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A powerful tool for the study of systems of equations: Bolzano type theorems

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1 Introduction and motivation

In secondary school, students are familiar with the study of equations. If they are of a particular kind, such as polynomials of degree two or special types of trigonometric, logarithmic or exponential equations, the teacher provides methods and ideas for obtaining the solutions explicitly. For example,

- The equation $2^{x-1} + 2^x + 2^{x+1} = 28$ has a unique solution, $x = 3$.
- The equation $x^4 - 5x^2 + 6 = 0$ has four solutions: $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{3}$, and $-\sqrt{3}$.
- The equation $4\sin x - \cos 2x + 1 = 0$ has infinitely many solutions: $x = k\pi$, $k \in \mathbb{Z}$ (the set of integer numbers).

In these trivial examples, some basic properties of the considered elementary functions are used to explicitly obtain the solutions: the change of variable $2^{x-1} = y$ in the first case, the change of variable $x^2 = y$ in the second case, and the use of the formula $-\cos 2x = 2\sin^2 x - 1$, in the third one.

However, too many equations which arise in applied sciences cannot be solved explicitly. For example,

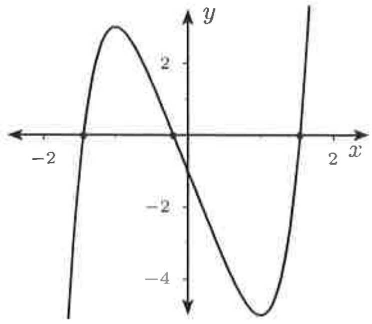

Figure 1: $f(x) = x^5 - 5x - 1$

1. The simple equation $x^5 - 5x - 1 = 0$ has three real but not rational solutions, i.e., solutions of the type
$$x = \frac{a}{b}, \; a \in \mathbb{Z}, \; b \in \mathbb{Z} \setminus \{0\},$$
since according to Fubini's rule, the only possible rational solutions of the previous equation are 1 and -1, and none of them is a solution of that equation.
2. For any given real number a, the equation $e^x + x^3 + x + \cos x = a$, has a unique solution, but it cannot be obtained explicitly.

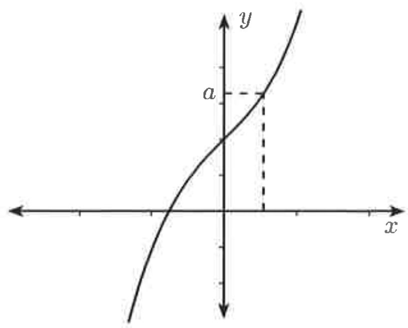

Figure 2: $f(x) = e^x + x^3 + x + \cos x$

In these situations, Bolzano's theorem provides a good method to prove the existence of solutions. This theorem, together with an additional study of the monotonicity of the given function, can provide an adequate and complete study on the solutions of the considered equation.

Bolzano's theorem contains all the conditions of a very good theorem: simple statement, affordable proof, and a large and wide applicability in the scientific world.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers.

Theorem 1 (Bolzano, 1817). *If for some real numbers $a < b$, $f : [a, b] \to \mathbb{R}$ is a continuous functions such that $f(a) < 0 < f(b)$, then there exists some point $c \in (a, b)$ satisfying the equation $f(x) = 0$.*

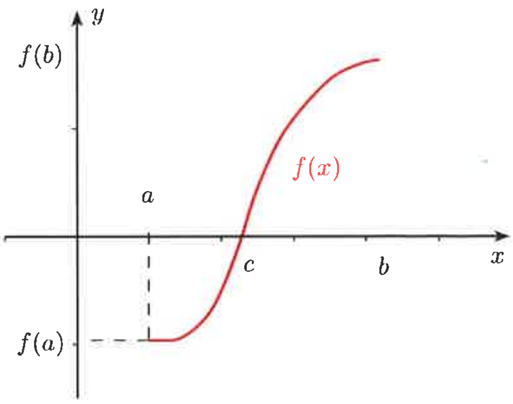

Figure 3

For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^k + h(x)$, with k an odd natural number and $h : \mathbb{R} \to \mathbb{R}$ continuous and satisfying $\lim\limits_{|x|\to+\infty} \dfrac{h(x)}{|x|^k} = 0$, then the equation $f(x) = 0$ has a solution. This is the case of a polynomial equation of odd degree $x^k + a_{k-1}x^{k-1} + \ldots + a_1 x + a_0 = 0$ (where $a_{k-1}, \ldots, a_1, a_0$ are given real numbers), as well as the case where the function h is continuous and bounded as in the equation $x^k + \sin^3(e^{x^2} + 7) = 0$.

In regard to the existence of solutions, similar ideas can be used if we consider scalar equations with several variables, i.e., f :

$\mathbb{R}^n \to \mathbb{R}$. If f is continuous and there exist some points $a, b \in \mathbb{R}^n$ such that $f(a) < 0 < f(b)$, then the equation in several variables $f(x_1, \ldots, x_n) = 0$ has at least one solution in the "open segment" of $\mathbb{R}^n$ defined as

$$(a, b)_{\mathbb{R}^n} = \{(1 - \lambda)a + \lambda b,\ \lambda \in (0, 1)\}.$$

The proof of this fact is trivial if we consider the continuous function $g : [0, 1] \to \mathbb{R}$, defined as $g(\lambda) = f((1 - \lambda)a + \lambda b)$ and we observe that $g(0) = f(a) < 0 < f(b) = g(1)$, and finally we apply Bolzano's theorem.

As an example, if $h : \mathbb{R}^3 \to \mathbb{R}$ is continuous and bounded, the equation $x_1 e^{x_2} + h(x_1, x_2, x_3) = a$ has a solution for each $a \in \mathbb{R}$. The proof of this fact is very easy, since

$$\lim_{x_1 \to +\infty} x_1 e^{x_2} + h(x_1, x_2, x_3) = +\infty$$

and

$$\lim_{x_1 \to -\infty} x_1 e^{x_2} + h(x_1, x_2, x_3) = -\infty.$$

For example, this is the case of the equation $x_1 e^{x_2} + x_1^2 e^{-x_1^2} + \sin(x_1 x_2^5 + \ln(1 + x_1^2)) = a$.

At this point, we should note that if we are considering not only the existence, but also the multiplicity of solutions, the situation may be completely different from the scalar case ($n = 1$). Let us clarify this statement: if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f'(x) > 0$, $\forall\ x \in \mathbb{R}$, then the equation $f(x) = 0$, has at most one solution, because f is strictly increasing. However if $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = x + y$, the equation $f(x, y) = 0$ has infinitely many solutions, although both partial derivatives, $f_x(x, y) = 1$ and $f_y(x, y) = 1$, are positive in $\mathbb{R}^2$. This example must not be surprising, since if $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, then its derivative f' is a function from $\mathbb{R}^n$ into $\mathbb{R}^n$ and we can not define, in an appropriate way, what $f'(x)$ being positive for $x \in \mathbb{R}^n$ means.

The situation is much more complicated in the case of systems of equations

$$f_1(x_1, \ldots, x_n) = 0, \ldots, f_n(x_1, \ldots, x_n) = 0,$$

i.e., the case where $f = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$.

Let us consider, for instance, the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = ((e^y + 1)\sin x, (e^y + 1)\cos x)$. The function f is continuous and its image, $f(\mathbb{R}^2)$, contains points of the four quadrants of $\mathbb{R}^2$, but the equation $f(x, y) = 0$ has no solutions, since $f(\mathbb{R}^2) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 > 1\}$ (see Figure 4).

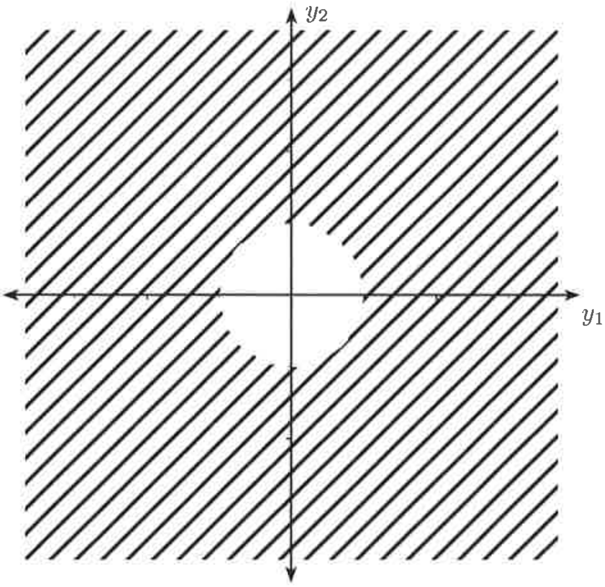

Figure 4

In Bolzano's theorem, the main hypothesis (besides the continuity of the considered function) is that the image of the function f takes values into the two sets $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}$. But it is clear from the previous example that the key idea to study systems of equations is not that the image of the function f takes values into the 2^n subsets

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, x_2 > 0, \dots, x_n > 0\},$$
$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, x_2 < 0, \dots, x_n > 0\},$$
$$\dots$$
$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 < 0, x_2 < 0, \dots, x_n < 0\}.$$

Just to prove the existence of solutions for systems of equations, the key idea is that the function f has an appropriate behavior at the topological boundary of the considered domain. Let us recall this topological concept.

For x, $y \in \mathbb{R}^n$, $d(x, y)$ denotes their euclidean distance. If Ω is a given subset of $\mathbb{R}^n$, the topological boundary of Ω, $\partial\Omega$, is defined as the set of points $x \in \mathbb{R}^n$ such that for each $r > 0$, the open euclidean ball of center x and radius r, $B_{\mathbb{R}^n}(x; r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$, contains points of Ω and points of the complementary set $\mathbb{R}^n \setminus \Omega$.

Turning to Bolzano's theorem, think that the hypotheses are given in terms of the behavior of the function f on the topological boundary of $\Omega = (a, b)$, since in this simple case, $\partial\Omega = \{a, b\}$ is a set with two points.

We finish this section with several reflections and questions:

1. Bolzano's theorem is stated for the case when $n = 1$ and $\Omega = (a, b)$ is an interval of real numbers. If, for example, $n = 2$, the most simple generalization is, perhaps, $\Omega = (a, b) \times (c, d)$, a rectangle. Then, $\partial\Omega$ is the set given by the union of its four sides:

   $$\partial\Omega = \{a\} \times [c, d] \cup \{b\} \times [c, d] \cup [a, b] \times \{c\} \cup [a, b] \times \{d\}.$$

   If $f : [a, b] \times [c, d] \to \mathbb{R}^2$, $(x, y) \to (f_1(x, y), f_2(x, y))$, is a continuous function, can we provide sign type conditions on the components f_1, f_2 on the corresponding opposite sides of Ω such that the system of equations $(f_1(x, y), f_2(x, y)) = (0, 0)$ has a solution in Ω?

2. If Ω is an open euclidean ball in $\mathbb{R}^2$, of center x and radius r, then there are no sides. Now, $\partial\Omega$ is a circumference. Is it possible to give sufficient conditions in terms of the behavior of the continuous function $f : \overline{\Omega} \to \mathbb{R}^2$ on $\partial\Omega$, such that the system of equations $f(x, y) = 0$ has a solution in Ω? (here $\overline{\Omega}$ is the closed euclidean ball of center x and radius r, $\overline{B}_{\mathbb{R}^n}(x; r) = \{y \in \mathbb{R}^n : d(x, y) \leq r\}$).
3. If Ω is a given "general" subset of $\mathbb{R}^n$, with n an arbitrary natural number, how can we prove that the system of n equations $f(x) = 0$ has a solution in Ω?
4. Is there some concept or theory that unifies all these previous cases?

The answers are given in the next section.

2 Systems of equations in rectangles, balls and...

2.1 The case of a rectangle

If we are considering systems of equations in a rectangle, the so called Poincaré-Miranda's theorem is an appropriate generalization of Bolzano's theorem. Roughly speaking, it can be stated as follows: *if each component function of the given system has opposite signs on the corresponding opposite sides of some rectangle, then the system of equations has at least one solution inside of such rectangle.* More precisely,

Theorem 2 (Poincaré-Miranda). *If*

$$f : [a, b] \times [c, d] \to \mathbb{R}^2, \ (x, y) \to (f_1(x, y), f_2(x, y)),$$

is a continuous function and

$$f_1(a, y) < 0 < f_1(b, y), \ \forall\, y \in [c, d],$$
$$f_2(x, c) < 0 < f_2(x, d), \ \forall\, x \in [a, b],$$

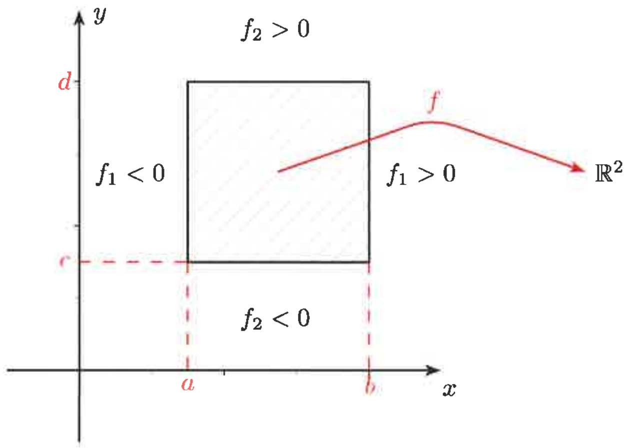

Figure 5

then the system of two equations

$$f_1(x,y) = 0,\ f_2(x,y) = 0,$$

has at least one solution in $(a, b) \times (c, d)$.

As a nontrivial example, if h, $g : \mathbb{R}^2 \to \mathbb{R}^2$ are bounded and continuous functions and $(a, b) \in \mathbb{R}^2$ is given, the system of equations

$$x^5 + h(x,y) = a, \quad \frac{y}{1+|y|}e^{y^2} + g(x,y) = b,$$

has at least one solution in the rectangle $[-r, r] \times [-r, r]$ for r a sufficiently large positive real number. To prove this fact, let us note that

$$\lim_{x\to+\infty} x^5 + h(x,y) = +\infty, \quad \lim_{x\to-\infty} x^5 + h(x,y) = -\infty$$

and that

$$\lim_{y\to+\infty} \frac{y}{1+|y|}e^{y^2} + g(x,y) = +\infty, \quad \lim_{y\to-\infty} \frac{y}{1+|y|}e^{y^2} + g(x,y) = -\infty.$$

As a consequence, the image of the function

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \to (x^5 + h(x,y), \frac{y}{1+|y|}e^{y^2} + g(x,y))$$

is the whole space $\mathbb{R}^2$.

The previous theorem was proved by Poincaré in 1886 and, obviously, there is a formulation for the general case of n equations in a rectangle contained in $\mathbb{R}^n$ (see [6]). In 1940 Miranda proved that it is equivalent to Brouwer's fixed point theorem [5].

2.2 The case of a euclidean ball

It can be affirmed that Poincaré-Miranda's theorem is very intuitive, since, on the one hand, the natural generalization of an interval of real numbers $[a, b]$ is a box in the euclidean space $\mathbb{R}^n$ given by $[a_1, b_1] \times \ldots \times [a_n, b_n]$ and, on the other hand, the sign type hypothesis of Bolzano's theorem $f(a) < 0 < f(b)$ is replaced by an appropriate sign type hypothesis on the component functions of $f = (f_1, \ldots, f_n)$ on the corresponding opposite sides of the n-dimensional rectangle. But, what happens if we are dealing with subsets of $\mathbb{R}^n$ which are not supposed to have sides? For instance, a ball. In this case we have the following result.

Theorem 3 (Systems of equations in euclidean balls). *Let $f : \overline{B}_{\mathbb{R}^n}(0; r) \to \mathbb{R}^n$ be a continuous function such that*

$$\langle f(x), x \rangle > 0, \ \forall\, x \in \partial \overline{B}_{\mathbb{R}^n}(0; r) \tag{1}$$

(where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^n$). Then the equation $f(x) = 0$ has a solution in the open ball $B_{\mathbb{R}^n}(0; r)$.

An example: if $h_1, h_2, h_3 : \mathbb{R}^2 \to \mathbb{R}$ are bounded and continuous functions, the system of equations

$$x + h_1(y, z) = 0, \ \ y + h_2(x, z) = 0, \ \ z + h_3(x, y) = 0$$

has a solution in $B_{\mathbb{R}^3}(0; r)$ for sufficiently large r. To see this,

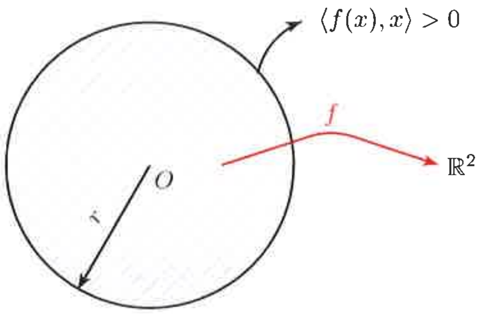

Figure 6

take into account that

$$\lim_{x^2+y^2+z^2\to+\infty} \langle (x,y,z),(x+h_1(y,z),y+h_2(x,z),z+h_3(x,y))\rangle$$
$$= \lim_{x^2+y^2+z^2\to+\infty} x(x+h_1(y,z))+y(y+h_2(x,z))+z(z+h_3(x,y))$$
$$= \lim_{x^2+y^2+z^2\to+\infty} (x^2+y^2+z^2)+xh_1(y,z)+yh_2(x,z)+zh_3(x,y)$$
$$= +\infty.$$

Let us think that, as in Poincaré-Miranda's theorem, Theorem 3 is also a generalization of the classical Bolzano's theorem, which is obtained if $n = 1$. To clarify this claim, take into account that, in Bolzano's theorem, it is clearly not restrictive to assume that $a = -r$, $b = r$, where r is a positive real number. Then, we can formulate Bolzano's theorem (Theorem 1) in the following equivalent manner:

Theorem 4 (Bolzano, revisited). *If $f : [-r,r] \to \mathbb{R}$ is continuous and*

$$f(-r)(-r) > 0, f(r)r > 0$$

(which is equivalent to $f(-r) < 0 < f(r)$), then the equation $f(x) = 0$ has a solution in $(-r,r)$, the open ball of center zero and radius r in $\mathbb{R}$.

In the previous lines, we have stated two generalizations of Bolzano's theorem which are, apparently, very different. It seems that they are not related. Nothing is further from reality, since Bolzano's theorem, Poincaré-Miranda's theorem and Theorem 3, on systems of equations on an euclidean ball, can be viewed from a unified point of view by using a powerful tool called the Brouwer degree theory (see [4]).

3 Some applications of Bolzano type theorems

The previous Bolzano type theorems are not only of interest to mathematicians. In this section, we briefly discuss some of their elementary applications. The first one uses the classical Bolzano's theorem in one and two variables to prove the simultaneous bisection of two given polygons [3]. The second one is about how to use Poincaré-Miranda's theorem to prove the existence of fixed points of a pair of functions of two variables. The fixed point theory has been of fundamental importance in the development of general equilibrium theory in Economy [7].

In the next theorem, we consider for simplicity the case of two polygons, but the same ideas may be used to deal with two bounded subsets of $\mathbb{R}^2$ with well defined area.

Theorem 5. *For any pair of given convex polygons P_1 and P_2, there exists a line $Ax + By = C$ which bisects them simultaneously, i.e., if*

$$\begin{aligned} P_1^+ &= P_1 \cap \{(x,y) \in \mathbb{R}^2 : Ax + By > C\}, \\ P_1^- &= P_1 \cap \{(x,y) \in \mathbb{R}^2 : Ax + By < C\}, \\ P_2^+ &= P_2 \cap \{(x,y) \in \mathbb{R}^2 : Ax + By > C\}, \\ P_2^- &= P_2 \cap \{(x,y) \in \mathbb{R}^2 : Ax + By < C\}, \end{aligned}$$

then $area(P_1^+) = area(P_1^-)$, $area(P_2^+) = area(P_2^-)$.

Main ideas of the proof. In the first step, we apply Theorem 1 for functions of one variable. It is clear that there exists a unique line in each direction which bisects the polygon P_1. That is, for each given $(a, b) \in S^1 = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$ there exists a unique $c = c(a, b)$ such that

$$area(P_1^+(a, b, c)) = area(P_1^-(a, b, c)),$$

where

$$P_1^+(a, b, c) = P_1 \cap \{(x, y) \in \mathbb{R}^2 : ax + by > c\},$$
$$P_1^-(a, b, c) = P_1 \cap \{(x, y) \in \mathbb{R}^2 : ax + by < c\}.$$

In fact, if $(a, b) \in S^1$ is fixed, then the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(c) = area(P_1^+(a, b, c)) - area(P_1^-(a, b, c))$$

is continuous and takes positive and negative values.

In the second step, we apply Bolzano's theorem for functions of two variables. More precisely, the function $H : S^1 \to \mathbb{R}$ defined by

$$H(a, b) = area(P_2^+(a, b, c(a, b))) - area(P_2^-(a, b, c(a, b)))$$

is continuous and takes positive and negative values. Therefore, there exists $(A, B) \in S^1$ such that $H(A, B) = 0$, i.e., the line $Ax + By = c(A, B)$ bisects P_2, and this line also bisects P_1, by the definition of $c(A, B)$. $\square$

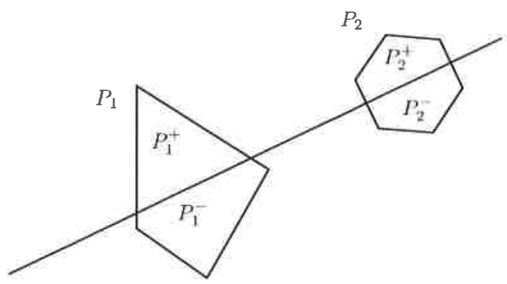

Figure 7

Finally, general equilibrium is a unified framework for studying the general interdependence of economic activities: consumption, production, exchange. Traditionally, proofs of the existence of equilibrium rely on fixed-point theorems such as Brouwer's fixed-point theorem [1]. In this regard, we can say that it is very intuitive that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, then the fixed point equation $x = f(x)$ has at least one solution, since we can apply Bolzano's Theorem 1 to the function $F : \mathbb{R} \to \mathbb{R}$ defined as $F(x) = x - f(x)$.

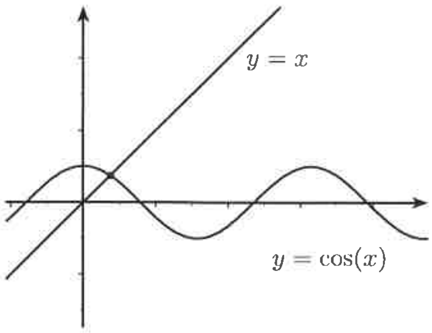

Figure 8

Perhaps, the intuition is difficult to use when we are in the case of systems of equations. In this case we show how to use the Poincaré-Miranda Theorem 2 to prove, very easily, the existence of a fixed point. Indeed, if $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \to (f(x, y), g(x, y))$ are continuous and bounded functions, then for some sufficiently large $r \in \mathbb{R}^+$, we can apply Poincaré-Miranda's theorem 2 to the function $(F, G) : [-r, r] \times [-r, r] \to \mathbb{R}^2$ defined as $(F, G)(x, y) = (x - f(x, y), y - g(x, y))$. As a consequence, $(F, G)(x_0, y_0) = (0, 0)$, for some $(x_0, y_0) \in [-r, r] \times [-r, r]$, and therefore $(x_0, y_0) = (f(x_0, y_0), g(x_0, y_0))$, for some $(x_0, y_0) \in [-r, r] \times [-r, r]$.

Finally, we comment that Theorem 3 is only a special case of the celebrated Gale-Nikaido-Debreu lemma [1], which, according to many authors, has been of great interest in the development of general equilibrium theory (especially in market equilibrium).

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References

[1] Ben-El-Mechaiekh, H., Bich, P., and Florenzano, M. "General equilibrium and fixed point theory: a partial survey". *J. Fixed Point Theory Appl.* 6.2 (2009), pp. 207–226. ISSN: 1661-7738. DOI: 10.1007/s11784-009-0129-2.

[2] Cañada, A. and Villegas, S. "¿El Teorema de Bolzano en varias variables?" *Gac. R. Soc. Mat. Esp.* 7.1 (2004), pp. 101–121. ISSN: 1138-8927.

[3] Courant, R. and John, F. *Introduction to calculus and analysis. Vol. I.* Classics in Mathematics. Reprint of the 1989 edition. Springer-Verlag, Berlin, 1999, pp. xxiv+661. ISBN: 3-540-65058-X. DOI: 10.1007/978-3-642-58604-0.

[4] Deimling, K. *Nonlinear functional analysis.* Springer-Verlag, Berlin, 1985, pp. xiv+450. ISBN: 3-540-13928-1. DOI: 10.1007/978-3-662-00547-7.

[5] Miranda, C. "Un'osservazione su un teorema di Brouwer". *Boll. Un. Mat. Ital. (2)* 3 (1940), pp. 5–7.

[6] Poincaré, H. "Sur les courbes définies par une équation différentielle IV". *J. Math. Pures Appl.* 85 (1886), pp. 151–217.

[7] Urai, K. *Fixed Points and Economic Equilibria.* Vol. 5. World Scientific, 2010.

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