TEOREMA DE GREEN Y TEOREMA DE LA DIVERGENCIA

HIPÓTESIS $\gamma : [a,b] \to \mathbb{R}^2$, cerrada, simple, regular a trozos
$\Gamma = \gamma[a,b]$, curva de Jordan, regular a trozos
Ω abierto de $\mathbb{R}^2$, $\Omega \supset \Gamma \cup D$, D "región interior a Γ", $P, Q \in C^1(\Omega, \mathbb{R})$

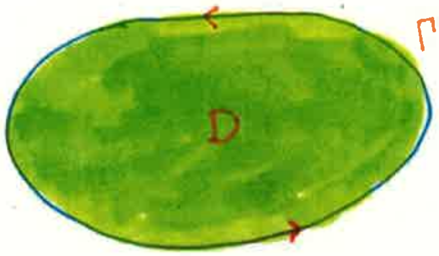
NOTACIÓN CLÁSICA

T. Green

$$\int_{(\partial D)^+} (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

T. Divergencia

$$\int_{(\partial D)^+} (-Q\,dx + P\,dy) = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy$$



NOTACIÓN MODERNA

$F : \Omega \to \mathbb{R}^2$, $\gamma : [a,b] \to \mathbb{R}^2$,
$(x,y) \to (P(x,y), Q(x,y))$ $\quad \gamma(t) = (x(t), y(t))$

T. Green

$$\int_a^b \langle F(\gamma(t)), \gamma'(t)\rangle\,dt = \int_{\gamma^+ = (\partial D)^+} F \cdot d\gamma = \iint_D (\text{rot}\,F)(x,y)\,dx\,dy$$

$\gamma'(t) = (x'(t), y'(t))$

T. Divergencia

$$\int_a^b \langle F(\gamma(t)), N(t)\rangle\,dt = \int_{\gamma^+ = (\partial D)^+} F \cdot N = \iint_D (\text{Div}\,F)(x,y)\,dx\,dy$$

$N(t) = (y'(t), -x'(t))$

A. Cañada, 31/10/2018

P. Valor medio para funciones armónicas. $(n=2)$ A. Cañada (13/5/2019) ①

$P=(p_1,p_2)\in\mathbb{R}^2$

$R>0 \qquad B_{\mathbb{R}^2}(P;R)\equiv\Omega$

$u\in C^2(\Omega)\cap C(\bar\Omega) \qquad \Delta u(x_1,x_2)=0$ en Ω

Sea $r>0,\ r<R$. Sabemos que $\Delta u=\operatorname{div}(\nabla u)$

Aplicamos el T. Divergencia a $D=B_{\mathbb{R}^2}(P;r)$

$$\int_{(\partial D)^+} F\cdot N=\int_D \operatorname{div}F \qquad F=(u_{x_1},u_{x_2})$$

↓ o bien

$$\int_{(\partial D)^+}(-Q\,dx+P\,dy)=\int_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy$$

Entonces $\operatorname{div}F(x_1,x_2)=\Delta u(x_1,x_2)=0 \Longrightarrow$

$$\int_{(\partial D)^+}F\cdot N=0 \qquad (\partial D)^+\begin{cases}x_1=p_1+r\cos\theta\\ x_2=p_2+r\operatorname{sen}\theta\end{cases} \quad 0\le\theta\le 2\pi$$

$$N=(\overset{x_2'(\theta)}{r\cos\theta},\ \overset{-x_1'(\theta)}{r\operatorname{sen}\theta})$$

$$\Rightarrow \int_{(\partial D)^+}F\cdot N=\int_{\theta=0}^{2\pi}u_{x_1}(p_1+r\cos\theta,p_2+r\operatorname{sen}\theta)\,r\cos\theta+$$
$$+u_{x_2}(p_1+r\cos\theta,p_2+r\operatorname{sen}\theta)\,r\operatorname{sen}\theta\,d\theta$$

$$=r\int_{\theta=0}^{2\pi}\overset{\partial/\partial r}{\frac{d}{dr}}\left[u(p_1+r\cos\theta,p_2+r\operatorname{sen}\theta)\right]d\theta=r\int^{2\pi}\frac{d}{dr}[u(\ \ )]\,d\theta=0$$

$$\Rightarrow\int_{r=0}^{R}\left[\int_{\theta=0}^{2\pi}\frac{d}{dr}\left[u(p_1+r\cos\theta,p_2+r\operatorname{sen}\theta)\right]d\theta\right]dr=0 \ =$$

$$=\int_{\theta=0}^{2\pi}\left(\int_{r=0}^{R}\frac{d}{dr}\left[u(p_1+r\cos\theta,p_2+r\operatorname{sen}\theta)\right]dr\right)d\theta \ =$$

$$= \int_{\theta=0}^{2\pi} \left(u(p_1 + R\cos\theta, p_2 + R\sin\theta) - u(p_1, p_2) \right) d\theta \Rightarrow$$

$$u(p_1,p_2) = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} u(p_1+R\cos\theta, p_2+R\sin\theta)\, d\theta}_{\substack{\| \\ \int_{\|P-Z\|=R} u(s)\,ds}} = \frac{1}{2\pi R} \int_{\|P-Z\|=R} u(s)\,ds$$

I. línea

Recordemos

$$\int_{\|P-Z\|=R} u(s)\,ds = \int_0^{2\pi} u(p_1+R\cos\theta, p_2+R\sin\theta)\, R\, d\theta$$

$\|(x_1'(\theta), x_2'(\theta))\|$

I. línea de un campo escalar $ds = \|(x_1', x_2')\|$

$$= \|(R(-\sin\theta), R\cos\theta)\| = R$$

$$\Rightarrow u(p_1,p_2) = \frac{1}{2\pi R} \int_{\|P-Z\|=R} u(s)\,ds \quad (*)$$

(P.V. Medio para esferas)
(Circunferencias $n=2$)

$\forall r \in (0,R)$

$$2\pi r\, u(p_1,p_2) = \int_{\|P-Z\|=r} u(s)\,ds \Rightarrow$$

$$\int_0^R 2\pi r\, u(p_1,p_2)\, dr = \int_{r=0}^R \left(\int_{\|P-Z\|=r} u(s)\,ds \right) dr \Rightarrow$$

$$\cancel{2}\pi \frac{R^2}{\cancel{2}} u(p_1,p_2) = \int_{\|P-Z\|\le R} u(x)\,dx \Rightarrow$$

$$u(p_1,p_2) = \frac{1}{\pi R^2} \int_{\|P-Z\|\le R} u(x)\,dx$$

c.q.d.

(P.V. Medio para bolas (círculos $n=2$)

Hemos deducido

P. valor medio para esferas

'' $\Rightarrow$ P. V. medio para bolas

Ahora bien, PVM para bola $\Rightarrow$ P.V. medio para esferas:

$$u(p_1, p_2) = \frac{1}{\pi r^2} \int_{\|P-\bar{z}\| \leq r} u(x)\,dx \Rightarrow$$

$$\pi r^2 u(p_1, p_2) = \int_{\|P-\bar{z}\| \leq r} u(x)\,dx = \int_{t=0}^{r} \left[\int_{\|P-\bar{z}\| = t} u(s)\,ds \right] dt$$

$$= \int_{t=0}^{r} v(t)\,dt, \quad \text{donde}$$

$$v(t) = \int_{\|P-\bar{z}\| = t} u(s)\,ds \qquad v \text{ es continua en } t$$

Aplicando el T.F. Cálculo, ~~tenem~~os

$$2\pi r\, u(p_1, p_2) = v(r) = \int_{\|P-\bar{z}\| = r} u(s)\,ds$$

$$\Rightarrow u(p_1, p_2) = \frac{1}{2\pi r} \int_{\|P-\bar{z}\| = r} u(s)\,ds, \quad \text{c.q.d.}$$