J. geom. 60 (1997) 80 - 84
0047-2468/97/020080-05 $ 1.50+0.20/0

Journal of Geometry

A NOTE ON H-SURFACES WITH BOUNDARY

Rafael López [1]

In this note, we prove that a constant mean curvature compact embedded surface with planar boundary, which is a graph near the boundary, over the compact planar domain determined by the boundary, is indeed a graph globally.

1. INTRODUCTION AND PRELIMINARIES

The structure of the space of compact constant mean curvature surfaces with prescribed boundary is not known, even in the simplest case: when the boundary is a round circle with, for instance, unit radius. Heinz [3] found that a necessary condition for existence in this situation is that $|H| \leq 1$. The only known examples, excluding the trivial minimal case, are the following: the two spherical caps with radius $1/|H|$, which are the only umbilical ones and some non-embedded surfaces of genus bigger than two whose existence was proved by Kapouleas in [4].

We shall refer to connected compact constant mean curvature surfaces as *H-surfaces*, H the constant value of the mean curvature. We note that if $H = 0$, the surface lies in the convex hull of its boundary, and therefore, if the boundary is planar, then the surface is also planar. Hence we assume in this paper that $H \neq 0$.

When the surface is embedded, the Alexandrov reflection method is a powerful tool [1]. So, if the surface Σ is embedded with planar convex boundary and it is over the plane P containing the boundary, then Σ inherits all symmetries of its boundary: see [5] for the compact case and [8] for the non-compact one. Then, if the boundary $\partial\Sigma$ is a circle, the surface is a spherical cap or a Delaunay surface.

[1]This paper has been partially supported by a DGICYT Grant No. PB94-0796

For this reason, it is important to put hypothesis to assure that the surface is over the plane P. For example, in [2] it is proved that an embedded H-surface with convex planar boundary that is in a halfspace near the boundary, is completely contained in this halfspace. Also, in [5] it has been proved that if the surface does not intersect the outside of the boundary in the plane P, then the surface is over P.

In this paper, we give sufficient conditions to get an embedded H-surface in a halfspace, more precisely, we shall give conditions to be a graph. We state

THEOREM. *Let Σ be an embedded H-surface with boundary $\partial\Sigma$ a Jordan curve included in a plane P. Let $D \subset P$ be the domain bounded by $\partial\Sigma$ in P. If $\Sigma \cap D = \emptyset$ and Σ is locally a graph over D around $\partial\Sigma$, then Σ is a graph.*

The proof uses the Alexandrov reflection method with planes parallel to the plane P, joined with a certain "balancing formula" for H-surfaces. A consequence of this theorem is the following result on embedded H-surfaces included in a halfspace, which is proved in [7]:

COROLLARY 1. *Let Σ be an embedded H-surface with boundary $\partial\Sigma$ a Jordan curve included in a plane P. Let $D \subset P$ be the bounded domain by $\partial\Sigma$ in P. If Σ is included in one of the two halfspaces determined by P and it is locally a graph over D around $\partial\Sigma$, then Σ is a graph.*

Finally, we get the result stated in the summary of this paper.

COROLLARY 2. *Let Σ be an embedded H-surface with boundary $\partial\Sigma$ a Jordan curve included in a plane P. Let $D \subset P$ be the bounded domain by $\partial\Sigma$ in P. If Σ is locally a graph over D around $\partial\Sigma$, then Σ is a graph.*

2. PROOF OF THE RESULTS

To prove the Theorem, we will need a certain flux formula due to Rob Kusner which appears in [6]:

BALANCING FORMULA. *Let Σ be an embedded H-surface in $\mathbb{R}^3$ with boundary a Jordan curve included in a plane P. Then, if $a \in \mathbb{R}^3$,*

$$2H \int_D \langle \eta_D, a \rangle = \int_{\partial\Sigma} \langle \nu(s), a \rangle ds,$$

where $H > 0$, $\nu(s)$ is the interior conormal to Σ along $\partial\Sigma$, D is the bounded domain in P by $\partial\Sigma$ and η_D is the unit normal vector field to D, induced by the orientation of the cycle $\Sigma \cup D$ when Σ is oriented by its mean curvature vector.

Without loss of generality, we assume that P is the plane $P = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; x_3 = 0\}$ and Σ is included in $\mathbb{R}^2 \times [0, \infty)$ in a neighbourhood of $\partial\Sigma$. Let a be the vector $(0, 0, 1)$.

We define the embedded and closed surface $T = \Sigma \cup D$ and let $W \subset I\!R^3$ be the bounded domain determined by T. We orient T to have $H > 0$ on Σ. If N is the Gauss map of Σ, then N points towards W. First, we prove that $\langle N, a\rangle < 0$ along $\partial\Sigma$. If ν is the interior conormal of Σ along $\partial\Sigma$, the balancing formula gives

$$2H \int_D \langle \eta_D, a\rangle = \int_{\partial\Sigma} \langle \nu(s), a\rangle ds.$$

Because $\langle \nu(s), a\rangle \geq 0$ and $H > 0$, then $\eta_D = a$. Since η_D points towards W and Σ is a graph near the boundary, then $\langle N, a\rangle \leq 0$. Remark that Σ is not tangent to D in any point $p \in \partial\Sigma$, because in this case, Σ and D have a common boundary arc near p, D is locally above Σ near p respect to $N(p) = -a$ and then the mean curvature of D should be bigger that the mean curvature of Σ, in contradiction with $H > 0$. Then we have $\langle N, a\rangle < 0$ along $\partial\Sigma$. Therefore, the domain W, near D, lies above D, i.e., there is $\epsilon > 0$ such that

$$W \cap (D \times (-\epsilon, \epsilon)) = W \cap (D \times (0, \epsilon)) \quad (*).$$

To use the Alexandrov reflection method, we introduce notation. For any $t \in I\!R$, we denote $P_t = \{x_3 = t\}$. If $A \subset I\!R^3$, let A_t^* be the reflection of A respect to P_t, i.e.,

$$A_t^* = \{(x_1, x_2, x_3) \in I\!R^3; (x_1, x_2, 2t - x_3) \in A\}.$$

Set

$$A_{t^+} = \{x \in A; x_3 > t\} \qquad A_{t^-} = \{x \in A; x_3 < t\}$$

and $A_{t^+}^* = (A_{t^+})_t^*$, $A_{t^-}^* = (A_{t^-})_t^*$. If B is a subset of $I\!R^3$, we say that $A \geq B$ is for $x \in A, y \in B$ with $x_1 = y_1$, $x_2 = y_2$, then $x_3 \geq y_3$.

Now let $b < 0$ such that

$$\Sigma_{0^-} \cup \Sigma_{0^+}^* \subset \{x_3 > b\} \qquad (**)$$

and we consider the part of the solid cylinder below D given by $X = \overline{D} \times [b, 0]$. Define a bounded domain Z in $I\!R^3$ by $Z = W \cup X$ and denote by Ω the boundary of Z. The set Ω is an embedded closed surface and included in the union of the sets Σ with the piece of cylinder $\partial\Sigma \times [b, 0]$ and the disc $\overline{D} \times \{b\}$.

We orient Ω by the Gauss map N of Σ and since N points towards W then N points inside Z. Now we apply to the surface Ω the Alexandrov reflection principle by horizontal planes P_t coming down from above. Because Ω is compact, let t be a large positive number such that $P_t \cap \Omega = \emptyset$. Translate P_t down until the first height $t_0 > 0$ at which it first touches Ω. Thus the plane P_{t_0} touches Ω in points of Σ and since Σ is embedded, for small $\delta > 0$ we have

$$\Omega^*_{(t_0-\delta)^+} \geq \Omega_{(t_0-\delta)^-} \qquad \text{and} \qquad \Omega^*_{(t_0-\delta)^+} \subset \overline{Z}.$$

Now we decrease t to get the first time $t_1 < t_0$ such that $\Omega^*_{t_1^+} \subset \overline{Z}$ but $\Omega^*_{t^+} \not\subset \overline{Z}$ for $t < t_1$. Then there exists a point $p \in \Omega_{t_1^+}$ such that the reflected point p^* of p with respect to the plane P_{t_1} is a touching point between $\Omega^*_{t_1^+}$ and $\Omega_{t_1^-}$.

Let us prove that $t_1 > 0$ is impossible. In this case, this implies that the point p lies in the surface Σ and according to the parts of Ω we have three possibilities about the point p^*:

1. The point p^* is an interior point of Σ. Then $\Omega^*_{t_1^+}$ and $\Omega_{t_1^-}$, around the point p^*, are domains of Σ and then, they are tangent at p^*; moreover the Gauss maps of both surfaces point towards Z and they agree at p. Then the maximum principle [9] gets that P_{t_1} is a plane of symmetry of Σ (if $p^* \in P_{t_1}$, we apply the boundary version of the maximum principle). But this is impossible, because t_1 is positive and the boundary of Σ is below P_{t_1}.

2. The point p^* belongs to $\partial\Sigma \times [b, 0]$. Since t_1 is the time of the first contact and we come from above then $p^* \in \partial\Sigma \times \{0\} = \partial\Sigma$. In this case, the segment joining p with p^* is included in $\overline{Z} \cap \{x_3 \geq 0\}$. But by (*) the piece of cylinder $\partial\Sigma \times (0, \epsilon)$ does not intersect the domain Z, getting a contradiction.

3. The point p^* belongs to $D \times \{b\}$. This case is impossible by (**).

Therefore $t_1 = 0$ and we can go reflecting the surface Σ until the height $t = 0$. As conclusion and since Σ is a locally a graph over $\partial\Sigma$ around the boundary, we have that Σ_{0^+} is a graph over D, $\Sigma^*_{0^+}$ is included in $\overline{D} \times [b, 0]$ and Σ does not intersect the outside of the domain $\overline{D}$ in the plane P. Then by [5] the surface Σ is over P, i.e., $\Sigma = \Sigma_{0^+}$ and we conclude that Σ is a graph over D.

To prove the Corollary 2, the proof of the Theorem gives that Σ is never tangent to D along $\partial\Sigma$. Then the surface Σ is transverse to P along $\partial\Sigma$. Now the Theorem 2 in [2] gives that the surface does not intersect D and we apply the Theorem.

Acknowledgement. The author thanks to the referee for the suggestions in the proof of the Theorem.

REFERENCES

[1] ALEXANDROV, A.D.: *Uniqueness theorems for surfaces in the large*, V. Vestnik Leningrad Univ., **13**, No. 19, A.M.S. (Series 2), **21** (1958), 412–416.

[2] BRITO, F., EARP, R., MEEKS III, W.H. and ROSENBERG, H.: *Structure theorems for constant mean curvature surfaces bounded by a planar curve*, Indiana Univ. Math. J. **40** (1991), 333–343.

[3] HEINZ, H.: *On the nonexistence of a surface of constant mean curvature with finite area and prescribed rectificable boundary*, Arch. Rational Mech. Anal., **35** (1969), 249–252.

[4] KAPOULEAS, N.: *Compact constant mean curvature surfaces in Euclidean three-space*, J. Diff. Geom. **33** (1991), 683–715.

[5] KOISO, M.: *Symmetry of hypersurfaces of constant mean curvature with symmetric boundary*, Math. Z., **(191)** (1986), 567–574.

[6] KOREVAAR, N., KUSNER, R. and SOLOMON, B.: *The structure of complete embedded surfaces with constant mean curvature*, J. Diff. Geom. **30** (1989), 465–503.

[7] ROS, A. and ROSENBERG, H.: *Constant mean curvature surfaces in a halfspace of $\mathbb{R}^3$ with boundary in the boundary of the halfspace*, to appear in J. Diff. Geom.

[8] ROSENBERG, H. and SA EARP R.: *Some structure theorems for complete constant mean curvature surfaces with boundary a convex curve*, Proc. A.M.S., **113** (1991), 1045–1053.

[9] SPIVAK, M.: A Comprehensive Introduction to Differential Geometry, Vol. 1–5, Publish or Perish Inc., Berkeley, 1979.

Departamento de Geometría y Topología
Universidad de Granada, 18071 Granada - SPAIN
e-mail: rcamino@ugr.es

Eingegangen am 28. Dezember 1995; in revidierter Form am 30. Januar 1997