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Surfaces of Constant Mean Curvature Bounded by Convex Curves

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Abstract. This paper proves that an embedded compact surface in the Euclidean space with constant mean curvature $H \neq 0$ bounded by a circle of radius 1 and included in a slab of width $1/|H|$ is a spherical cap. Also, we give partial answers to the problem when a surface with constant mean curvature and planar boundary lies in one of the halfspaces determined by the plane containing the boundary, exactly, when the surface is included in a slab.

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0. Introduction

A constant mean curvature surface immersed in Euclidean three-space $\mathbb{R}^3$ can be viewed as a surface where the exterior pressure and the surface tension forces are balanced. Until 1986, the only known examples of closed (compact without boundary) constant mean curvature surfaces were the round spheres. Wente [17] constructed examples of constant mean curvature tori which are non-embedded. One year later, Kapouleas [10] did the same for genera bigger than 2. These results activated in a remarkable way the research in this subject and gave their exact value to the two principal theorems about closed constant mean curvature surfaces which were known at that moment: the Hopf theorem, which asserts that the sphere is the only example of genus zero [9] and the Alexandrov theorem, which says us that the sphere is the only embedded example [1].

When the considered surface Σ is compact and with non-empty boundary $\partial\Sigma$, and particulary if $\partial\Sigma$ is a Jordan curve, the problem of existence has been studied by many authors. The existence of small solutions is due to Hildebrandt [8] and the search of a second solution was culminated by Brézis and Coron [5].

With respect to the study of the space of compact constant mean curvature surfaces with prescribed boundary, we do not know its structure even in the simplest case: when $\partial\Sigma$ is a round circle with, for instance, unit radius. Heinz found that a necessary condition for existence in this situation is that $|H| \leq 1$, where H is

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the mean curvature. The only known examples, excluding the trivial minimal case, are the following: the two spherical caps with radius $1/|H|$, which are the only umbilical ones and some non-embedded surfaces of genus bigger than 2 whose existence was showed by Kapouleas in [10].

The lack of examples and the analogy with the closed case allow us to believe that the following statements are true.

CONJECTURE 1. *An embedded compact surface with non-zero constant mean curvature bounded by a circle is a spherical cap.*

CONJECTURE 2. *An immersed disc with non-zero constant mean curvature bounded by a circle is a spherical cap.*

Partial answers to Conjecture 1 have been given in [4] and [6]. With respect to Conjecture 2, some progress was done by S. Montiel and the author in [13] and about the study of constant mean curvature surfaces with small volume was done in [14].

We shall consider a connected compact surface Σ and $\phi\colon \Sigma \to \mathbb{R}^3$ an immersion of constant mean curvature $H \neq 0$ such that ϕ takes $\partial\Sigma$ diffeomorphically onto $\phi(\partial\Sigma)$. We will say in this situation that Σ is an *H-surface with boundary* Γ, where $\Gamma = \phi(\partial\Sigma)$ (we note that if $H = 0$, the surface lies in the convex hull of its boundary and therefore, if the boundary is planar, the surface is also planar). If there is no confusion, we identify Σ with $\phi(\Sigma)$.

This paper is motived by a height estimate due to Meeks. In [15] he gets, using the Alexandrov reflection method, the following estimate: if Σ is an embedded H-surface with boundary contained in a plane P, it can rise at most $2/|H|$ above P, i.e. Σ is contained in a slab symmetric to P with width $4/|H|$. We study embedded H-surfaces included in an arbitrary slab, not necessarily parallel to the plane P.

Barbosa studied the case when the boundary is a circle and the surface Σ, assumed only immersed, is included in a ball of radius $1/|H|$ and he showed, without any further hypothesis, that the surface must be a spherical cap ([2]). Later ([3]) he established an extension of this result for the case in which the surface is contained in a solid cylinder of radius $1/|H|$. In both results, three or two independent directions of the immersion are bounded. So it is natural to ask about immersions with only one bounded direction, i.e. surfaces included in a slab. In this direction we state.

> *'The spherical caps are the only embedded H-surfaces with boundary a circle of radius* 1 *and included in a slab of width* $1/|H|$.'

The width of a slab is the distance between the planes which define the slab. As a consequence of the proof of this statement, we also get some results about embedded H-surfaces with boundary a planar convex curve and on *immersed* H-surfaces with boundary a circle and included in a slab of width $1/|H|$ parallel to the boundary.

1. The Main Theorem

An important ingredient is a result established by Koiso [11]. She proved that an embedded H-surface with boundary a Jordan curve Γ contained in a plane P which does not intersect P outside of the region bounded by Γ, it is included in one of the two halfspaces determined by the plane P. Then the Alexandrov reflection technique immediately proves that the surface inherits the symmetries of its boundary. Hence, if Γ is a circle, Σ is a surface of revolution and, therefore, is a spherical cap (by Delaunay's classification of constant mean curvature surfaces of revolution). Therefore, it is interesting to obtain natural geometric conditions that forces an embedded H-surface to be contained in a halfspace. It is still an opened question (see [4]) whether an embedded H-surface bounded by a plane convex curve and contained in one halfspace defined by the plane containing the boundary, has genus zero or not.

Another preliminary result in this paper is a kind of uniqueness for embedded surfaces included in a right cylinder. Exactly, if Ω is a bounded domain included in a plane P, it is classical that if there is a graph on Ω with constant mean curvature and boundary $\partial\Omega$, then there are no other graphs in Ω with the same boundary and mean curvature ([7]). From the Alexandrov reflection method and the Koiso's result, it is easy to show that an embedded H-surface with boundary $\partial\Omega$, and included in the solid cylinder C orthogonal to P determined by Ω, is a graph (a detailed proof of this fact, together with several related results, can be viewed in [12]).

In [14], using a flux formula, it is proved that if there is an H-graph G on Ω and an embedded H-surface Σ included in C, with the same boundary $\partial\Omega$, then $\Sigma = G$.

Now we prove the main result of this paper.

THEOREM 1. *Let Σ be an embedded H-surface with boundary a circle Γ of radius* 1. *If Σ is included in a solid slab with width $1/|H|$, then Σ is a spherical cap of radius $1/|H|$.*

Remark. Among the two possible spherical caps, the only one contained in a slab with width $1/|H|$ is the small spherical cap.

Proof. The proof is similar to Theorem 3.1 in [3]. For completeness, we shall follow the analogous steps. We choose the normal vector field N of Σ to have $H > 0$. We denote by S the slab containing the surface. Let v be a unit vector parallel to S. Let C_v be the closed halfcylinder not bounded of radius $1/(2H)$ with axis parallel to S, perpendicular to the vector v and $\partial C_v \subset \partial S$. The set C_v splits S in two components. We suppose that v points towards the non-convex component and we consider the direction of v as upward and the direction of $-v$ as downward. First of all, move C_v upward until it does not intersect Σ. This is possible since Σ is a compact surface. Now, we move C_v downward until it touches Σ for the first time. In this position, Σ lies completely in the closed convex region of S below C_v. We want to apply the Hopf maximun principle to compare C_v and Σ (see [16]).

LEMMA 2. *There is no point p interior to Σ such that p belongs to C_v and Σ lies below C_v.*

Proof (*of the lemma*). Because Σ lies below C_v and p belongs to $C_v \cap \Sigma$, then C_v and Σ are tangent at p. This is true even when $p \in \partial C_v$, because in this case, the point p will also belong to the boundary of the slab S. Hence Σ will be tangent to the boundary of the slab at p and, hence, will be tangent to C_v at p. To apply the maximun principle, the unit normal vector fields of Σ and C_v must agree at p. If they do, then C_v and Σ must intersect along an open set. By analyticity of the solutions of the equation $H =$ constant, we conclude that Σ must be a subset of the halfcylinder C_v of radius $1/(2H)$, which is not possible. If the Gauss maps of Σ and C_v do not agree at p, then they must be opposite. In this case, we consider in Σ the Gauss map $-N$ and then the mean curvature is negative, in contradiction with the maximum principle. $\square$

LEMMA 3. *Under the hypothesis of Theorem* 1, *there is no point p of $\partial\Sigma$ such that p belongs to the interior of C_v where Σ and C_v are tangent at p and assuming that Σ lies below C_v.*

Proof (*of the lemma*). This lemma can be proved in the same way as the previous one, because the extra hypothesis guarantees that Σ and C_v are comparable as the above lemma ([16]). $\square$

From both lemmas it follows that C_v touches Σ only at points of Γ. These points are either points of ∂C_v or points of C_v where Σ and C_v are not tangent. If we take all vectors v in the set of parallel vectors to S, we conclude that Σ lies in the convex K determined by the sets C_v (we remark that C_v and C_{-v} can intersect). In this moment, we have two possibilities:

(A) The plane P is orthogonal to S. Let v be an orthogonal direction to P and then, parallel to S. We consider the halfcylinders C_v and C_{-v}. Then Σ lies in the convex domain K_v determined by C_v and C_{-v} in S. Because v is orthogonal to P, by symmetry with respect to the plane P, the points of contact between C_v and C_{-v} with Σ are the same. Thus, C_v and C_{-v} intersect too. Therefore K_v is included in a solid cylinder of radius $1/(2H)$ parallel to C_v. As $1/(2H) < 1/H$, Theorem 3.1 of [3] asserts that Σ is a spherical cap and the theorem has been proved.

(B) If the plane P is not orthogonal to S and because Γ is a circle, the contact points between C_v and C_{-v} with Γ are opposite points in the circle Γ. Therefore, for any parallel direction, we have two opposite points in Γ. Hence if we take all parallel directions to S we have that $\Gamma \subset \partial K$. Let Ω be the disc bounded by Γ in the plane P. Because $\Sigma \subset K$ and K is a convex set, then $(P - \overline{\Omega}) \cap \Sigma = \emptyset$.

On the other hand, by [11], one knows that $\Sigma \cap \Omega = \emptyset$. Thus, the surface is included in one of the two halfspaces determined by P. Therefore, by [11], it is a spherical cap. □

Remark 1. In step (A), the hypothesis of embeddedness has not been used, i.e. this part is true for immersed surfaces.

Remark 2. To use the result of Koiso, the hypothesis of embeddedness has been used only at the end of step (B).

With these remarks, we have the following consequence:

COROLLARY 4. *Let Σ be an H-surface with boundary a circle of radius 1 included in a plane P. If the surface is included in a slab S of width $1/|H|$, then in any next cases the immersion describes a spherical cap:*

(1) *S is orthogonal to P.*
(2) $|H| \leq \frac{1}{2}$.

Proof. (1) This is step (A) in the proof of Theorem 1 and the above Remark 1.

(2) In this case, the convex set K_v in the proof of Theorem 1 is included in a right cylinder orthogonal to S and with radius $1 + 1/(2|H|)$. Because $|H| \leq \frac{1}{2}$

$$1 + \frac{1}{2|H|} = \frac{2|H| + 1}{2|H|} \leq \frac{2}{2|H|} = \frac{1}{|H|}$$

and then this radius is less than $1/|H|$. Now we can apply [3]. □

2. Other Related Results

The proof of Theorem 1 also gives the following generalization when the boundary Γ is not a circle.

COROLLARY 5. *Let Σ be an H-surface in $\mathbb{R}^3$ with boundary a convex curve Γ in a plane P. If Σ lies in a slab of width $1/|H|$ symmetric with respect to P, then the surface is included in the solid cylinder determined by Γ and orthogonal to P. In the case that Σ is embedded, then it is a graph*.

Proof. If we repeat the proof of Theorem 1, we get, with the same notation, that for any parallel direction v, the halfcylinders C_v touch at Σ in boundary points. Moreover, since Γ is convex, for any point in Γ there is one direction v such that C_v touches Σ in this point.

Taking all v, using that the slab is symmetric with respect to P and Γ is a planar convex curve, we conclude that Σ is included in the solid cylinder orthogonal to S and determined by Γ.

If the surface is embedded, it is easy to show that it is a graph using the Alexandrov reflection method with planes parallel to P ([1]). $\square$

When the surface is embedded, we can further state: from the proof of Corollary 5, if S is not necessarily symmetric with respect to P, but only parallel, the surface Σ has no points in the outside of the bounded domain defined by Γ in P. Thus, from [11], the surface lies in one of the two halfspaces determined by P.

It is possible to obtain uniqueness in the last corollary for *immersed* surfaces with small H. Exactly

COROLLARY 6. *Let Γ be a closed convex curve in a plane P. Then there is $H_0 > 0$, depending only on Γ, such that, for any $H \in \mathbb{R}$ with $0 < |H| \leq H_0$, the only H-surface with boundary Γ and included in the slab symmetric with respect to P and with width $1/|H|$ is a graph.*

Proof. Let Ω be the domain determined by Γ in P. By the Implicit Function Theorem [7], there is $H_0 > 0$ (depending only on Γ) such that for any $H \in \mathbb{R}$, $|H| \leq H_0$, there is an H-graph on Ω with boundary $\partial\Omega = \Gamma$.

Let Σ be an H-surface with boundary Γ and included in the slab S symmetric with respect to P and with width $1/|H|$. Also, let G be an H-graph on Ω with boundary Γ. From the above corollary, Σ is contained in the cylinder determined by Ω and orthogonal to P. It remains to prove that Σ is G or the reflection G^* of G with respect to P: that is a uniqueness result given in [13] and, for completeness, we give the proof.

Let a be a unit vector orthogonal to P. We suppose that G is over P (with respect to a) and we orient it by a Gauss map N_G so that $H > 0$. Then N_G points downwards: $\langle N_G, a\rangle < 0$. We move G up so that it does not touch Σ and then, we move it down to touch Σ. If there is a point of contact (interior or boundary) and because N_G points downwards, then $\Sigma = G$ (maximum principle). We can do similar arguments with G^*. Then we have two possibilities: Σ is G or G^* or, otherwise, Σ is between G and G^*. In this case, if we compare the interior conormal of Σ and G, ν_Σ, ν_G, along Γ, we have the inequality

$$|\langle \nu_\Sigma, a\rangle| < \langle \nu_G, a\rangle.$$

But a 'balancing' formula given for H-surfaces (see for instance [4], [10]) asserts that

$$\left|\int_\Gamma \langle \nu_\Sigma, a\rangle\right| = \int_\Gamma \langle \nu_G, a\rangle,$$

giving a contradiction. Therefore $\Sigma = G$ or $\Sigma = G^*$. $\square$

We remark that if H is small, the width of the slab in the above corollary is large. Fixing a slab, we can paraphrase the above result in the following way:

'Let Γ be a closed planar convex curve included in a plane P and let S be a slab containing P. Then there is a number $H_0 = H_0(\Gamma, S) > 0$ such that, for any $H \in \mathbb{R}$, $0 < |H| \leq H_0$, the only H-surface with boundary Γ and included in S is a graph.'

To end this paper, we return to Theorem 1. The difference between this theorem and the results of [2] and [3] is the extra hypothesis about the embeddedness of the surface. It would be nice to change in Theorem 1 the words 'embedded H-surface' to *immersed H-surface*. We study immersed surfaces bounded by a circle included in a slab parallel to the plane which contains the circle. Changing coordinates, we assume that the slab S is defined by the planes $z = -1/(2|H|)$ and $z = 1/(2|H|)$.

THEOREM 7. *Let Σ be an H-surface such that $\phi(\Sigma) \subset S$, denoting by ϕ the immersion and*

$$\phi(\partial\Sigma) = \Gamma = \left\{ \left(x, y, -\frac{1}{2|H|} \right) \in \mathbb{R}^3; x^2 + y^2 = 1 \right\}.$$

If $H^2 \leq \frac{3}{4}$, then $\phi(\Sigma)$ is a spherical cap of radius.

Proof. We assume the orientation in the surface $\phi(\Sigma)$ is chosen to get a positive mean curvature H. From Corollary 4, it is only necessary to prove the theorem for values of H in the interval $(\frac{1}{2}, \sqrt{3}/2]$.

We are going to consider a family of arcs of circles in the plane xz passing by the points $(1, 1/(2H))$ and $(1, -1/(2H))$, starting from the original circle of radius $1/(2H)$ centred at the point $(1, 0)$ and becoming less and less curved until reaching the circle whose centre is at the origin. Each of these surfaces can be parametrized by a real variable t as

$$t \mapsto (\mu, 0, 0) + r(\cos t, 0, \sin t),$$

where $r = \sqrt{(1-\mu)^2 + 1/4H^2}$ and $t \in [-t_0, t_0]$, in which t_0 is the solution of the equation $r \sin t_0 = 1/(2H)$, and μ is any number in the interval $[0, 1]$.

For each of these curves, one considers the revolution surface generated by its rotation around the z-axis, and the three-dimensional region K_μ bounded by this surface and the planes $z = 1/(2H)$ and $z = -1/(2H)$. The mean curvature function $\bar{H}_\mu$ of the non-flat part of ∂K_μ is given by

$$\bar{H}_\mu = \frac{\mu + 2r\cos t}{2r(\mu + r\cos t)} = \frac{1}{2r}\left(2 - \frac{\mu}{\mu + r\cos t}\right).$$

We know that $\phi(\Sigma)$ is contained in K_1 and that the contact points among $\phi(\Sigma)$ and the boundary ∂K_1 of K_1 belong either to Γ or to the flat part of ∂K_1. We also know that, at such points, ∂K_1 and $\phi(\Sigma)$ are not tangent.

Under our hypothesis, as we move t from 1 to 0, we are going to show the existence of a number μ_0 such that, for $1 \geq \mu \geq \mu_0$ we have that $\bar{H}_\mu$ is kept larger than or equal to H. The centre of the arc of radius r which determines the surface of revolution for the variable μ goes from $\mu = 1$ to $\mu_0 = (3 - 4H^2)/8H(1 - H)$. Exactly, it will prove that we can consider all cases up to $\mu_0 + r_0 = 1/H$. Because $\mu + r\cos t \geq 1$,

$$2 - \frac{\mu}{\mu + r\cos t} \geq 2 - \mu.$$

Thus, to show that $\bar{H}_\mu \geq H$ we need to prove $2 - \mu \geq 2Hr$. Squaring, we have to show that $1 - \mu_0 \leq 2/(4H^2 - 1)$. Because $1 - \mu \leq 1 - \mu_0$, we get the wanted inequality if $1 - \mu_0 \leq 2/(4H^2 - 1)$. But this is true for any $H \in (\frac{1}{2}, \sqrt{3}/2]$.

Because of the maximum principle it can be deduced that $\phi(\Sigma)$ is never tangent to the non-flat part of ∂K_μ, for each $1 \geq \mu \geq \mu_0$, thus $\phi(\Sigma) \subset K_{\mu_0}$. Since $\mu_0 + r_0 = 1/H$, K_{μ_0} and so, $\phi(\Sigma)$ is contained in a cylinder of radius $1/H$. Thus, by [3], it must be a spherical cap. □

The question is: What happens when $\frac{3}{4} < H^2 \leq 1$? Related with the Theorems 1 and 7 we state the following

CONJECTURE. *An H-surface immersed in $\mathbb{R}^3$ included in a slab of width $1/|H|$ bounding a circle of radius 1 is a spherical cap.*

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