MR3443424 34-02 34B05 34C10 34L15 35J20 35J61 35J62 58E50
Cañada, Antonio (E-GRAN-MA); **Villegas, Salvador** (E-GRAN-MA)
★**A variational approach to Lyapunov type inequalities.**
From ODEs to PDEs.
With a foreword by Jean Mawhin.
SpringerBriefs in Mathematics.
Springer, Cham, 2015. *xviii*+120 *pp. ISBN* 978-3-319-25287-2; 978-3-319-25289-6

More than one hundred years ago, in his doctoral thesis [*The general problem of the stability of motion* (Russian), Univ. Moscow, 1892], A. M. Lyapunov established a famous stability criterion for Hill's equation

$$x'' + a(t)x = 0. \tag{1}$$

It asserts that if $a(t)$ is a continuous L-periodic function satisfying

$$a(t) \ge 0\ \forall t \quad \text{and} \quad \int_0^L a(t)\,dt \le \frac{4}{L}, \tag{2}$$

then all solutions $x(t)$ of Equation (1) are bounded on $\mathbb{R}$, together with their derivatives $x'(t)$. Moreover, the constant $4/L$ in (2) is optimal in a certain sense in order to guarantee the stability of Equation (1). Nowadays (2) is called the *Lyapunov inequality*.

Though originally motivated by the stability problem, the Lyapunov inequality and its extensions are related with many problems for linear or nonlinear, ordinary or partial differential equations, like oscillation, disconjugacy, disfocality, eigenvalues, nonresonance, isoperimetric inequalities and solvability of nonlinear boundary value problems from the point of view of mathematics, and crystallography and inverse problems from the point of view of applied sciences. Because of certain progress achieved, the study of Lyapunov type inequalities is still a hot topic at present. The extensions mainly involve the following ingredients: (i) the types of differential equations and systems, linear or nonlinear, ordinary or partial, (ii) the types of boundary conditions, and (iii) the types of potentials or weights in equations.

The monograph under review gives a unified and elegant treatment of this interesting topic, mainly based on the use of a variational method. It is also a relatively systematic summary of the authors' research works over the last decade. The treatment of Lyapunov type inequalities related with higher eigenvalues, partial differential equations and systems of equations is quite inspired.

The monograph contains five chapters. In Chapter 1, the authors give a brief introduction by explaining the main idea of their approaches. In Chapter 2, the second-order scalar linear ordinary differential equations, with the Neumann, Dirichlet, periodic or antiperiodic boundary conditions, are considered. For example, for the Neumann condition

$$x'(0) = x'(L) = 0, \tag{3}$$

it is proved that the optimal constant β_p in the L^p Lyapunov inequality is characterized as

$$\beta_p = \inf_{u \in X_p \setminus \{0\}} \left\{ \frac{1}{\|u\|^2_{p/(p-1)}} \int_0^L u'^2(t)\,dt \right\} \tag{4}$$

for any exponent $p \in [1, \infty]$. Here $\|\cdot\|_q$ is the L^q norm on $[0, L]$ and the set X_p is

$$X_\infty = \left\{u \in H^1(0, L) : \int_0^L u = 0\right\},$$

$$X_1 = \left\{u \in H^1(0, L) : \min_{x\in[0,L]} u(x) + \max_{x\in[0,L]} u(x) = 0\right\},$$

$$X_p = \left\{u \in H^1(0, L) : \int_0^L |u|^{2/(p-1)} u = 0\right\} \quad \text{for } 1 < p < \infty.$$

In particular, after finding $\beta_1 = 4/L$ from the minimization problem (4) with $p = 1$, one recovers the classical Lyapunov inequality (2).

In Chapter 3, the Lyapunov type inequalities for equation (1) at higher eigenvalues are discussed. For example, with the Neumann condition (3) again, the classical L^∞-L^∞ type Lyapunov inequality is

$$(n\pi/L)^2 =: \lambda_n \prec a \prec \lambda_{n+1}. \tag{5}$$

Here n may be any non-negative integer. It is then shown that the optimal Lyapunov constant in the L^∞-L^1 version of (5) can be characterized as

$$\beta_{1,n} = \inf_{a\in\Lambda_n} \|a\|_1,$$

$$\Lambda_n = \left\{a \in L^1(0, L) : \lambda_n \prec a, \text{ problem (1)–(3) has a non-zero solution}\right\},$$

and it is explicitly given by

$$\beta_{1,n} = L\lambda_n + \frac{2\pi n(n+1)}{L} \cot \frac{\pi n}{2(n+1)}.$$

In Chapter 4, the L^p Lyapunov inequalities for partial differential equations

$$\begin{cases} -\Delta u = a(x)u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

are studied. Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded and regular domain. The corresponding optimal Lyapunov constant is characterized as

$$\beta_p = \inf_{a\in\Lambda\cap L^p(\Omega)} \|a^+\|_p, \quad 1 \leq p \leq \infty,$$

$$\Lambda = \left\{a \in L^{N/2}(\Omega) \setminus \{0\} : \int_\Omega a \geq 0, \text{ problem (6) has a non-zero solution}\right\}$$

if $N \geq 3$, and

$$\Lambda = \Big\{a \in L^q(\Omega) \setminus \{0\} \text{ for some } q \in (1, \infty] : \int_\Omega a \geq 0, \text{ problem (6) has a non-zero solution}\Big\}$$

if $N = 2$. It has been shown that $\beta_p > 0$ if and only if $1 < p \leq \infty$ for $N = 2$, and $N/2 \leq p \leq \infty$ for $N \geq 3$. These results are compatible with those deduced from the Sobolev imbedding theorem. The corresponding Lyapunov constants resulting from the radial higher eigenvalues are also characterized.

In Chapter 5, the extensions of Lyapunov inequalities to systems of ordinary and partial differential equations are studied. The optimal Lyapunov constants are also characterized using the variational method.

The monograph is self-contained and well written. In each chapter, some main references, most of which are up-to-date, are given. The monograph will be a useful reference for mathematicians and mathematical physicians working on differential equations, eigenvalue problems, the variational method, optimal control, etc. *Meirong Zhang*