}^{d^i(k)} & m \neq i \neq k, \\ \alpha_i^k \alpha_{i+1}^k & m = i, \\ \alpha_{m+1}^i \circ \alpha_{m+1}^{i+1}\ k = i & 0 < k < m \leqslant n-1; \end{cases}$$

and the degeneracy operators $s_i : \overline{W}_n\mathcal{C} \to \overline{W}_{n+1}\mathcal{C}$, $0 \leqslant i \leqslant n$, *are given by* $s_i(x, f, \alpha) = (y, g, \beta)$, *where*

$$y_k = x_{s^i(k)} \quad 0 \leqslant k \leqslant n+1,$$

$$g^k_m = \begin{cases} f^{s^i(k)}_{s^i(m)} & m \neq i+1, \\ \mathrm{Id}_{x_i} & m = i+1, \ 0 \leqslant k < m \leqslant n+1, \end{cases}$$

$$\beta^k_m = \begin{cases} \alpha^{s^i(k)}_{s^i(m)} & m \neq i+1 \neq k, \\ \mathrm{Id}_{\mathrm{Id}_{x_i}} & m = i+1, \\ \mathrm{Id}_{f^i_{i+1}} & k = i+1, \ 0 < k < m \leqslant n+1. \end{cases}$$

Proof. The classifying space of a small category is the geometric realization of its nerve [16]. Then, by replacing the hom-categories $\mathcal{C}(x, y)$ by their classifying spaces $\mathrm{B}\mathcal{C}(x, y) = |N\mathcal{C}(x, y)|$, any given 2-category $\mathcal{C}$ gives rise to a topological category $\mathfrak{B}\mathcal{C}$ (with discrete space of objects) whose nerve, $N\mathfrak{B}\mathcal{C}$, is then a simplicial topological space. The Segal's realization of this simplicial space is just defined to be the *classifying space* $\mathrm{B}\mathcal{C}$ of the 2-category $\mathcal{C}$, that is, $\mathrm{B}\mathcal{C} = |N\mathfrak{B}\mathcal{C}|$.

By the compatibility of the classifying space construction with products of small categories, for each $p \geqslant 0$, we have $N_p\mathfrak{B}\mathcal{C} = \mathrm{B}N_p\mathcal{C} = |NN_p\mathcal{C}|$, where

$$N_p\mathcal{C} = \coprod_{x_0,\ldots,x_p \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(x_1, x_0) \times \mathcal{C}(x_2, x_1) \times \cdots \times \mathcal{C}(x_p, x_{p-1})$$

($N_0\mathcal{C} = \mathrm{Ob}(\mathcal{C})$, as a discrete category). Therefore, there is a natural homeomorphism $\mathrm{B}\mathcal{C} \cong |\operatorname{diag} NN\mathcal{C}|$, where $NN\mathcal{C} : ([p], [q]) \mapsto N_qN_p\mathcal{C}$ is the bisimplicial set *double nerve* of the 2-category.

Now, let us write $\overline{W}\mathcal{C}$ by $\overline{W}NN\mathcal{C}$. Then $\overline{W}\mathcal{C}$ is precisely the simplicial set whose simplices have the pleasing description made in Corollary 6.1. Since Theorem 1.1 implies that the spaces $\mathrm{B}\mathcal{C} = |\operatorname{diag} NN\mathcal{C}|$ and $|\overline{W}\mathcal{C}| = |\overline{W}NN\mathcal{C}|$ are naturally homotopy equivalent, it follows that $\overline{W}\mathcal{C}$ is a simplicial set model for the classifying space of the 2-category $\mathcal{C}$. $\square$

Remark 6.2. As it is proved in [3, Theorem 1.1], the classifying space $\mathrm{B}\mathcal{C}$ of a 2-category $\mathcal{C}$ can also be realized by another simplicial set $\Delta\mathcal{C}$, the so-called geometric nerve of the 2-category [24,9]. This simplicial set $\Delta\mathcal{C}$ is different from $\overline{W}\mathcal{C}$ —but naturally weak homotopy equivalent to it—except in the case where $\mathcal{C}$ is a 2-groupoid (i.e., when all arrows and deformations of $\mathcal{C}$ are invertible). Indeed, $\Delta\mathcal{C} \cong \overline{W}\mathcal{C}$ for any 2-groupoid $\mathcal{C}$, and our Corollary 6.1 can be seen as a generalization of a result by Moerdijk and Svensson [20, Section 2, Remark] stating that the classifying space of a 2-groupoid $\mathcal{C}$ is homotopy equivalent to the geometric realization of the simplicial set $\Delta\mathcal{C}$ ($= \mathrm{N}\mathcal{C}$ in the notation of [20]).

Remark 6.3. Corollary 6.1 may quite possible be of interest not only to researchers in higher dimensional category theory but also to K-theorists, since strict monoidal ("tensor") categories can be identified with 2-categories having a single object. In effect, a strict monoidal category $(\mathcal{C}, \otimes, I)$ can be viewed as a 2-category with only one object, say I, the objects x of $\mathcal{C}$ as arrows $x : I \to I$ and the arrows of $\mathcal{C}$ as deformations. The horizontal composition of arrows and deformations is given by the strict tensor functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and the vertical composition of deformations is given by the composition of arrows in $\mathcal{C}$. Then, the classifying space $\mathrm{B}(\mathcal{C}, \otimes, I)$ of the monoidal category [23] is just the classifying space of the 2-category it defines. Hence, by Corollary 6.1, the simplicial set $\overline{W}(\mathcal{C}, \otimes, I)$ realizes the classifying space of the strict monoidal category $(\mathcal{C}, \otimes, I)$ (for readers interested in the non-strict case, we refer to [4]).

On the other hand, a strict monoidal category $(\mathcal{C}, \otimes, I)$ is also a monoid object in **Cat** and therefore, since the nerve functor preserves products, its nerve $N\mathcal{C}$ is a simplicial monoid. Observe now that the simplicial set $\overline{W}(\mathcal{C}, \otimes, I)$ is reduced, that $\overline{W}_n(\mathcal{C}, \otimes, I) = N_0\mathcal{C} \times N_1\mathcal{C} \times \cdots \times N_{n-1}\mathcal{C}$, for each $n \geqslant 1$, and then finally conclude that $\overline{W}(\mathcal{C}, \otimes, I) = \overline{W} N\mathcal{C}$, where $\overline{W} N\mathcal{C}$ is the classical $\overline{W}$-construction on the simplicial monoid $N\mathcal{C}$.

7. Dec, $\overline{W}$ and weak equivalences

Thanks to Theorem 1.1, it is easy to specify the behavior of the Dec and $\overline{W}$ constructions on weak equivalences. Indeed, we can show how these adjoint functors induce an equivalence between the homotopy categories of simplicial and bisimplicial sets as follows.

Recall that a bisimplicial map $f : X \to X'$ is a weak equivalence if the induced cellular map between the corresponding Segal's geometric realizations is a homotopy equivalence of spaces. From the fact that the geometric realization of a bisimplicial set is homeomorphic to the geometric realization of its diagonal, it follows that $f : X \to X'$ is a weak equivalence if and only if $\mathrm{diag}(f) : \mathrm{diag}\, X \to \mathrm{diag}\, X'$ is a weak equivalence of simplicial sets. Since the comparison map (1) is natural, any bisimplicial map f as above gives rise to a commutative square

$$\begin{array}{ccc} \mathrm{diag}\, X & \xrightarrow{\mathrm{diag}(f)} & \mathrm{diag}\, X' \\ \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_{X'}} \\ \overline{W} X & \xrightarrow{\overline{W}(f)} & \overline{W} X' \end{array}$$

where both ϕ_X and $\phi_{X'}$ are weak equivalences (by Theorem 1.1) and, therefore, we conclude that *a bisimplicial map $f : X \to X'$ is a weak equivalence if and only if the induced simplicial map $\overline{W}(f) : \overline{W} X \to \overline{W} X'$ is a weak equivalence*.

Write $\mathbf{S}$ for the category of simplicial sets and $\mathbf{S}^2$ for the category of bisimplicial sets, and recall from Section 2.1 that the functor $\overline{W} : \mathbf{S}^2 \to \mathbf{S}$ is right adjoint to Illusie's functor $\mathrm{Dec} : \mathbf{S} \to \mathbf{S}^2$. The unit and the counit of the adjunction

$$u : Y \to \overline{W}\,\mathrm{Dec}\, Y \quad (Y \in \mathbf{S}),$$
$$v : \mathrm{Dec}\, \overline{W} X \to X \quad (X \in \mathbf{S}^2)$$

are respectively defined by

$$u(y) = (s_0 y, \ldots, s_n y) \qquad (y \in Y_n),$$
$$v(x_0, \ldots, x_{p+q+1}) = d_0^v x_{p+1} \quad \big((x_0, \ldots, x_{p+q+1}) \in \mathrm{Dec}_{p,q}\, \overline{W} X\big).$$

Proposition 7.1. *For any simplicial set Y, the unit map $u : Y \to \overline{W}\,\mathrm{Dec}\,Y$ is a weak equivalence.*

Proof. Consider the canonical simplicial augmentation $d_0 : \mathrm{Dec}\,Y \to Y$, which, regarding Y as a bisimplicial set constant in the horizontal direction, is a weak equivalence of simplicial sets (see Section 2.1). Then, the induced simplicial map $\overline{W}(d_0) : \overline{W}\,\mathrm{Dec}\,Y \to \overline{W}Y \cong Y$, $(y_0, \dots, y_n) \mapsto d_0 y_0$, is a weak equivalence. Since $\overline{W}(d_0)u = id_Y$, the proposition follows. □

Proposition 7.1 implies that *a simplicial set map $f : Y \to Y'$ is a weak equivalence if and only if the induced bisimplicial map* $\mathrm{Dec}(f) : \mathrm{Dec}\,Y \to \mathrm{Dec}\,Y'$ *is a weak equivalence* (from the natural equality $\overline{W}\,\mathrm{Dec}(f)\,u_Y = u_{Y'}\,f$). Furthermore, a simplicial map $f : Y \to \overline{W}X$ is a weak equivalence if and only if its adjoint bisimplicial map $\tilde{f} : \mathrm{Dec}\,Y \to X$ is a weak equivalence (from the adjoint equality $\overline{W}(\tilde{f})\,u_Y = f$). In particular, *the counit $v : \mathrm{Dec}\,\overline{W}X \to X$ is a weak equivalence* of bisimplicial sets.

Corollary 7.2. *The adjoint functors* $\mathrm{Dec} \dashv \overline{W} : \mathbf{S}^2 \leftrightarrows \mathbf{S}$ *induce an equivalence of categories*

$$\mathrm{Ho}\big(\mathbf{S}^2\big) \simeq \mathrm{Ho}(\mathbf{S})$$

between the homotopy category $\mathrm{Ho}(\mathbf{S}^2)$ *of bisimplicial sets and the homotopy category* $\mathrm{Ho}(\mathbf{S})$ *of simplicial sets.*

8. Algebraic models for (not necessarily connected) homotopy 3-types

To help motivate the reader with the data of our algebraic models for homotopy 3-types (see Definition 8.2 below), we shall begin this section with a commentary on 2-groupoids. Let $\mathcal{C}$ be a small 2-category. Identifying any 1-cell with the corresponding identity 2-cell, we can describe the 2-category as a system $\mathcal{C} = (C, A, s, t, \circ)$ where C is a small category (whose objects are the 0-cells and whose arrows are the 2-cells), $A \subseteq C$ is a subcategory with the same objects (whose arrows are the 1-cells), and $s, t : C \to A$ are functors (those assigning the corresponding source and target 1-cells to any 2-cell) such that their restriction to A are the identity and $\circ$ is a partial composition functor $C \times_A C \xrightarrow{\circ} C$, which is associative and satisfies $\alpha \circ s(\alpha) = \alpha = t(\alpha) \circ \alpha$, $s(\alpha \circ \beta) = s(\beta)$ and $t(\alpha \circ \beta) = t(\alpha)$.

Note that the condition of $\circ$ being a functor implies that the usual interchange law holds, namely that

$$(\alpha\alpha') \circ (\beta\beta') = (\alpha \circ \beta)(\alpha' \circ \beta')$$

whenever both sides are defined.

Lemma 8.1. *Let $\mathcal{C} = (C, A, s, t, \circ)$ be a 2-category such that the categories C and A are groupoids. Then $\mathcal{C}$ is a 2-groupoid and the composition $\circ : C \times_A C \to C$ is determined by the equalities*

$$\alpha(s\alpha)^{-1}\beta = \alpha \circ \beta = \beta(t\beta)^{-1}\alpha. \tag{26}$$

Proof. By the interchange law we have

$$\alpha(s\alpha)^{-1}\beta = (\alpha \circ s\alpha)\big(s\alpha^{-1} \circ s\alpha^{-1}\big)(s\alpha \circ \beta) = \alpha \circ \beta,$$
$$\beta(t\beta)^{-1}\alpha = (t\beta \circ \beta)\big(t\beta^{-1} \circ t\beta^{-1}\big)(\alpha \circ t\beta) = \alpha \circ \beta.$$

Furthermore, the inverse α^* of a deformation α always exists and it is given by the formula $\alpha^* = (s\alpha)\alpha^{-1}(t\alpha)$. □

According to the above Lemma 8.1, we see that a 2-groupoid is given by the data (C, A, s, t), where C is a groupoid, $A \subseteq C$, is a *wide* subgroupoid (i.e., with the same objects) and $s, t : C \to A$ are functors satisfying $s|_A = \mathrm{id}_A = t|_A$, and such that the map $\circ : C \times_A C \to A$ given by the formula (26) is a functor. Now, it is not difficult to see that this last condition is equivalent to the commutativity condition

$$\alpha\beta = \beta\alpha$$

for any automorphisms $\alpha, \beta \in C(x) = C(x, x)$, $x \in \mathrm{Ob}\, C$, such that $s\alpha = 1_x = t\beta$. This leads to our notion of 1-*cat-groupoid* (C, A, s, t) and, as suggested by the Loday's work [17], we extend this definition to the higher dimension as follows.

Definition 8.2. A 2-*cat-groupoid* $\mathcal{C}$ is a system

$$\mathcal{C} = (C, C_h, C_v, s_h, t_h, s_v, t_v), \tag{27}$$

where C is a small groupoid, $C_h, C_v \subseteq C$ are wide subgroupoids, and $s_h, t_h : C \to C_h$ and $s_v, t_v : C \to C_v$ are functors, such that

(1) $s_h a = a = t_h a$ for all $a \in C_h$ and $s_v b = b = t_v b$ for all $b \in C_v$,
(2) $\alpha\beta = \beta\alpha$, for any two automorphisms $\alpha, \beta \in C(x)$, $x \in \mathrm{Ob}\, C$, such that $s_h\alpha = 1_x = t_h\beta$ or $s_v\alpha = 1_x = t_v\beta$,
(3) $s_h s_v = s_v s_h$, $t_h t_v = t_v t_h$, $s_h t_v = t_v s_h$ and $s_v t_h = t_h s_v$.

A *morphism* $f : \mathcal{C} \to \mathcal{C}'$ between 2-cat-groupoids is a functor $f : C \to C'$ such that $f(C_h) \subseteq C'_h$, $f(C_v) \subseteq C'_v$ and the following equalities hold: $f s_h = s'_h f$, $f t_h = t'_h f$, $f s_v = s'_v f$ and $f t_v = t'_v f$. The category of 2-cat-groupoids is denoted by 2-**Cat-Gd**.

Let $\mathcal{C}$ be a 2-cat-groupoid as above. There is associated to each object $x \in \mathrm{Ob}\, C$ a (not necessarily Abelian) chain complex

$$\widetilde{\mathcal{C}}(x) = \cdots \to 0 \to \cdots \to 0 \to \widetilde{\mathcal{C}}_2(x) \xrightarrow{\tilde{\partial}_2} \widetilde{\mathcal{C}}_1(x) \xrightarrow{\tilde{\partial}_1} \widetilde{\mathcal{C}}_0(x) \to 0, \tag{28}$$

in which

- $\widetilde{\mathcal{C}}_0(x) = C_h(x) \cap C_v(x)$,
- $\widetilde{\mathcal{C}}_1(x) = \{(\alpha, \beta) \in C_h(x) \times C_v(x) \mid s_v\alpha = t_h\beta,\ s_h\beta = 1_x\}$,
- $\widetilde{\mathcal{C}}_2(x) = \{\alpha \in C(x) \mid s_h\alpha = 1_x = s_v\alpha\}$,

and the boundary maps are defined by

- $\tilde{\partial}_1(\alpha,\beta) = t_v\alpha$,
- $\tilde{\partial}_2\alpha = ((t_h\alpha)^{-1}t_h t_v\alpha, t_v\alpha)$.

Then, associated to $\mathcal{C}$, we take the invariant made up of:

- $\pi_0\mathcal{C} = \pi_0 C$, the set of isomorphism classes of the objects in C,

and for each object $x \in \operatorname{Ob} C$ and $i = 1, 2, \ldots,$

- $\pi_i(\mathcal{C}, x) = H_{i-1}\widetilde{\mathcal{C}}(x)$.

Clearly, each morphism $f : \mathcal{C} \to \mathcal{C}'$ of 2-cat-groupoids induces a map $\pi_0 f : \pi_0\mathcal{C} \to \pi_0\mathcal{C}'$ and group homomorphisms $\pi_i f : \pi_i(\mathcal{C}, x) \to \pi_i(\mathcal{C}', fx)$ for $i > 0$, $x \in \operatorname{Ob} C$. Call a morphism $f : \mathcal{C} \to \mathcal{C}'$ of 2-cat-groupoids a *weak equivalence* if it induces bijections $\pi_i f$ for $i \geqslant 0$. The localization of 2-Cat-Gd with respect to the family of weak equivalences will be denoted as Ho(2-**Cat-Gd**).

We now state our result on homotopy 3-types, whose proof the rest of the section is mainly dedicated to.

Theorem 8.3. *There are adjoint functors* $B : 2\text{-}\mathbf{Cat\text{-}Gd} \to \mathbf{S}$, *the right adjoint, and* $P : \mathbf{S} \to 2\text{-}\mathbf{Cat\text{-}Gd}$, *the left adjoint, that induce an equivalence of categories*

$$\operatorname{Ho}(2\text{-}\mathbf{Cat\text{-}Gd}) \simeq \operatorname{Ho}\big(X \in \mathbf{S} \mid \pi_i(X, x) = 0,\ i \geqslant 4,\ x \in X_0\big).$$

The pair of adjoint functors $P \dashv B$ in the above Theorem 8.3 is obtained by composition of three adjoint pairs of functors

$$2\text{-}\mathbf{Cat\text{-}Gd} \underset{\mathcal{N}}{\overset{\wp}{\rightleftarrows}} s\mathbf{Gd}^2 \underset{\overline{W}}{\overset{\mathrm{Dec}}{\rightleftarrows}} s\mathbf{Gd} \underset{\overline{W}}{\overset{G}{\rightleftarrows}} \mathbf{S},$$

that is, B and P are respectively given by

$$B = \overline{W}\,\overline{W}\mathcal{N}, \qquad P = \wp\,\mathrm{Dec}\,G,$$

where

- $s\mathbf{Gd}$ denotes the category of simplicial groupoids (with a discrete simplicial set of objects) and $G \dashv \overline{W} : s\mathbf{Gd} \leftrightarrows \mathbf{S}$ is the pair of adjoint functors defined by Dwyer and Kan in [10].
- $s\mathbf{Gd}^2$ denotes the category of bisimplicial groupoids (with discrete bisimplicial set of objects) and $\mathrm{Dec} \dashv \overline{W} : s\mathbf{Gd}^2 \leftrightarrows s\mathbf{Gd}$ is the pair of adjoint functors canonically induced by the pair of adjoint functors $\mathrm{Dec} \dashv \overline{W} : \mathbf{S}^2 \leftrightarrows \mathbf{S}$.
- the functor $\mathcal{N} : 2\text{-}\mathbf{Cat\text{-}Gd} \to s\mathbf{Gd}^2$ is defined by a double nerve construction as follows. Let $\mathcal{C}$ be a 2-cat-groupoid as in (27). Then, the bisimplicial groupoid $\mathcal{N}\mathcal{C}$ has the same objects as the groupoid C. An arrow $\alpha \in \mathcal{N}_{p,q}\mathcal{C}$, for $p, q \geqslant 1$, is a $(q \times p)$-matrix

  $$\bar{\alpha} = (\alpha_{ij})_{1 \leqslant i \leqslant q, 1 \leqslant j \leqslant p}$$

  of arrows $\alpha_{ij} \in C$ such that $s_h\alpha_{ij} = t_h\alpha_{ij+1}$ and $s_v\alpha_{ij} = t_v\alpha_{i+1j}$. Furthermore, $\mathcal{N}_{0,q}\mathcal{C} = \mathcal{N}_{1,q}\mathcal{C} \cap C_h^q$, $\mathcal{N}_{p,0}\mathcal{C} = \mathcal{N}_{p,1}\mathcal{C} \cap C_v^p$ and $\mathcal{N}_{0,0}\mathcal{C} = C_h \cap C_v$.

For $q \geqslant 2$, the vertical face functors $d_k^v : \mathcal{N}_{p,q}\mathcal{C} \to \mathcal{N}_{p,q-1}\mathcal{C}$ are given by $d_k^v \bar\alpha = \bar\beta$, where

$$\beta_{ij} = \begin{cases} \alpha_{ij} & i < k, \\ \alpha_{kj}(s_v\alpha_{kj})^{-1}\alpha_{k+1j} & i = k, \\ \alpha_{i+1j} & i > k, \end{cases}$$

and for $q = 1$ they are defined by

$$d_0^v(\alpha_1, \ldots, \alpha_p) = (s_v\alpha_1, \ldots, s_v\alpha_p), \qquad d_1^v(\alpha_1, \ldots, \alpha_p) = (t_v\alpha_1, \ldots, t_v\alpha_p).$$

The horizontal face functors $d_k^h : \mathcal{N}_{p,q}\mathcal{C} \to \mathcal{N}_{p-1,q}\mathcal{C}$ are similarly defined. Thus, for $p \geqslant 2$, $d_k^h \bar\alpha = \bar\beta$, where

$$\beta_{ij} = \begin{cases} \alpha_{ij} & j < k, \\ \alpha_{ik}(s_h\alpha_{ik})^{-1}\alpha_{ik+1} & j = k, \\ \alpha_{ij+1} & j > k, \end{cases}$$

and for $p = 1$ they are given by

$$d_0^h \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} = \begin{pmatrix} s_h\alpha_1 \\ \vdots \\ s_h\alpha_q \end{pmatrix}, \qquad d_1^h \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} = \begin{pmatrix} t_h\alpha_1 \\ \vdots \\ t_h\alpha_q \end{pmatrix}.$$

The degeneracy functors $s_0^v : \mathcal{N}_{p,0}\mathcal{C} \to \mathcal{N}_{p,1}\mathcal{C}$ and $s_0^h : \mathcal{N}_{0,q}\mathcal{C} \to \mathcal{N}_{1,q}\mathcal{C}$ are inclusions and for $p, q \geqslant 1$ and an arrow $\bar\alpha \in \mathcal{N}_{p,q}\mathcal{C}$, then $s_k^v \bar\alpha = \bar\beta$ and $s_k^h \bar\alpha = \bar\gamma$, where

$$\beta_{ij} = \begin{cases} \alpha_{ij} & i \leqslant k, \\ s_v\alpha_{kj} & i = k+1, \\ \alpha_{i-1j} & i > k+1, \end{cases} \qquad \gamma_{ij} = \begin{cases} \alpha_{ij} & j \leqslant k, \\ s_h\alpha_{ik} & j = k+1, \\ \alpha_{ij-1} & j > k+1, \end{cases}$$

for $k > 0$, and

$$\beta_{ij} = \begin{cases} t_v\alpha_{1j} & i = 1, \\ \alpha_{i-1j} & i > 1, \end{cases} \qquad \gamma_{ij} = \begin{cases} t_h\alpha_{i1} & j = 1, \\ \alpha_{ij-1} & j > 1, \end{cases}$$

for $k = 0$.

- the functor $\wp : s\mathbf{Gd}^2 \to 2\text{-}\mathbf{Cat}\text{-}\mathbf{Gd}$ is given by a double Poincaré groupoid construction as follows. Let $\mathbb{G}$ be a bisimplicial groupoid. For any objects $x, y \in \mathrm{Ob}\,\mathbb{G}$, two arrows $\alpha, \beta \in \mathbb{G}_{1,1}(x, y)$ are *bihomotopic*, written as $\alpha \approx \beta$, if there exists a pair of arrows $(a, b) \in \mathbb{G}_{1,2} \times \mathbb{G}_{2,1}$ such that the following equalities hold:

$$d_0^h a = s_0^h d_0^h \alpha, \qquad d_0^v b = s_0^v d_0^v \beta, \qquad d_1^h a = \alpha, \qquad d_1^v b = \beta, \qquad d_2^h a = d_2^v b.$$

Such a pair (a, b) is a *bihomotopy* from α to β and is denoted by $(a, b)\colon \alpha \approx \beta$. A straightforward verification proves that the four assertions below hold.

- If $(a, b)\colon \alpha \approx \beta$ and $(a', b')\colon \alpha' \approx \beta'$, then $(aa', bb')\colon \alpha\alpha' \approx \beta\beta'$ (whenever the compositions are meaningful).
- $(s_1^h\alpha, s_1^v\alpha)\colon \alpha \approx \alpha$.

- If $(a,b)\colon \alpha\approx\beta$ and $(c,d)\colon \beta\approx\gamma$, then $(a(s_1^h\beta)^{-1}c, b(s_1^v\beta)^{-1}d)\colon \alpha\approx\gamma$.
- If $(a,b)\colon \alpha\approx\beta$, then $(a',b')\colon \beta\approx\alpha$, where $a'=(s_1^h d_2^h a)a^{-1}s_1^h(\alpha(d_2^h a)^{-1}\beta)$ and $b'=s_1^v(\alpha(d_2^v b)^{-1}\beta)b^{-1}(s_1^v d_2^v b)$.

Hence, bihomotopy is a congruence on the groupoid $\mathbb{G}_{1,1}$ and we shall denote by

$$\mathrm{C}\mathbb{G}=\mathbb{G}_{1,1}/\approx$$

the corresponding quotient groupoid. Thus, $\mathrm{C}\mathbb{G}$ has the same objects as the bisimplicial groupoid $\mathbb{G}$ and its arrows $[\alpha]\in\mathrm{C}\mathbb{G}(x,y)$ are bihomotopy classes of arrows $\alpha\in\mathbb{G}_{1,1}(x,y)$, $x,y\in\mathrm{Ob}\,\mathbb{G}$. Multiplication is given by $[\alpha][\beta]=[\alpha\beta]$.

The 2-cat-groupoid $\wp\mathbb{G}$, which we shall call the *fundamental* 2-*cat-groupoid* of the bisimplicial groupoid $\mathbb{G}$, is then given by

$$\wp\mathbb{G}=(\mathrm{C}\mathbb{G},\mathrm{C}_h\mathbb{G},\mathrm{C}_v\mathbb{G},s_h,t_h,s_v,t_v),$$

where $\mathrm{C}_h\mathbb{G},\mathrm{C}_v\mathbb{G}\subseteq\mathrm{C}\mathbb{G}$ are the wide subgroupoids with arrows

$$\mathrm{C}_h\mathbb{G}(x,y)=\big\{\big[s_0^h\alpha\big],\ \alpha\in\mathbb{G}_{0,1}(x,y)\big\},$$
$$\mathrm{C}_v\mathbb{G}(x,y)=\big\{\big[s_0^v\alpha\big],\ \alpha\in\mathbb{G}_{1,0}(x,y)\big\},\quad x,y\in\mathrm{Ob}\,\mathbb{G},$$

and the functors $s_h,t_h:\mathrm{C}\mathbb{G}\to\mathrm{C}_h\mathbb{G}$ and $s_v,t_v:\mathrm{C}\mathbb{G}\to\mathrm{C}_v\mathbb{G}$ are respectively defined by

$$s_h[\alpha]=\big[s_0^h d_0^h\alpha\big],\qquad t_h[\alpha]=\big[s_0^h d_1^h\alpha\big],$$
$$s_v[\alpha]=\big[s_0^v d_0^v\alpha\big]\quad\text{and}\quad t_v[\alpha]=\big[s_0^v d_1^v\alpha\big],$$

for any $\alpha\in\mathbb{G}_{1,1}$. Note that if $(a,b)\colon\alpha\approx\beta$, then $(s_1^h s_0^h d_i^h\alpha, s_0^h d_i^h b)\colon s_0^h d_i^h\alpha\approx s_0^h d_i^h\beta$, for $i=0,1$, and therefore s_h and t_h are well-defined. And similarly we see that both s_v and t_v are well defined.

The only difficulty in proving that the so-defined $\wp\mathbb{G}$ is actually a 2-cat-groupoid consists in verifying condition (2) in Definition 8.2. For let $\alpha,\beta\in\mathbb{G}_{1,1}(x)$ be automorphisms such that $s_h[\alpha]=[1_x]=t_h[\beta]$ (the discussion in the case in which $s_v[\alpha]=[1_x]=t_v[\beta]$ is parallel). Then, $s_h[\alpha^{-1}]=[1_x]=t_h[\beta^{-1}]$ also, and therefore $s_0^h d_0^h\alpha^{-1}\approx 1\approx s_0^h d_1^h\beta^{-1}$. Taking $\alpha'=\alpha\ s_0^h d_0^h\alpha^{-1}$ and $\beta'=\beta\ s_0^h d_1^h\beta^{-1}$, we deduce that $\alpha'\approx\alpha$ and $\beta'\approx\beta$. Since there is the bihomotopy

$$\big(\big(s_0^h\beta'\big)\big(s_1^h\alpha'\big)\big(s_0^h\beta'\big)^{-1}\big(s_1^h\beta'\big),s_1^v(\alpha'\beta')\big)\colon\ \beta'\alpha'\approx\alpha'\beta',$$

we conclude that $[\alpha][\beta]=[\alpha'][\beta']=[\alpha'\beta']=[\beta'\alpha']=[\beta'][\alpha']=[\beta][\alpha]$, as required.

The construction of $\wp\mathbb{G}$ is clearly functorial, with $\wp(f):\wp\mathbb{G}\to\wp\mathbb{G}'$, for $f:\mathbb{G}\to\mathbb{G}'$ being a bisimplicial groupoid morphism, the induced functor on the corresponding bihomotopy quotient groupoids by the functor $f_{1,1}:\mathbb{G}_{1,1}\to\mathbb{G}'_{1,1}$, that is, $\wp(f)[\alpha]=[f_{1,1}\alpha]$, $[\alpha]\in\mathrm{C}\mathbb{G}=\mathbb{G}_{1,1}/\approx$.

It is easy to see that, for any 2-cat-groupoid $\mathcal{C}$ as in (27), the bihomotopy relation on $\mathcal{N}_{1,1}\mathcal{C}=C$ is trivial and then that $\wp\mathcal{N}\mathcal{C}=\mathcal{C}$, that is,

$$\wp\mathcal{N}=\mathrm{id}_{\mathbf{2\text{-}Cat\text{-}Gd}}.$$

Moreover, there is a natural transformation

$$u:\mathrm{id}_{s\mathbf{Gd}^2}\to\mathcal{N}\wp \tag{29}$$

that takes a bisimplicial groupoid $\mathbb{G}$ to the bisimplicial groupoid morphism $u_{\mathbb{G}}:\mathbb{G}\to\mathcal{N}\wp\mathbb{G}$, which is the identity map on objects and on an arrow $\alpha\in\mathbb{G}_{p,q}$ is defined by

$$u_{\mathbb{G}}(\alpha)=\begin{cases}\big(\big[(d_2^h)^{p-j}(d_0^h)^{j-1}(d_2^v)^{q-i}(d_0^v)^{i-1}\alpha\big]\big)_{1\leqslant i\leqslant q,\ 1\leqslant j\leqslant p} & p,q\geqslant 1,\\ \big(\big[(d_2^h)^{p-j}(d_0^h)^{j-1}s_0^v\alpha\big]\big)_{1\leqslant j\leqslant p} & p\geqslant 1,\ q=0,\\ \big(\big[(d_2^v)^{q-i}(d_0^v)^{i-1}s_0^h\alpha\big]\big)_{1\leqslant i\leqslant q} & p=0,\ q\geqslant 1,\\ \big(\big[s_0^h s_0^v\alpha\big]\big) & p=0,\ q=0.\end{cases}$$

Since $\wp(u_{\mathbb{G}})=\mathrm{id}_{\wp\mathbb{G}}$, for any bisimplicial groupoid $\mathbb{G}$, and $u_{\mathcal{NC}}=\mathrm{id}_{\mathcal{NC}}$, for any 2-cat-groupoid $\mathcal{C}$, it follows that $\mathcal{N}$ is a right adjoint to $\wp$, with u and the identity being the unit and the counit of the adjunction respectively. In fact, the functor $\mathcal{N}:2\text{-}\mathbf{Cat}\text{-}\mathbf{Gd}\to s\mathbf{Gd}^2$ embeds the category of 2-cat-groupoids into the category of bisimplicial groupoids as a reflexive subcategory whose reflector functor is $\wp$.

Theorem 8.3 will be proved as a consequence of Propositions 8.4 and 8.5 below.

Proposition 8.4. *Let $\mathcal{C}=(C,C_h,C_v,s_h,t_h,s_v,t_v)$ be a 2-cat-groupoid. Then,*

$$\pi_0\mathcal{C}=\pi_0B\mathcal{C}$$

and, for any $x\in\mathrm{Ob}\,C=B_0\mathcal{C}$,

$$\pi_i(\mathcal{C},x)=\pi_i(B\mathcal{C},x),\quad i\geqslant 1.$$

In particular, $\pi_i(B\mathcal{C},x)=0$ for all $i\geqslant 4$ and all $x\in B_0\mathcal{C}$.

Proof. Recall that the normalized (o Moore) complex $\mathrm{N}G$ of a simplicial group G has the group

$$\mathrm{N}_mG=\bigcap_{i=0}^{m-1}\mathrm{Ker}(d_i:G_m\to G_{m-1})\ (=G_0\quad\text{if } m=0)$$

as m-chains, and has as a boundary $\partial:\mathrm{N}_mG\to\mathrm{N}_{m-1}G$ the induced homomorphism by $d_m:G_m\to G_{m-1}$ (0, if $m=0$). If $\mathbb{G}$ is a simplicial groupoid, then

- $\pi_0\overline{W}\mathbb{G}=$ the set of path components of $\mathbb{G}=$ the set of path components of $\mathbb{G}_{p,q}$ for any $p,q\geqslant 0$,
- $\pi_{i+1}(\overline{W}\mathbb{G},x)=H_i\mathrm{N}\mathbb{G}(x)$, for all $i\geqslant 0$ and $x\in\mathrm{Ob}\,\mathbb{G}=\overline{W}_0\mathbb{G}$.

Therefore, for a 2-cat-groupoid $\mathcal{C}$ as in the proposition, we have

$$\begin{aligned}\pi_0B\mathcal{C}&=\pi_0\overline{W}\,\overline{W}\mathcal{NC}\\&=\text{the set of components of the simplicial groupoid }\overline{W}\mathcal{NC}\\&=\text{the set of components of the bisimplicial groupoid }\mathcal{NC}\\&=\text{the set of components of the groupoid }\mathcal{N}_{1,1}\mathcal{C}\ (=C)\\&=\pi_0\mathcal{C}.\end{aligned}$$

Furthermore, for any object $x \in \mathrm{Ob}\,\mathcal{C}$ and $i \geqslant 0$,

$$\pi_{i+1}(B\mathcal{C}, x) = \pi_{i+1}\big(\overline{W}\,\overline{W}\mathcal{N}\mathcal{C}, x\big) = H_i\big(\mathrm{N}\overline{W}\mathcal{N}\mathcal{C}(x)\big) = H_i\big(\widetilde{\mathcal{C}}(x)\big) = \pi_{i+1}(\mathcal{C}, x),$$

since $\mathrm{N}\overline{W}\mathcal{N}\mathcal{C}(x) \cong \widetilde{\mathcal{C}}(x)$. □

Proposition 8.5. *Suppose that X is a simplicial set. Then the unit of the adjunction $u' : X \to BPX$ induces a bijection*

$$\pi_0 X \cong \pi_0 BPX$$

and, for any vertex $x \in X_0$, isomorphisms

$$\pi_i(X, x) \cong \pi_i(BPX, x), \quad i = 1, 2, 3.$$

Proof. By [10, Theorem 3.3] and Proposition 7.1, both natural maps $X \to \overline{W}GX$ and $GX \to \overline{W}\,\mathrm{Dec}\,GX$ are weak homotopy equivalences. Hence it suffices to prove that the unit bisimplicial groupoid map $u : \mathrm{Dec}\,GX \to \mathcal{N}\wp\,\mathrm{Dec}\,GX$ induces bijections

$$\pi_0\overline{W}\,\overline{W}\,\mathrm{Dec}\,GX \cong \pi_0\overline{W}\,\overline{W}\mathcal{N}\wp\,\mathrm{Dec}\,GX, \tag{30}$$

$$\pi_i(\overline{W}\,\overline{W}\,\mathrm{Dec}\,GX, x) \cong \pi_i(\overline{W}\,\overline{W}\mathcal{N}\wp\,\mathrm{Dec}\,GX, x), \quad i = 1, 2, 3,\ x \in X_0. \tag{31}$$

The bijection (30) is easy to prove. Note that the set $\pi_0\overline{W}\,\overline{W}\,\mathrm{Dec}\,GX$ coincides with the set of components of the groupoid $\mathrm{Dec}_{1,1}\,GX = G_3X$ while $\pi_0\overline{W}\,\overline{W}\mathcal{N}\wp\,\mathrm{Dec}\,GX$ coincides with the set of components of the groupoid $\mathcal{N}_{1,1}\wp\,\mathrm{Dec}\,GX = G_3X/_{\approx}$. Since the projection functor to the quotient groupoid $u_{1,1} : G_3X \to G_3X/_{\approx}$ induces a bijection on the corresponding sets of path components, the bijection (30) follows.

The proof of the isomorphisms (31) is rather hard and some preliminary work is needed. To begin with, hereafter we will write $\mathbb{G}$ to denote the simplicial groupoid GX and $\mathcal{C}$ to denote the 2-cat-groupoid $\wp\,\mathrm{Dec}\,\mathbb{G}$. Thus, $\mathcal{C} = (C, C_h, C_v, s_h, t_h, s_v, t_v)$, where $C = \mathbb{G}_3/_{\approx}$ is the groupoid with the same objects as $\mathbb{G}$ whose arrows $[\alpha] \in C(x, y)$ are bihomotopy classes of arrows $\alpha \in \mathbb{G}(x, y)$. So that, for $\alpha, \beta \in \mathbb{G}_3$, we have $[\alpha] = [\beta]$ whenever there is a pair $(a, b) \in \mathbb{G}_4 \times \mathbb{G}_4$ such that $(a, b) : \alpha \approx \beta$, which means that the equalities below hold.

$$d_0a = s_0d_0\alpha, \qquad d_2b = s_2d_2\beta, \qquad d_1a = \alpha, \qquad d_3b = \beta, \qquad d_2a = d_4b.$$

Furthermore, the wide subgroupoids $C_h, C_v \subseteq C$ have arrows

$$C_h(x, y) = \big\{[s_0\sigma],\ \sigma \in \mathbb{G}_2(x, y)\big\},$$

$$C_v(x, y) = \big\{[s_2\sigma],\ \sigma \in \mathbb{G}_2(x, y)\big\},$$

and the functors $s_h, t_h : C \to C_h$ and $s_v, t_v : C \to C_v$ are respectively given by

$$s_h[\alpha] = [s_0d_0\alpha], \qquad t_h[\alpha] = [s_0d_1\alpha], \qquad s_v[\alpha] = [s_2d_2\alpha], \qquad t_v[\alpha] = [s_2d_3\alpha].$$

Next we shall show a handy description of the group chain complexes $\widetilde{\mathcal{C}}(x) = \mathrm{N}\overline{W}\mathcal{N}\mathcal{C}(x)$ (see (28)), $x \in X_0$, and, to do that, we first observe the existence of the following identity on object groupoid isomorphisms:

(a) $C_h \cap C_v \cong \mathbb{G}_1$, by the functor $\phi : [\alpha] \mapsto d_1 d_0 \alpha$.
(b) $C_h \cong \mathbb{G}_1 \times_{d_1} \mathbb{G}_1$, by the functor $\phi_h : [\alpha] \mapsto (d_1 d_0 \alpha, d_2 d_0 \alpha)$. Here, we are writing $\mathbb{G}_1 \times_{d_1} \mathbb{G}_1$ for the pullback groupoid of $\mathbb{G}_1 \xrightarrow{d_1} \mathbb{G}_0 \xleftarrow{d_1} \mathbb{G}_1$.
(c) $C_v \cong \mathbb{G}_1 \times_{d_0} \mathbb{G}_1$, by the functor $\phi_v : [\alpha] \mapsto (d_0 d_2 \alpha, d_1 d_2 \alpha)$.
(d) $C \cong \mathbb{G}_2 \times_{d_2} \mathbb{G}_2/\sim$, by the functor $\Phi : [\alpha] \mapsto [d_2\alpha, d_3\alpha]$. Here, $\mathbb{G}_2 \times_{d_2} \mathbb{G}_2/\sim$ denotes the quotient of the pullback groupoid $\mathbb{G}_2 \times_{d_2} \mathbb{G}_2$ by the congruence $\sim$ defined on it by $(\sigma, \tau) \sim (\sigma', \tau')$ if and only if there exists a pair of arrows $(\alpha, \beta) \in \mathbb{G}_3 \times_{d_3} \mathbb{G}_3$ satisfying $d_0\alpha = s_0 d_0 \sigma,\ d_0 \beta = s_0 d_0 \tau,\ d_1 \alpha = \sigma,\ d_1 \beta = \tau,\ d_2 \alpha = \sigma'$ and $d_2 \beta = \tau'$. In such a case we write $(\alpha, \beta)\colon (\sigma, \tau) \sim (\sigma', \tau')$ and also $[\sigma, \tau] = [\sigma', \tau']$.

Proof of isomorphism (a). It is easy to see that ϕ is a well defined functor. That ϕ is injective, follows from the fact that for any $[\alpha] \in C_h \cap C_v$, the equality $[\alpha] = [s_0 s_1 d_1 d_0 \alpha]$ holds. In effect, if $[\alpha] \in C_h \cap C_v$, then we can write $[\alpha] = [s_0 \sigma] = [s_2 \tau]$ for certain $\sigma, \tau \in \mathbb{G}_2$. By choosing $(a, b)\colon s_0 \sigma \approx s_2 \tau$, any bihomotopy, we obtain $(s_1 s_0 \sigma, s_0 d_0 b)\colon s_0 \sigma \approx s_0 d_0 s_2 \tau$ and therefore $[\alpha] = [s_0 s_1 d_0 \tau] = [s_0 s_1 d_1 d_0 \alpha]$. Since, for any $w \in \mathbb{G}_1$, we have $\phi[s_0 s_1 w] = w$, the functor ϕ is surjective.

Proof of isomorphism (b). That ϕ_h is well defined is easily seen. To prove that ϕ_h is injective, let us suppose that $\phi_h[s_0\sigma] = \phi_h[s_0 \tau]$, for certain $\sigma, \tau \in \mathbb{G}_2$. Then, $d_1 \sigma = d_1 \tau$ and $d_2 \sigma = d_2 \tau$ and a straightforward verification shows that we have the bihomotopy

$$\big(s_1 s_0 \sigma, s_0\big((s_2\sigma)(s_1\sigma)^{-1}(s_1\tau)(s_0\tau)^{-1}(s_0\sigma)\big)\big)\colon s_0\sigma \approx s_0 \tau.$$

Therefore, $[s_0\sigma] = [s_0\tau]$ and ϕ_h is injective. The functor ϕ_h is surjective, since for any $(v, w) \in \mathbb{G}_1 \times_{d_1} \mathbb{G}_1$ we have $\phi_h[s_0((s_1 w)(s_0 w)^{-1}(s_0 v))] = (v, w)$.

Proof of isomorphism (c). This is parallel to the given proof of the isomorphism (b).

Proof of isomorphism (d). The functor Φ is well defined, since, if $(a, b)\colon \alpha \approx \beta$, then $(d_3 a, d_4 a)\colon (d_2\alpha, d_3\alpha) \sim (d_2\beta, d_3\beta)$. That Φ is injective follows from the fact that if $(\gamma, \delta)\colon (d_2\alpha, d_3\alpha) \sim (d_2\beta, d_3\beta)$, for certain $\alpha, \beta \in \mathbb{G}_3$, then there is a bihomotopy $(a, b)\colon \alpha \approx \beta$, defined by

$$\begin{aligned}
a &= (s_1\alpha)(s_1 s_1 d_2 \alpha)^{-1}(s_1 s_1 d_3 \alpha)(s_1 s_2 d_3 \alpha)^{-1}(s_3\delta)(s_2\delta)^{-1}(s_2\gamma),\\
b &= (s_3\alpha)(s_1 s_2 d_2 \alpha)^{-1}(s_1\alpha)(s_2\alpha)^{-1}(s_2\beta)(s_1\beta)^{-1}(s_1 s_2 d_3 \beta)(s_3\delta)^{-1}\\
&\quad (s_3\gamma)(s_2\gamma)^{-1}(s_2\delta)(s_1\delta)^{-1}(s_1\gamma).
\end{aligned}$$

Finally, we see that Φ is surjective, since for any $[\sigma, \tau] \in \mathbb{G}_2 \times_{d_2} \mathbb{G}_2/\sim$ we have $\Phi[(s_2\tau)(s_1\tau)^{-1}(s_1\sigma)] = [\sigma, \tau]$.

We now return to the proof of isomorphisms (31). Let $x \in X_0$ be any fixed vertex of the simplicial set X. Then, by taking into account the above isomorphisms (a)–(d), one easily checks that the group chain complex $\widetilde{\mathcal{C}}(x) = \mathrm{N}\overline{W}\mathcal{N}\mathcal{C}(x)$ is described as

$$\widetilde{\mathcal{C}}(x) = \cdots \to 0 \to \cdots \to 0 \to \widetilde{\mathcal{C}}_2(x) \xrightarrow{\tilde{\partial}_2} \widetilde{\mathcal{C}}_1(x) \xrightarrow{\tilde{\partial}_1} \widetilde{\mathcal{C}}_0(x) \to 0,$$

where

- $\widetilde{\mathcal{C}}_0(x) = \mathbb{G}_1(x)$,
- $\widetilde{\mathcal{C}}_1(x) = \{(v, w) \in \mathbb{G}_1(x) \times_{d_1} \mathbb{G}_1(x) \mid d_0 v = 1_x\}$,
- $\widetilde{\mathcal{C}}_2(x) = \{[\sigma, \tau] \in \mathbb{G}_2(x) \times_{d_2} \mathbb{G}_2(x)/_\sim \mid d_0\sigma = 1_x = d_1\sigma = d_0\tau\}$,
- $\tilde{\partial}_1(v, w) = w$, $\tilde{\partial}_2[\sigma, \tau] = (d_1\tau, 1_x)$.

Thus, $\pi_i(\overline{W}\,\overline{W}\mathcal{N}\wp\,\mathrm{Dec}\,\mathbb{G}, x) = H_{i-1}\widetilde{\mathcal{C}}(x)$, for $i \geqslant 1$.

On the other hand, if we write $\mathrm{N}(x) = \mathrm{N}\overline{W}\,\mathrm{Dec}\,\mathbb{G}(x)$, then

$$\mathrm{N}(x) = \cdots \to \mathrm{N}_3(x) \xrightarrow{\partial_3} \mathrm{N}_2(x) \xrightarrow{\partial_2} \mathrm{N}_1(x) \xrightarrow{\partial_1} \mathrm{N}_0(x) \to 0,$$

where

- $\mathrm{N}_0(x) = \mathbb{G}_1(x)$,
- $\mathrm{N}_1(x) = \{(\sigma, \tau) \in \mathbb{G}_2(x) \times_{d_1} \mathbb{G}_2(x) \mid d_0\tau = 1_x\}$,
- $\mathrm{N}_2(x) = \{(\alpha, \beta, \gamma) \in \mathbb{G}_3(x) \times_{d_1} \mathbb{G}_3(x) \times_{d_2} \mathbb{G}_3(x) \mid d_0\beta = 1_x = d_0\gamma = d_2\alpha = d_1\gamma\}$,
- $\mathrm{N}_3(x) = \{(a, b, c, d) \in \mathbb{G}_4(x) \times_{d_1} \mathbb{G}_4(x) \times_{d_2} \mathbb{G}_4(x) \times_{d_3} \mathbb{G}_4(x) \mid d_0 b = 1_x = d_0 c = d_0 d = d_2 a = d_1 c = d_1 d = d_3 a = d_3 b = d_2 d\}$,
- $\partial_1(\sigma, \tau) = d_2\sigma$, $\partial_2(\alpha, \beta, \gamma) = (d_3\alpha, d_3\beta)$, $\partial_3(a, b, c, d) = (d_4 a, d_4 b, d_4 c)$.

Thus, $\pi_i(\overline{W}\,\overline{W}\,\mathrm{Dec}\,\mathbb{G}, x) = H_{i-1}\mathrm{N}(x)$, for $i \geqslant 1$.

Since the unit bisimplicial groupoid map $u : \mathrm{Dec}\,\mathbb{G} \to \mathcal{N}\wp\,\mathrm{Dec}\,\mathbb{G}$ induces the chain complex map

$$\begin{array}{ccccccccccc}
\mathrm{N}(x)\cdots & \longrightarrow & \mathrm{N}_3(x) & \xrightarrow{\partial_3} & \mathrm{N}_2(x) & \xrightarrow{\partial_2} & \mathrm{N}_1(x) & \xrightarrow{\partial_1} & \mathrm{N}_0(x) & \longrightarrow & 0 \\
\downarrow{\scriptstyle u} & & \downarrow & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \\
\widetilde{\mathcal{C}}(x)\cdots & \longrightarrow & 0 & \longrightarrow & \widetilde{\mathcal{C}}_2(x) & \xrightarrow{\tilde{\partial}_2} & \widetilde{\mathcal{C}}_1(x) & \xrightarrow{\tilde{\partial}_1} & \widetilde{\mathcal{C}}_0(x) & \longrightarrow & 0
\end{array}$$

given by

$$u_0(w) = w, \qquad u_1(\sigma, \tau) = (d_1\sigma, d_2\sigma), \qquad u_2(\alpha, \beta, \gamma) = [d_2\beta, d_3\beta],$$

we are now ready to complete the proof of the proposition, that is, to prove that the homology maps $u_* : H_i\mathrm{N}(x) \to H_i\widetilde{\mathcal{C}}(x)$ are isomorphisms for $0 \leqslant i \leqslant 2$. The proof is naturally divided into five parts.

(1) $u_* : H_0\mathrm{N}(x) \to H_0\widetilde{\mathcal{C}}(x)$ *is an isomorphism.* In effect, note that the homomorphism $u_1 : \mathrm{N}_1(x) \to \widetilde{\mathcal{C}}_1(x)$ is surjective, since, given any $(v, w) \in \widetilde{\mathcal{C}}_1(x)$, we have $((s_1 w)(s_0 w)^{-1}(s_0 v), s_1 v) \in \mathrm{N}_1(x)$ and $u_1((s_1 w)(s_0 w)^{-1}(s_0 v), s_1 v) = (v, w)$. Therefore, the result follows since $u_0 = \mathrm{id}_{\mathbb{G}_1(x)}$.

(2) $u_* : H_1\mathrm{N}(x) \to H_1\widetilde{\mathcal{C}}(x)$ *is surjective.* Any element in $\mathrm{Ker}\,\tilde{\partial}_1$ is of the form $(v, 1_x)$, for some $v \in \mathbb{G}_1(x)$ such that $d_0 v = 1_x = d_1 v$. For any such $(v, 1_x)$, we see that the element $(s_0 v, s_1 v)$ belongs to $\mathrm{Ker}\,\partial_1$ and it satisfies that $u_1(s_0 v, s_1 v) = (v, 1_x)$. Thus, the restricted homomorphism $u_1 : \mathrm{Ker}\,\partial_1 \to \mathrm{Ker}\,\tilde{\partial}_1$ is surjective, whence the induced homomorphism in homology is also surjective.

(3) $u_* : H_1\mathrm{N}(x) \to H_1\widetilde{\mathcal{C}}(x)$ *is injective.* If we suppose that $(\sigma, \tau) \in \mathrm{Ker}\,\partial_1$ is an element such that $u_1(\sigma, \tau) = \tilde{\partial}_2[\sigma', \tau']$, for some $[\sigma', \tau'] \in \widetilde{\mathcal{C}}_2(x)$, then we have $(\sigma, \tau) = \partial_2(\alpha, \beta, \gamma)$, where $(\alpha, \beta, \gamma) \in \mathrm{N}_2(x)$ is defined by

$$\alpha = (s_2\sigma)(s_1\sigma)^{-1}(s_0\sigma)(s_0\tau')^{-1}(s_0\sigma'),$$
$$\beta = (s_2\tau)(s_1\tau')^{-1}(s_1\sigma'),$$
$$\gamma = (s_2\tau)(s_2\tau')^{-1}(s_2\sigma').$$

(4) $u_*: H_2\mathrm{N}(x) \to H_2\widetilde{\mathcal{C}}(x)$ *is surjective*. Let $[\sigma, \tau] \in \operatorname{Ker}\tilde{\partial}_2 = H_2\widetilde{\mathcal{C}}(x)$ be any element. The triple $((s_0\sigma)(s_0\tau)^{-1}, (s_1\sigma)(s_1\tau)^{-1}, (s_2\sigma)(s_2\tau)^{-1})$ represents an element of the group $\operatorname{Ker}\partial_2$ and it is mapped by u_2 to the element $[\sigma\tau^{-1}, 1_x]$ of $H_2\widetilde{\mathcal{C}}(x)$. Then, the result follows since $[\sigma\tau^{-1}, 1_x] = [\sigma, \tau]$. Indeed, we have $((s_1\sigma)(s_1\tau)^{-1}(s_2\tau), s_2\tau)$: $(\sigma\tau^{-1}, 1_x) \sim (\sigma, \tau)$.

(5) $u_*: H_2\mathrm{N}(x) \to H_2\widetilde{\mathcal{C}}(x)$ *is injective*. Let $[\alpha, \beta, \gamma]$ denote the homology class in $H_2\mathrm{N}(x)$ of an element $(\alpha, \beta, \gamma) \in \operatorname{Ker}\partial_2$, and suppose that $[\alpha, \beta, \gamma] \in \operatorname{Ker} u_*$. Then, observe that $(\alpha, \beta, \gamma) = \partial_3(a, b, c, d)(s_0d_1\alpha, s_1d_1\alpha, s_2d_1\alpha)$, where

$$a = (s_3\alpha)(s_3s_0d_1\alpha)^{-1}(s_2s_0d_1\alpha)(s_2\alpha)^{-1}(s_1\alpha)(s_0\alpha)^{-1},$$
$$b = (s_3\beta)(s_3s_1d_1\alpha)^{-1}(s_2s_1d_1\alpha)(s_2\beta)^{-1},$$
$$c = (s_3\gamma)(s_3s_2d_1\alpha)^{-1}(s_2s_1d_1\alpha)(s_2\beta)^{-1},$$
$$d = (s_3\gamma)(s_3s_2d_1\alpha)^{-1}(s_3s_1d_1\alpha)(s_3\beta)^{-1},$$

and thus that the equality $[\alpha, \beta, \gamma] = [s_0d_1\alpha, s_1d_1\alpha, s_2d_1\alpha]$ holds. Now, this element belongs to the kernel of $u_*: H_2\mathrm{N}(x) \to H_2\widetilde{\mathcal{C}}(x)$, that is, $[d_1\alpha, 1_x] = [1_x, 1_x]$, which means that there is $(\alpha', \beta') \in \mathbb{G}_3 \times_{d_3} \mathbb{G}_3$ such that (α', β'): $(1_x, 1_x) \sim (d_1\alpha, 1_x)$. But then, the equality

$$(s_0d_1\alpha, s_1d_1\alpha, s_2d_1\alpha) = \partial_3(a', b', c', d'),$$

where

$$a' = (s_0s_2d_1\alpha)(s_0\alpha')^{-1}(s_0\beta'), \qquad b' = (s_1s_2d_1\alpha)(s_1\alpha')^{-1}(s_1\beta'),$$
$$c' = (s_2s_2d_1\alpha)(s_2\alpha')^{-1}(s_2\beta'), \qquad d' = (s_3s_2d_1\alpha)(s_3\alpha')^{-1}(s_3\beta'),$$

shows that $[\alpha, \beta, \gamma]$ is the zero homology class.

Hence the proof of Proposition 8.5 is complete. □

We are now ready to prove the main result in this section:

Proof of Theorem 8.3. From Proposition 8.4, it follows that a 2-cat-groupoid morphism $f: \mathcal{C} \to \mathcal{C}'$ is a weak equivalence if and only if the induced simplicial map $B(f): B\mathcal{C} \to B\mathcal{C}'$ is a weak equivalence. If X is any simplicial set such that $\pi_i(X, x) = 0$ for all $i \geqslant 4$ and all base point $x \in X_0$, then Proposition 8.5 implies that the unit of the adjunction $u'_X: X \to BPX$ is a weak equivalence. Furthermore, for any 2-cat-groupoid $\mathcal{C}$, the counit $v'_{\mathcal{C}}: PB\mathcal{C} \to \mathcal{C}$ is a weak equivalence of 2-cat-groupoids, from the adjunction equality $B(v'_{\mathcal{C}})\, u'_{B\mathcal{C}} = \mathrm{id}_{B\mathcal{C}}$. Finally, suppose that $f: X \to Y$ is any simplicial map between simplicial sets X, Y such that $\pi_i(X, x) = 0 = \pi_i(Y, y)$ for all $i \geqslant 4$ and base points $x \in X_0$, $y \in Y_0$. Then, from the natural commutativity $BP(f)u'_X = u'_Y\, f$, it follows that f is a weak equivalence if and only if the induced $P(f): PX \to PY$ is a weak equivalence of 2-cat-groupoids. □

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