A. Cañada (Análisis Funcional 2014/2015) (1)

$(E, \|\cdot\|)$ t.q. se verifica la I. Paralelogramo

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Entonces $\|\cdot\|$ deriva del producto escalar

$$\langle x, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right] \quad (*)$$

Demostración.

a) $\langle y, x\rangle = \frac{1}{4}\left[\|y+x\|^2 - \|y-x\|^2\right] = \langle x, y\rangle$

b) $\langle x, x\rangle = \frac{1}{4}\|2x\|^2 = \|x\|^2 \geq 0$, $\langle x, x\rangle = 0 \iff x = 0$.

c) ¿Es $\langle \cdot, \cdot \rangle$ lineal respecto de la 1ª variable?

Pasos

c.1) $\langle x, y\rangle + \langle z, y\rangle = \frac{1}{2}\langle x+z, 2y\rangle$, pues

$$\langle x, y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$$

$$\langle z, y\rangle = \frac{1}{4}\|z+y\|^2 - \frac{1}{4}\|z-y\|^2$$

$$\frac{1}{2}\langle x+z, 2y\rangle = \frac{1}{2}\frac{1}{4}\left[\|x+z+2y\|^2 - \|x+z-2y\|^2\right]$$

pues

$$\|x+z+2y\|^2 = \|\underbrace{x+y}_{u} + \underbrace{z+y}_{v}\|^2 \overset{\text{Paral.}}{=} 2\|x+y\|^2 + 2\|z+y\|^2 - \|x-z\|^2$$

$$\|x+z-2y\|^2 = \|\underbrace{x-y}_{u} + \underbrace{(z-y)}_{v}\|^2 = 2\|x-y\|^2 + 2\|z-y\|^2 - \|x-z\|^2$$

Así

$$\frac{1}{2}\langle x+z, 2y\rangle = \frac{1}{\cancel{2}}\frac{1}{4}\left[\cancel{2}\|x+y\|^2 + \cancel{2}\|z+y\|^2 - \cancel{\|x-z\|^2} - \cancel{2}\|x-y\|^2 - \cancel{2}\|z-y\|^2 + \cancel{\|x-z\|^2}\right]$$

$$\langle x, y\rangle + \langle z, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right] + \frac{1}{4}\left[\|z+y\|^2 - \|z-y\|^2\right]$$

c.2) Si $z = 0$ en c.1 $\Rightarrow \langle x, y\rangle = \frac{1}{2}\langle x, 2y\rangle$. Así

$$\langle \underbrace{u+v}_{x}, y\rangle = \frac{1}{2}\langle u+v, 2y\rangle \overset{c.1}{=} \langle u, y\rangle + \langle v, y\rangle$$

c.3) Así, $\langle u+v, y\rangle = \langle u,y\rangle + \langle v,y\rangle$. Si $u=v$

$\langle rx, y\rangle = r\langle x,y\rangle$, $\forall r\in\mathbb{R}$, $\forall x,y\in E$

$\langle 2u,y\rangle = 2\langle u,y\rangle$, $\langle 3u,y\rangle = \langle 2u+u,y\rangle = \langle 2u,y\rangle + \langle u,y\rangle = \langle 3u,y\rangle$

$\forall n\in\mathbb{N} \Rightarrow \langle nu,y\rangle = n\langle u,y\rangle$

¿$n=0$ trivial?

¿$n<0$? $\langle -nu,y\rangle = \langle n(-u),y\rangle = n\langle -u,y\rangle =$

$$= n\left[\frac{1}{4}\left(\|-u+y\|^2 - \|-u-y\|^2\right)\right] = -n\frac{1}{4}\left[\|u+y\|^2 - \|u-y\|^2\right]$$

$$= -n\langle u,y\rangle$$

Así $\forall n\in\mathbb{Z}$ $\langle nu,y\rangle = n\langle u,y\rangle$

$\forall n\in\mathbb{Z}\setminus\{0\}$ $n\langle x/n, y\rangle = \langle x,y\rangle \Rightarrow \langle x/n,y\rangle = \frac{1}{n}\langle x,y\rangle$

Si $r = p/q$ con $p,q\in\mathbb{Z}$, $q\neq 0$

$$\left\langle \frac{p}{q}x, y\right\rangle = \frac{1}{q}\langle px,y\rangle = \frac{p}{q}\langle x,y\rangle$$

Si $r\in\mathbb{R}$ $\exists \{r_k\}\subset\mathbb{Q} / \{r_k\}\to r$

$\Downarrow$

$\langle rx,y\rangle = \lim \langle r_k x,y\rangle =$
(a partir de la def. (*))
$= \lim r_k\langle x,y\rangle = r\langle x,y\rangle$

c.q.d.