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# An exterior boundary value problem in Minkowski space

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In three-dimensional Lorentz–Minkowski space $\mathbb{L}^3$, we consider a spacelike plane $\Pi$ and a round disc $\Omega$ over $\Pi$. In this article we seek the shapes of unbounded surfaces whose boundary is $\partial\Omega$ and its mean curvature is a linear function of the distance to $\Pi$. These surfaces, called stationary surfaces, are solutions of a variational problem and governed by the Young–Laplace equation. In this sense, they generalize the surfaces with constant mean curvature in $\mathbb{L}^3$. We shall describe all axially symmetric unbounded stationary surfaces with special attention in the case that the surface is asymptotic to $\Pi$ at the infinity.



## 1 Formulation of the problem

Let $\mathbb{L}^3$ denote the 3-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the metric $\langle\,,\rangle = dx_1^2 + dx_2^2 - dx_3^2$, where $x = (x_1, x_2, x_3)$ are the canonical coordinates in $\mathbb{R}^3$. An immersion $x : M \to \mathbb{L}^3$ of a smooth surface $M$ is called spacelike if the induced metric on the surface is positive definite. We identify $\mathcal{S} =: x(M)$ with $M$. For spacelike surfaces, the notions of the first and second fundamental form, and the mean curvature are defined in the same way as in Euclidean space. Constant mean curvature (spacelike) surfaces are obtained as solutions of a variational problem, exactly, they are critical points of the area functional for variations which preserve a suitable volume function. In general, constant mean curvature spacelike submanifolds of a Lorentzian manifold are interesting in relativity theory. In this setting, there is interest of finding real-valued functions on a given spacetime, all of whose level sets have constant mean curvature. Then the mean curvature function may then be used as a global time coordinate which has many applications. See [4, 11].

In this work, we generalize constant mean curvature surfaces in $\mathbb{L}^3$ by surfaces whose mean curvature is a linear function of the time coordinate and that follow being solutions of a more general variational problem. We explicit our mise in scene. Consider the horizontal plane $\Pi = \{x_3 = 0\}$ and let us fix a round disc $\Omega$ of radius $R > 0$ at distance $b$ from $\Pi$, namely,

$$\Omega := \Omega_{R,b} = \big\{x \in \mathbb{L}^3; x_3 = b, x_1^2 + x_2^2 \leq R^2\big\}.$$

We assume the existence of a timelike potential $Y = \kappa x_3 + \lambda$, $\kappa > 0$, $\lambda \in \mathbb{R}$, which measures at each point, up constants, the distance to $\Pi$. Let $\mathcal{S}$ be an unbounded spacelike surface whose boundary is the circle $\partial\Omega$ and consider all the perturbations of $\mathcal{S}$ in such way that $\mathcal{S}$ remains adhered to $\Omega$ along its boundary. For each relatively compact domain $D$ in $M$ such that $\partial M \subset \partial D$, we define the energy functional

$$E = |D| - \cosh\beta\,|\Omega| + \int_D Y\, dM,$$

where $|D|$ and $|\Omega|$ denote the areas of $D$ and $\Omega$ respectively. As in the case of constant mean curvature surface, we are interested in those configurations where the energy of the physical system is critical under any perturbation of the system that does not change the volume of $x(D)$ and the adherence of $x(D)$ on $\Omega$ along the circle $\partial\Omega$. We

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say then that $\mathcal{S}$ is a *stationary surface*. According to the principle of virtual work, and when the equilibrium of the system is achieved, the possible configurations that adopts a stationary surface are characterized as follows:

**Proposition 1.1** *With the above assumptions, the surface $\mathcal{S}$ is stationary if and only if*

1. *The mean curvature $H$ of $\mathcal{S}$ satisfies the relation*

$$2H(p) = \kappa x_3(p) + \lambda, \quad p \in \mathcal{S}, \quad \kappa > 0, \quad \lambda \in \mathbb{R}. \tag{1.1}$$

2. *The surface $\mathcal{S}$ meets the support disc $\Omega$ in a constant hyperbolic angle $\beta$, that is,* $\cosh\beta = -\langle N, N_\Omega\rangle$ *along $\partial\mathcal{S}$, when $N$ and $N_\Omega$ denote future-directed orientations on $\mathcal{S}$ and $\Omega$ respectively.*

We address the reader to [1, 2, 3, 10] for a specific background of the problem. As a particular case, if $\kappa = 0$, the mean curvature of $\mathcal{S}$ is constant, namely, $H = \lambda/2$. Although the study of the possible shapes of a stationary surface presents difficulty of analysis in all its generality, at least in the situation that $\mathcal{S}$ is an embedded surface, that is, without self-intersections, one can expect that $\mathcal{S}$ inherits the symmetries of its boundary, that is, $\mathcal{S}$ is a surface of revolution. For this reason, we shall focus in the case of axially symmetric configurations of the problem: $\mathcal{S}$ is rotational symmetric with respect to the $x_3$-axis. See Figure 1.

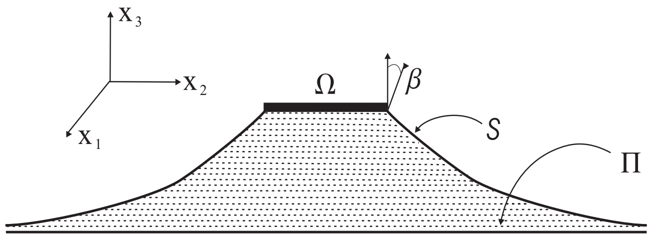


**Fig. 1** Description of the physical system

Moreover, we are interested by those stationary surfaces, if there exist, that are almost flat at the infinity of $\Pi$. This means that we seek *unbounded stationary surfaces $\mathcal{S}$* such that

$$\lim_{\substack{p\in\mathcal{S} \\ |\pi(p)|\to+\infty}} x_3(p) = 0,$$

where $\pi : \mathbb{R}^3 \to \Pi$ is the orthogonal projection onto $\Pi$. In the present work, we will show that such surface exist.

Besides the results that we shall state in the next section, it is worthwhile to announce here the following two conclusions. The first one shows that part of the space $\mathbb{L}^3$ can be parametrized by slices that are stationary surfaces. Exactly

[**C. 1**] *Given $R, \kappa > 0$ and $b \in \mathbb{R}$, it is possible to foliate the set of spacelike directions that go out from $\partial\Omega_{R,b}$ by a uniparametric family of unbounded stationary surfaces that are solutions of Equation* (1.1)*. Each leaf of the foliation is rotational symmetric with respect to the $x_3$-axis.*

The second consequence informs us that one slice of the above family is close to the level reference $\Pi$ at inflinity.

[**C. 2**] *Let us fix $R, \kappa > 0$. For each $b \in \mathbb{R}$, there exists an unbounded stationary rotational symmetric surface whose boundary is $\partial\Omega_{R,b}$ and asymptotic to $\Pi$ at infinity.*

## 2 Preliminaries and statement of results

For a spacelike surface $M$ in $\mathbb{L}^3$, the mean curvature $H$ of $M$ is given by $2H = \text{trace}\left(\mathrm{I}^{-1}\mathrm{II}\right)$. When $M$ is the graph of a smooth function $u = u(x_1, x_2)$, the spacelike condition is equivalent to $|Du| < 1$ and $H$ writes as

$$\text{div}(Tu) = 2H, \quad Tu = \frac{Du}{\sqrt{1-|Du|^2}}, \tag{2.1}$$

 

where the orientation $N$ on $M$ is $N = (Du, 1)/\sqrt{1-|Du|^2}$. According to (2.1), the Euler equation (1.1) converts into

$$\mathrm{div}(Tu) = \kappa u + \lambda, \quad \kappa > 0, \quad \lambda \in \mathbb{R}. \tag{2.2}$$

We will orient all the surfaces by the Gauss map $N$ according to $\langle N, (0,0,1)\rangle < 0$, that is, $N$ points upwards, that is, $N$ is *future-directed*. In particular, if $M$ is the graph of a smooth function $u$ defined in $\mathbb{R}^2 \setminus \Omega_{R,0}$, then $u$ satisfies (2.2). Let us remark that Equation (2.2) is a quasilinear elliptic equation of divergence-type due to the spacelike condition on the interface $\mathcal{S}$. This allows us, for example, to use classical maximum and comparison principles. On the other hand, the condition on the constancy of the angle of contact between the disc $\Omega$ and $\mathcal{S}$ along $\partial\mathcal{S}$ writes now as

$$-\langle N, N_\Omega\rangle = \frac{1}{\sqrt{1-|Du|^2}} = \cosh\beta \quad \text{on} \quad \partial\Omega_{R,0}. \tag{2.3}$$

In this article, we discuss the case that $\mathcal{S}$ is a graph on the plane $\Pi$ and rotational symmetric with respect to a straight-line $\mathcal{L}$ orthogonal to $\Pi$. This hypothesis on axial symmetry comes from some evidences in other similar configurations. For example, we have:

**Theorem** ([9]) *We consider a spacelike plane $\Pi$ in the Minkowski space $\mathbb{L}^3$. Then any bounded stationary surface whose boundary lies in $\Pi$ is rotational with respect to a straight-line orthogonal to $\Pi$ and the surface is a graph over $\Pi$. Moreover, any (nonempty) intersection with a parallel plane to $\Pi$ is a round circle. A similar statement is obtained for bounded stationary surfaces trapped in parallel (spacelike) planes.*

Under these assumptions, $\mathcal{S}$ is determined by the rotation of the profile of a function $u : [R,\infty) \to \mathbb{R}$ with respect to $\mathcal{L} = \{x \in \mathbb{R}^3; x_1 = x_2 = 0\}$ and $\mathcal{S}$ can be represented as $\mathcal{S} = \{(r\cos\theta, r\sin\theta, u(r)); r \in [R,m), \theta \in \mathbb{R}\}$, for some $m \leq \infty$. Equation (2.2) for the profile curve that defines the interface $\mathcal{S}$ become an ordinary differential equation given by

$$\frac{d}{dr}\left(\frac{ru'(r)}{\sqrt{1-u'(r)^2}}\right) = \kappa r\, u(r) + \lambda, \quad R < r < m, \tag{2.4}$$

and (2.3) is now

$$u'(R^+) = \tanh\beta. \tag{2.5}$$

Without loss of generality, and after a vertical translation, we assume in the present work that $\lambda = 0$. Given a positive number $\kappa$, we begin in Section 3 with the study of the exterior boundary value problem $\mathcal{P}$:

$$\left(\frac{ru'(r)}{\sqrt{1-u'(r)^2}}\right)' = \kappa r\, u(r), \quad r > R, \tag{2.6}$$

$$u(R) = b, \quad u'(R^+) = c. \tag{2.7}$$

Here $R > 0$, $b \in \mathbb{R}$ and $c \in (-1,1)$. The first result assures existence and uniqueness of the initial value problem $\mathcal{P}$.

**Theorem 2.1** *For each $(R,b,c)$, there exists a unique solution of $\mathcal{P}$. The maximal domain is $[R,\infty)$.*

We denote by $u = u(r;b,c)$ the dependence on the initial conditions. Next, we compare the solutions with respect to the constant $\kappa$ and the initial values.

**Theorem 2.2** *Let $\kappa_1$ and $\kappa_2$ be two positive constants. Denote $u_i = u_i(r)$ two solutions of (2.6)–(2.7) for $\kappa = \kappa_i$, $i = 1,2$. If $\kappa_1 < \kappa_2$, then*

$$u_1(r;b,c) < u_2(r;b,c) \quad \textit{and} \quad u_1'(r;b,c) < u_2'(r;b,c)$$

*for $r > R$.*

**Theorem 2.3** *Let $R > 0$. Consider $u = u(r; b, c)$ be the solution of $\mathcal{P}$. If $\eta > 0$, then*

$$u(r; b + \eta, c) - \eta > u(r; b, c),$$

*for $r > R$.*

It follows that if the curve $u(r; b + \eta, c)$ is moved rigidly downward a distance $\eta$, it will lie completely above the curve $u(r; b, c)$ except at the single point $(R, b)$ of contact.

In the last Section 4, we return our interest in those solutions that correspond with unbounded stationary surfaces that are almost horizontal at infinity. For this reason, we impose an aditional condition given by

$$\lim_{r\to\infty} u(r) = 0. \tag{2.8}$$

For a fixed number $b$, there exists a (unique) solution asymptotic to the plane at infinity.

**Theorem 2.4** *Let $R > 0$ and let $b \in \mathbb{R}$. There exists a unique value $\delta$, $\delta \in (-1, 1)$, such that the solution $u(r; b, \delta)$ of the initial value problem $\mathcal{P}$ satisfies*

$$\lim_{r\to\infty} u(r) = 0.$$

*Moreover, if $b > 0$, then*

$$\frac{-b(1+\sqrt{1+2\kappa R^2})}{\sqrt{R^2 + b^2(1+\sqrt{1+2\kappa R^2})^2}} < \delta < 0.$$

As a consequence of the classical maximum principle for elliptic equations, we conclude:

**Theorem 2.5** *Let $\kappa > 0$. Consider a solution $v$ of (2.2)–(2.3) on $\Omega_{R,0}$ such that $v(x) = b$ for all $x \in \partial\Omega_{R,0}$ and*

$$\lim_{|x|\to\infty} v(x) = 0.$$

*Then $v(x)$ depends only on $|x|$ and $v(|x|) = u(r; b, \delta)$, $r = |x|$, where $u$ is the function obtained in Theorem 2.4.*

**Remark 2.6** The boundary value problem (2.2) with $v = b$ along $\Omega_{R,0}$ is equivalent to the minimum problem

$$F(v) = \int_{\mathbb{R}^2\setminus\Omega_{R,0}} \left(\frac{\kappa}{2}v^2 - \sqrt{1-|Dv|^2}\right) dx \longrightarrow \min_{v\in\mathcal{C}},$$

where

$$\mathcal{C} = \left\{ v;\ \int_{\mathbb{R}^2\setminus\Omega_{R,0}} \left(v^2 + |Dv|^2\right) dx < \infty,\ v_{|\partial\Omega_{R,0}} = b;\ |Dv| \le 1 \text{ in } \mathbb{R}^2 \setminus \Omega_{R,0} \right\}.$$

The strong convexity of the functional $F$ guarantees an unique solution $v$ such that $v(x) \to 0$ for $|x| \to \infty$. Thus, $v$ must agree with the solution $u$ found in Theorem 2.4.

We continue in Section 4 obtaining results of monotony and continuity, as well as, estimates of the height of an unbounded stationary surface in terms of the angle of contact with the round disc. We hope that rotational symmetric stationary surfaces shall allow in the future the study for other more general configurations of the domain $\Omega$. We show an example of this giving an estimate of the contact angle in the case that $\Omega$ satisfies an interior sphere condition.

In Euclidean space, analogous problems have been studied by a number of authors: see [6] and references therein. While the analysis to follow is carried out using methods appropriate to the Euclidean case, the theorems yield new results when we consider Lorentzian setting. We point out that some differences appear in the Lorentzian setting, mainly due to the spacelike character of our surfaces. It is worthwhile to bring out two of them:

 

1. We prove that the maximal interval of a solution of (2.4)–(2.5) is $[R,\infty)$. On the contrary, the solutions of the (analogous) exterior boundary value problem are not defined in all real line, and we stop at points $r_1 < \infty$ such that $u'(r_1) = \infty$.
2. Our surfaces do not present vertical points. In particular, each solution goes out from the right solid cylinder $C = \Omega \times \mathbb{R}$ defined by $\Omega$. In contrast to this, in Euclidean setting, there exist stationary surfaces that reenter in $C$ and next, they go out from $C$ [12, 13, 14]. In these surfaces appear vertical points.

## 3 Existence of solutions of the exterior boundary value problem

Proof of Theorem 2.1. Equation (2.6) also writes as

$$u'' = \kappa u\big(1-u'^2\big)^{3/2} - \frac{u'\big(1-u'^2\big)}{r}. \tag{3.1}$$

As usual, set $x = u$, $y = u'$ and $f(r) = (x(r), y(r))$. Then a solution of the value problem $\mathcal{P}$ is equivalent to

$$f'(r) = F(r, f(r)), \quad F(r,x,y) = \left(y, \kappa x(1-y^2)^{3/2} - \frac{y(1-y^2)}{r}\right), \quad f(R) = (b,c)$$

where $F$ is $C^1$ in $(R,\infty)\times\mathbb{R}\times(-1,1)$. It follows from usual existence theorems for O.D.E. that we have local existence and uniqueness of solutions for each $(b,c)\in\mathbb{R}\times(-1,1)$. We prove that the maximal interval of existence is $[R,\infty)$. Integration of (2.4) from $R$ to $r$ leads to

$$\frac{u'(r)}{\sqrt{1-u'(r)^2}} = \frac{R}{r}\frac{c}{\sqrt{1-c^2}} + \frac{\kappa}{r}\int_R^r t u(t)\,dt.$$

Denote $v(r)$ the right side of this expression, that is,

$$v(r) = \frac{R}{r}\frac{c}{\sqrt{1-c^2}} + \frac{\kappa}{r}\int_R^r t u(t)\,dt, \tag{3.2}$$

for each $r > R$. Then

$$u'(r) = \frac{v(r)}{\sqrt{1+v(r)^2}}. \tag{3.3}$$

Consider $u$ a solution of (2.4)–(2.5) on the maximal interval $[R,m)$. If $m<\infty$, and combining (3.2) and (3.3), we would have $|v(r)| \to \alpha < 1$ as $r \to m$. Since $u(r) = b + \int_R^r u'(t)\,dt$, it follows that

$$|u(r)| \le |b| + \frac{|\alpha|}{\sqrt{1+\alpha^2}}|r-R|$$

and so, $|u(r)| \to M < \infty$ as $r \to m$. Standard argument says us then that we can extend the solution $u$ beyond of the point $r = m$. This contradiction implies that $m = \infty$. Moreover, we have continuity on the initial parameters. □

On the other hand, it is immediate that $u(r;-b,-c) = -u(r;b,c)$. Without loss of generality, throughout the remainder of the paper we shall assume that $b \ge 0$ is fulfilled. When $b = 0$, we also suppose $c \ge 0$. Notice that $u(r;0,0) = 0$. Next lemma will be useful for further results. The statement is the same that in Euclidean ambient [8]. We do the proof by completeness.

**Lemma 3.1** *Let $u_1$ and $u_2$ be two solutions of the O.D.E.* (2.6) *with $u_1(r_0) \le u_2(r_0)$, $u_1'(r_0) \le u_2'(r_0)$ and $u_1(r_0) + u_1'(r_0) < u_2(r_0) + u_2'(r_0)$ for some $r_0 \ge R$. Then $u_1 < u_2$ and $u_1' < u_2'$ for $r > r_0$.*

Proof. We have two possibilities. If $u_1'(r_0) < u_2'(r_0)$, then $u_1 < u_2$ and $u_1' < u_2'$ in some interval $(r_0, r_1)$. If $u_1'(r_0) = u_2'(r_0)$, then $u_1(r_0) < u_2(r_0)$. Taking into account (3.1), $u_1''(r_0) < u_2''(r_0)$ and, again, $u_1 < u_2$ and $u_1' < u_2'$ in some interval $(r_0, r_1)$. Anyway, we consider the maximal domain $(r_0, r_1)$ such that $u_1 < u_2$ and $u_1' < u_2'$. We prove that $r_1 = \infty$. On the contrary case, the function $u = u_2 - u_1$ satisfies $u(r_0) = 0$ and $u' > 0$ in the interval $(r_0, r_1)$. Thus, at the point $r = r_1$, $u'(r_1) = 0$, that is, $u_1'(r_1) = u_2'(r_1)$. Then (3.1) implies $u_1''(r_1) < u_2''(r_1)$, which means $u_1'(r_1) < u_2'(r_1)$: a contradiction. □

By setting $u_1 = 0$, we conclude from Lemma 3.1

**Corollary 3.2** *Let $u$ be a solution of the O.D.E.* (2.6) *such that for some $r_0 \geq R$, $u(r_0) \geq 0$, $u'(r_0) \geq 0$ and $u(r_0) + u'(r_0) > 0$. Then $u(r) > 0$ and $u'(r) > 0$ for $r > r_0$.*

We study the profiles of solutions of the initial value problem $\mathcal{P}$.

**Theorem 3.3** *Let $u(r; b, c)$ be a solution of $\mathcal{P}$.*

1. *If $c \geq 0$ (with $c > 0$ if $b = 0$), then $u(r; b, c)$ is strictly increasing.*
2. *If $c < 0$ and for some $r_0 > R$, $u'(r_0) \geq 0$, then $u(r; b, c)$ is positive and there exists $r_1$, $R < r_1 \leq r_0$, such that $u$ is strictly decreasing on $(R, r_1)$ and strictly increasing in $(r_1, \infty)$. At $r = r_1$, $u$ presents a minimum.*

*In both cases, $u$ is a convex function and*

$$\lim_{r\to\infty} \frac{u(r)}{r} = \lim_{r\to\infty} u'(r) = 1.$$

P r o o f. The first item is a consequence of Corollary 3.2 by taking $r_0 = R$. For the second one, we know that $u$ is strictly decreasing of some interval on the right of $r = R$. By contradiction, we assume that $u$ is not positive and set $s > R$ the first zero of $u$. By uniqueness of solutions, $u'(s) \neq 0$. If $u'(s) > 0$, Corollary 3.2 implies that $u$ is positive on $(s, \infty)$. Thus $s$ would be a minimum of $u$ and so, $u''(s) \geq 0$: this is a contradiction to Equation (3.1). Consequently $u'(s) < 0$. The same argument as in Lemma 3.1 and Corollary 3.2 for the function $-u$ yields that the both $u$ and $u'$ are negative in $(s, \infty)$. Thus $r_0 \in (R, s)$. In particular, $u(r_0) > 0$. By applying Corollary 3.2 again, $u$ is positive in $(r_0, \infty)$: a contradiction. As a consequence, $u$ is always positive in the interval of definition.

Therefore, $u(r_0) > 0$ and $u$ is strictly increasing in $(r_0, \infty)$ by Corollary 3.2. Since $u'(R) < 0$, there exists $r_1$, $R < r_1 \leq r_0$, such that $u'(r_1) = 0$. Let $r_1$ be the first critical point of $u$. As $u(r_1) > 0$, Corollary 3.2 yields $u' > 0$ in $(r_1, \infty)$, proving that $r_1$ is the only zero of $u$ and so, the (unique) minimum of $u$.

We show now that $u'' > 0$. We treat both cases, with $r_1 = R$ if $c \geq 0$. It is clear that in the interval $[R, r_1]$, $u' \leq 0$ and (3.1) yields $u'' > 0$. Thus, it suffices to study the case that $r > r_1$. This occurs if $c \geq 0$ but an integration of (2.6) between $r_1$ and $r$ gives

$$v(r) = \frac{\kappa}{r} \int_{r_1}^{r} t u(t)\, dt < \kappa u(r) \frac{r^2 - r_1^2}{2r}, \tag{3.4}$$

where we have bounded the integrand by $u(r)$ since $u$ is monotone increasing. On the other hand, equation (2.6) writes $(rv)' = \kappa r u(r)$. Then

$$v'(r) + \frac{v(r)}{r} = \kappa u(r).$$

From this equation and the inequality (3.4) we obtain

$$v'(r) = \kappa u(r) - \frac{v(r)}{r} > \kappa u(r) - \kappa u(r) \frac{r^2 - r_1^2}{2r^2} > \kappa u(r) - \frac{\kappa u(r)}{2} > 0.$$

Since $v' = u''/\left(1 - u'^2\right)^{3/2}$, we conclude that $u'' > 0$.

For the last statement, consider $u$ in $(r_1, \infty)$. Since $u$ is increasing, we bound (3.4) from below by $t u(r_1)$. This leads to

$$v(r) > \kappa u(r_1) \frac{r^2 - r_1^2}{2r} \longrightarrow \infty$$

as $r \to \infty$. Thus (3.3) implies $u' \to 1$ as $r \to \infty$. Now L'Hôpital theorem yields the other limit. □

As consequence, we have

**Corollary 3.4** *Let $u(r;b,c)$ be a solution of $\mathcal{P}$. Assume $b > 0$. If there exists some point $r_0$ such that $u(r_0) = 0$, then $u$ is monotone decreasing in $(R,\infty)$, $u$ has a unique zero and*

$$\lim_{r\to\infty} = \frac{u(r)}{r} = \lim_{r\to\infty} = u'(r) = -1.$$

Proof. We know from Theorem 3.3 that $c < 0$ and $u' < 0$ in $[R,\infty)$. If $u(r_0) = 0$, then the monotonicity of $u$ yields that $r_0$ is the unique zero. Corollary 3.2 implies that $u < 0$ in $(r_0,\infty)$. With similar arguments as in Theorem 3.3 for the function $-u$, we conclude the asymptotic behaviour. □

**Remark 3.5** Some positive solutions of Theorem 3.3 show examples of rotational symmetric compact stationary surfaces $M$ in $\mathbb{L}^3$ whose boundary is formed by two axial concentric circles in parallel planes but $M$ does not lie completely in the slab domain determined by these planes: it suffices to consider a positive solution $u(r;b,c)$ that initially decreases near the value $r = R$ and to restrict $u$ in the interval $[R,m]$, where $m$ is any value with $u(m) > b$.

Let $u = u(r;b,c)$ be a positive solution of $\mathcal{P}$ and assume that $c < 0$. We know from Theorem 3.3 that either $u' < 0$ in $[R,r_1)$ with $u'(r_1) = 0$, or $u'(r) < 0$ for all $r \geq R$. Consider the interval $[R,r_1)$, with $r_1 = \infty$ eventually. As $u$ is a convex function, $u'$ is monotone increasing, and it follows

$$\frac{u''(r)}{(1-u'(r)^2)^{3/2}} = \kappa u(r) - \frac{u'(r)}{r\sqrt{1-u'(r)^2}} < \kappa b - \frac{c}{R\sqrt{1-c^2}} := M.$$

Integrating between $R$ and $r_1$, we obtain

$$\frac{u'(r)}{\sqrt{1-u'(r)^2}} < \frac{c}{\sqrt{1-c^2}} + M(r-R).$$

The right side in the above inequality is negative if $r < r_2$ with

$$r_2 = R - \frac{c}{M\sqrt{1-c^2}}. \tag{3.5}$$

Hence $R < r_2 \leq r_1$. Because $0 < \sqrt{1-u'^2} < 1$, in the interval $(R,r_2)$ we have

$$u'(r) < \frac{c}{\sqrt{1-c^2}} + M(r-R).$$

We do a new integration in this inequality. For any $r$, with $R < r \leq r_2$, one obtains

$$u(r) < b + \frac{c}{\sqrt{1-c^2}}(r-R) + \frac{M}{2}(r-R)^2.$$

Setting $r = r_2$, we infer

$$b + \frac{c}{\sqrt{1-c^2}}(r_2-R) + \frac{M}{2}(r_2-R)^2 > 0.$$

With the value of $r_2$ defined in (3.5), put

$$\delta_-(b) = \frac{-\alpha}{\sqrt{1+\alpha^2}}, \quad \alpha = \frac{b}{R}\left(1+\sqrt{1+2\kappa R^2}\right).$$

Then a necessary condition for $u(r)$ to be positive in its domain is that

$$c > \delta_-(b).$$

Assume now that $u(r)$ is negative in some point. In this case, we know that $c < 0$ and $u(r_0) = 0$, for some $r_0 > R$. Furthermore, $r_0$ is the unique zero of $u$ and $u'(r_0) < 0$. Since $u'$ is negative, one obtains from equation (2.6) that

$$\frac{u''(r)}{(1-u'(r)^2)^{3/2}} > \kappa u(r).$$

Multiplying by $u'(r)$ and integrating between $R$ and $r_0$, we deduce

$$\frac{1}{\sqrt{1-u'(r_0)^2}} < \frac{1}{\sqrt{1-c^2}} - \frac{\kappa b^2}{2}.$$

Since the left side is greater than 1, we conclude $c < \delta_+(b)$ with

$$\delta_+(b) := \frac{-b}{2+\kappa b^2}\sqrt{\kappa b^2 + 4\kappa}.$$

This gives a necessary condition for $u$ to be not positive in the maximal interval. Consequently,

**Proposition 3.6** *Let $b > 0$ be a fixed number and let $u = u(r; b, c)$ be a solution of $\mathcal{P}$.*

1. *If $c \geq \delta_+(b)$, the solution $u(r; b, c)$ is positive in its domain.*
2. *If $c \leq \delta_-(b)$, the solution $u(r; b, c)$ is not positive in some point.*

We finish this section proving Theorems 2.2 and 2.3.

Proof of Theorem 2.2. We consider again the function $u = u_2 - u_1$ defined in $[R, \infty)$. Then $u$ satisfies $u(R) = u'(R) = 0$. From (3.1) and $0 < \kappa_1 < \kappa_2$, we obtain $u''(R) > 0$. This implies that $u$ is positive on the left of $r = R$. There exists a maximal $r_1 \leq \infty$ such that $u_1 < u_2$ and $u_1' < u_2'$ in the interval $(R, r_1)$. If $r_1 < \infty$, and because $u$ is positive in $r = r_1$, we conclude that $u_1(r_1) < u_2(r_1)$ and $u_1'(r_1) = u_2'(r_1)$. By using (3.1) again at $r = r_1$, one obtains $u_1''(r_1) < u_2''(r_1)$, that is, $u''(r_1) > 0$. Consequently, $u'$ is strictly increasing around the point $r = r_1$, in contradiction to the maximal property of $r_1$. This implies that $r_1 = \infty$. □

Proof of Theorem 2.3. Lemma 3.1 says that $u(r; b, c) < u(r; b+\eta, c)$ and $u'(r; b, c) < u'(r; b+\eta, c)$ for $r > R$. Define the function $w(r) = u(r; b+\eta, c) - u(r; b, c) - \eta$. Then $w(R) = w'(R) = 0$. By using (3.1), we have

$$u''(R; b+\eta, c) = u''(R; b, c) + \kappa\eta\big(1 - c^2\big) > u''(R; b),$$

and thus $w''(R) > 0$. This implies that $w$ is strictly increasing on the right of $R$. Let $(R, r_1)$ be the maximal interval where $w$ is positive. If $r_1 < \infty$, then $w'(r_1) = 0 < w(r_1)$, which is false. This contradiction proves that $r_1 = \infty$, and consequently, $w > 0$, as was to be shown. □

## 4 Unbounded stationary surfaces asymptotic to a spacelike plane

As a consequence of the above results, it remains the study of case $c < 0$ and that $u$ is positive and monotone decreasing.

**Theorem 4.1** *Given $R, b > 0$ and $c < 0$ such that the solution $u(r; b, c)$ satisfies*

$$u(r) > 0, \quad \textit{and} \quad u'(r) < 0, \quad \textit{for} \quad r \geq R. \tag{4.1}$$

*Then*

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} u'(r) = 0. \tag{4.2}$$

Proof. We know that $u' < 0$ and that $u$ is monotone decreasing. By (3.1) and (4.1), $u'' > 0$, that is, $u'$ is strictly increasing on $(R, \infty)$. Moreover we know that the number 0 is a lower and upper bound for $u$ and $u'$ respectively. From the theory of ordinary differential equations, the limits

$$\lim_{r\to\infty} u(r) = l, \quad \lim_{r\to\infty} u'(r) = L$$

exist and are finite, with $L \leq 0 \leq l$. In particular, $u$ is a bounded function and so, $L = 0$. From (3.1), $u''(\infty) = \kappa l$. Again, as $u'$ is bounded, $\lim_{r\to\infty} u''(r) = 0$ and since $\kappa > 0$, $l = 0$. □

 

We summarize the above results as follows. Let us fix $b > 0$ and we take values $c$ varying from $c = +1$ to $c = -1$. Notice that the solution $u(r; b, c)$ is defined for all $r \geq R$. For nonnegative values of $c$, we know that $u$ is monotone increasing on $r$ and $u$ is positive in all its domain. As we are going to decrease the value of $c$, that is, the shooting slope, $u$ decreases on the right of $r = R$ until a minimum. Next, $u$ increases again toward infinity. The function remains positive. However, it arrives a critical slope $\delta$ such that the function $u$ decreases monotonically towards $-\infty$. See Figure 2. By using a shooting argument, we shall prove that for the slope $c = \delta$, the function $u(r; b, \delta)$ is positive but asymptotic to the axis $r$ according (4.2). This is contained in Theorem 2.4. If $\mathcal{S}_c$ denotes the surface obtained by rotating $u(r; b, c)$ with respect to the $x_3$-axis, then the family $\mathcal{S}_c$, $c \in (-1, 1)$, is a foliation of the set of spacelike directions from $\partial\Omega_{R,b}$, where $c$ indicate the slope of each spacelike direction. Only the surface $\mathcal{S}_\delta$ is asymptotic to $\Pi$ at infinity. This was stated in [C. 1] and [C. 2] of Section 1.

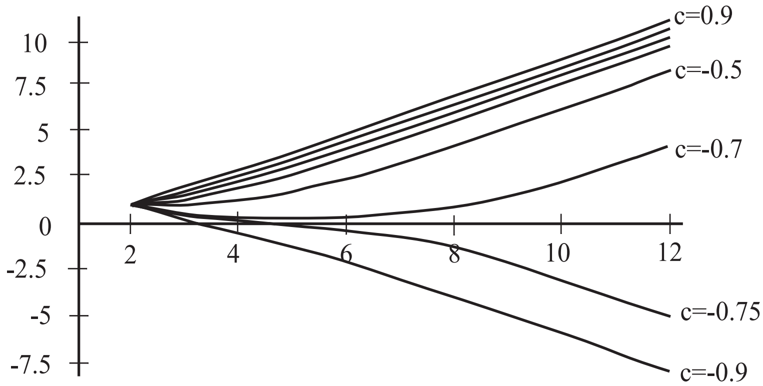


**Fig. 2** For $c \in \{0.9, 0.6, 0.3, 0, -0.5, -0.7, -0.75, -0.9\}$, we show the family of solutions $u = u(r; 1, c)$, for $R = 2$ and $\kappa = 1/2$. The profile curves corresponding to $c = -0.5$ and $c = -0.7$ have negative shooting slope but they lie over the $r$-axis

P r o o f o f T h e o r e m 2.4. If $b = 0$, then $\delta = 0$. Consider $b > 0$. Set

$$A^+ = \{c \in (-1, 1);\ \text{the solution } u(r; b, c) \text{ is positive in } [R, \infty)\},$$
$$A^- = \{c \in (-1, 1);\ \text{the solution } u(r; b, c) \text{ is negative in some point of } [R, \infty)\}.$$

Proposition 3.6 assures that both sets are not empty, with $\delta_+(b) \in A^+$, $\delta_-(b) \in A^-$. Furthermore, and a consequence of the preceding results, $(-1, 1) = A^+ \cup A^-$, $A^+ \cap A^- = \emptyset$. Also, we have $[-\epsilon, 1) \subset A^+$, for some $\epsilon > 0$. On the other hand, Lemma 3.1 implies that if $c \in A^-$ and $c' < c$, then $c' \in A^-$. In the same way, if $c \in A^+$ and $c' > c$, then $c' \in A^+$. This shows that both $A^+$ and $A^-$ are intervals of the real line. Set $\delta = \sup A^-$.

*Claim.* The number $\delta$ satisfies $\delta \in A^+$.

On the contrary case, assume $\delta \in A^-$. Then there exists $r_1 > R$ such that $u(r_1; b, \delta) < 0$. By the continuity of parameter for O.D.E., there exists $c > \delta$ such that $u(r_1; b, c)$ is also negative, proving then that $c \in A^-$, which is a contradiction with the definition of $\delta$. This proves the claim.

*Claim.* The solution $u(r; b, \delta)$ satisfies $u' < 0$ for any $r \geq R$.

By contradiction, we suppose that there exists $r_0 > R$ such that $u'(r_0; b, \delta) = 0$. Since $u(r_0) > 0$, Corollary 3.2 and Theorem 3.3 imply $u' > 0$ for any $r > r_0$, and a consequence, the point $r_0$ is the unique minimum of $u$. Therefore, $u(r) \geq u(r_0) > 0$ for any $r \geq R$. By using the continuity on the initial values, there exists $c$ near to $\delta$, and $c < \delta$ such that $u(r; b, c)$ is positive in all its domain. Thus $c \in A^+$: this contradicts the fact that $\delta$ is the supremum of $A^-$.

As conclusion of both claims, the solution $u = u(r; b, \delta)$ satisfies (4.1) and consequently, by Theorem 4.1, it satisfies (4.2). This shows the existence.

We prove the uniqueness. Let $u(r; b, c)$ be other solution of (2.6) that satisfies (4.2). Then $c \notin A^-$ because in such case, $u$ would be strictly decreasing and goes to $-\infty$ as $r \to \infty$: see the argument of Theorem 3.3 for the function $-u$. Therefore $c \in A^+$. If $c > \delta$, then $u'(R; b, c) > u'(R; b, \delta)$ and Lemma 3.1 implies that

$$u(r; b, c) > u(r; b, \delta), \quad u'(r; b, c) > u'(r; b, \delta),$$

for any $r > R$. Then the function $u(r; b, c) - u(r; b, \delta)$ vanishes at $r = R$, it is positive for $r > R$ and strictly increasing. Thus $u(r; b, c)$ is bounded away from 0, in contradiction with (4.2). □

Numerically (for example with the Mathematica software), it is difficult to obtain the exact value $\delta$ due to the discretization in the computations. For fixed positive numbers $R, b$ and $\kappa$, we can approximate values of $c$ near to $\delta = \delta(R, b, \kappa)$, where the solution $u(r; b, c)$ is very close to the $r$-axis. See Figure 3. The value obtained for $c$ is greater than $\delta$. For the same values as above, in Figure 4 we show as $u(r)$ comes back to increase to $\infty$ as $r \to \infty$. However, $u$ is very near to the $r$-axis in the interval $(10, 24)$. In fact, it would be possible to refine the computations to obtain the profile as close to 0 as we desire.

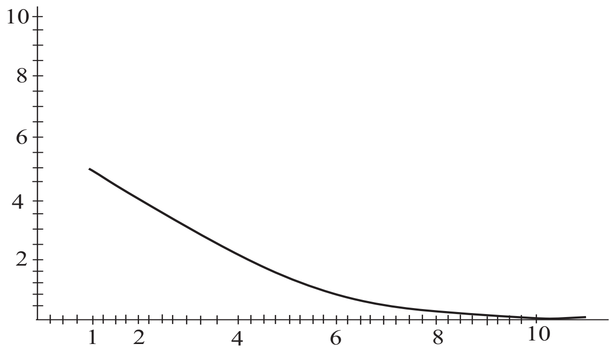


**Fig. 3** Here $R = 1$ and $\kappa = 1/4$. The solution is $u(r; 5, -0.99679848)$. We show the curve in the interval $[1, 10]$.

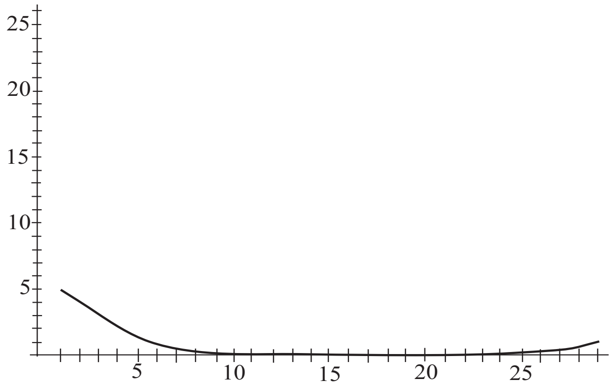


**Fig. 4** The same solution as Fig. 3 in the interval $[1, 29]$

P r o o f o f T h e o r e m 2.5. The proof is a consequence of the maximum principle (see for example, [7] as general reference, and [5] in the capillary context). We do it from a geometric viewpoint. Let $u = u(r; b, \delta)$ be the solution given by Theorem 2.4 and denote $\Sigma_u$ and $\Sigma_v$ the surfaces determined by the functions $u$ and $v$ respectively. In the same way, let us denote $H_u$ and $H_v$ their mean curvatures. Recall that the spacelike surfaces are future-directed oriented. By a reasoning by the absurd, we assume that $\Sigma_v \neq \Sigma_u$. Without loss of generality, we suppose that there are points of $\Sigma_v$ below $\Sigma_u$.

First, we claim that the surface $\Sigma_v$ lies completely over the plane $\Pi$. On the contrary case, and because $\Sigma_v$ is close to $\Pi$ at infinity, there would be points $p \in \Sigma_v$ with horizontal tangent plane and with non-positive third coordinate $x_3$. We compare then $\Sigma_v$ with the plane $\Pi_o = \{x_3 = x_3(p)\}$ in a neighborhood of $p$. If $x_3(p) < 0$, then $H_v(p) = \kappa v(p) < 0$: this is impossible by comparing with $\Pi_o$. If $x_3(p) = 0$, the Hopf maximum principle implies that an open set of $\Sigma_v$ around $p$ is included in $\Pi_o = \Pi$, that is, $v = 0$ in some open disc: contradiction.

Once proved that $\Sigma_v \subset \{x_3 > 0\}$, we move down $\Sigma_u$ until $\partial\Sigma_u$ lies in the halfspace $\{x_3 < 0\}$. Then we move up $\Sigma_u$ until to arrive the original position. Since we have assumed the existence of points of $\Sigma_v$ with lower height than points of $\Sigma_u$ (with the same orthogonal projection on $\Pi$), we infer that before $\Sigma_u$ arrives to the height $x_3 = b$, $\Sigma_u$ intersects $\Sigma_v$ at a first time. Let $p$ be an intersection point between both surfaces ($p \notin \Omega$). Then around $p$, $\Sigma_v$ would lie strictly over $\Sigma_u$ but both surfaces have the same mean curvature at $p$: the maximum principle together a continuation argument proves that both surfaces agree, which is false. This contradiction proves the theorem. □

The same arguments used in this proof allows to estimate the hyperbolic angle of contact between a stationary surface and a domain $\Omega$ more general than a round disc.

**Corollary 4.2** *Let $\kappa > 0$. Let $b$ and $\sigma$ be positive numbers. Then there exists a constant $\beta_0 = \beta_0(\kappa, b, \sigma)$ such that the following holds. Let $D$ be a bounded smooth domain included in the plane $P = \{x_3 = b\}$ that satisfies an interior circle condition $\sigma > 0$. Then any unbounded stationary surface between $D$ and $\Pi$ obtained as the graph of a solution $v$ of* (2.2)–(2.3) *makes an contact angle $\beta$ with $D$ along its boundary that satisfies $\beta \leq \beta_0$.*

A similar lower bound for $\beta$ is obtained if $D$ satisfies an exterior circle condition.

P r o o f. The interior circle property means that given $p \in \partial D$, there exists a round disc $\Omega_\sigma \subset D$ of radius $\sigma$ with $p \in \partial\Omega_\sigma$. Consider a round disc of radius $\sigma$ in the plane $P$ whose center lies at the line $\mathcal{L}$. Denote by $\Sigma_u$ the corresponding unbounded surface obtained by Theorem 2.4. This gives an angle $\beta_0$ of contact, with $\tanh\beta_0 = \delta$, according to the notation in Theorem 2.4. Given $p \in \partial D$, and by a horizontal translation, we put $\Sigma_u$ so $\partial\Sigma_u = \partial\Omega_\sigma$. An argument as in Theorem 2.5 proves that $\Sigma_u$ lies below $\Sigma_v$, and this yields the desired estimate for the angle of contact at the point $p$. □

We prove the monotony of solutions of the problem $\mathcal{P}$ and the condition (4.2) with respect to the initial values:

**Corollary 4.3** *Consider $\kappa, R > 0$. Let $b_1$ and $b_2$ be two positive numbers and $u(r; b_1, \delta_1)$ and $u(r; b_2, \delta_2)$ the corresponding solutions of $\mathcal{P}$ with the condition* (4.2)*. If $b_1 < b_2$, then*

$$u(r; b_1, \delta_1) < u(r; b_2, \delta_2), \quad u'(r; b_1, \delta_1) > u'(r; b_2, \delta_2),$$

*for $r > R$.*

P r o o f. Denote $u_i = u(r; b_i, \delta_i)$, $i = 1, 2$. We claim that there is not $r_0 \geq R$ such that $u_1(r_0) \leq u_2(r_0)$ and $u_1'(r_0) \leq u_2'(r_0)$. On the contrary case, some one of both inequalities is strict by the uniqueness of solutions of $\mathcal{P}$. In such case, Lemma 3.1 assures $u_1 < u_2$ and $u_1' < u_2'$ in $(r_0, \infty)$. Thus $u_2$ is bounded away from 0, in contradiction with (4.2). Since $b_1 < b_2$, we conclude $u_1 < u_2$ and $u_1' > u_2$ for any $r > R$. □

If we denote $\delta = \delta(R, b, \kappa)$ the dependence of the shooting angle on the initial value, we prove the continuity on their variables. Since we have dependence of the solutions of (2.6) with respect to $r$, we only consider the dependence of $\delta$ with respect to $(b, \kappa)$. With the above notations, we have

**Theorem 4.4** *The function $\delta = \delta(b, \kappa)$ is continuous on $(b, \kappa)$.*

P r o o f. It suffices to consider $b \geq 0$. Let $b_n \to b$ and $\kappa_n \to \kappa$. If $b = 0$, then $\delta_+(b_n), \delta_-(b_n) \to 0$. By the proof of Theorem 2.4, $\delta(b_n, \kappa_n) \to 0$ and $0 = \delta(0, \kappa)$. Thus, we assume that $b > 0$. Again, the same proof yields $\delta_-(b_n, \kappa_n) < \delta(b_n, \kappa_n) \leq \delta_+(b_n, \kappa_n)$. In particular, the sequence $\delta(b_n, \kappa_n)$ is bounded. Let $\lambda$ be a limit point, that is, assume there exists a subsequence (labeling in the same way) such that $\delta(b_n, \kappa_n) \to \lambda$. Assume by contradiction that $\lambda \neq \delta(b, \kappa)$ and consider the solution $u(r; b, \lambda, \kappa)$, assured by Theorem 2.1. By Theorems 3.3 and 2.4, there exists $r_o$ such that $u(r_o; b, \lambda, \kappa) < 0$ or $u'(r_o; b, \lambda, \kappa) > 0$. In the first case, and by the continuity with respect to the initial conditions, there exists $(b_n, \delta(b_n, \kappa_n), \kappa_n)$ close to $(b, \lambda, \kappa)$ such that $u(r_o; b_n, \delta(b_n, \kappa_n), \kappa_n) < 0$, which contradicts the properties of the solutions of Theorem 2.4. With a similar argument, it is impossible the second possibility. This contradiction shows the theorem. □

 

Now, we shall give an estimate of the rise for an unbounded stationary surface. The next calculations are similar as in Euclidean space [6, 12, 13]. Consider $u = u(r; b, c)$ a solution of (2.6)–(2.7) with the condition (4.2). Denote by $\psi(r)$ the hyperbolic angle that makes $u$ with the $r$-axis at each point $u(r)$. Setting

$$\sinh\psi(r) = v(r) = \frac{u'(r)}{\sqrt{1 - u'(r)^2}},$$

Equation (2.6) writes as

$$\frac{\sinh\psi}{r} + (\sinh\psi)' = \kappa u.$$

As $u' < 0$, we can take $u$ a new variable, $\psi = \psi(u)$ and so,

$$\frac{\sinh\psi}{r} + (\cosh\psi)_u = \kappa u.$$

Let us integrate between $u(r)$ and $u(R) = b$:

$$\kappa\frac{b^2 - u(r)^2}{2} = \int_{u(r)}^{b} \frac{\sinh\psi}{r}\, du + \cosh\beta - \cosh\psi(r).$$

Letting $r \to \infty$, and by (4.2), we conclude:

$$\frac{\kappa b^2}{2} = \int_0^b \frac{\sinh\psi}{r}\, du + \cosh\beta - 1. \tag{4.3}$$

We bound the integrand in two ways. Because $u' < 0$, the integrand is negative and so, we have

$$\frac{\kappa b^2}{2} < \cosh\beta - 1.$$

Hence that

$$b < \sqrt{\frac{2}{\kappa}(\cosh\beta - 1)}.$$

On the other hand, $(\sinh\psi)' > 0$ since $u'' > 0$. Thus $\sinh\beta < \sinh\psi(r)$ and because $\sinh\psi < 0$, we conclude

$$\frac{\sinh\beta}{r} < \frac{\sinh\psi}{R} < \frac{\sinh\psi}{r}.$$

By introducing this inequality into (4.3), we arrive

$$\frac{\kappa b^2}{2} > \frac{\sinh\beta}{R} b + \cosh\beta - 1.$$

Hence we can obtain a lower bound for $b = u(R)$. As conclusion,

**Theorem 4.5** *Let $\kappa > 0$ and let $\Omega$ be a round disc of radius $R$. Consider an unbounded stationary surface obtained by moving up $\Omega$ a distance $h > 0$ from the plane $\Pi$. Assume that $\beta$ is the hyperbolic angle of contact with $\Omega$ along its boundary. Then the height $h$ of the disc satisfies the inequalities:*

$$\frac{\sinh\beta}{\kappa R} + \sqrt{\frac{2}{\kappa}(\cosh\beta - 1) + \left(\frac{\sinh\beta}{\kappa R}\right)^2} < h < \sqrt{\frac{2}{\kappa}(\cosh\beta - 1)}.$$

*As consequence,*

$$b = \sqrt{\frac{2}{\kappa}(\cosh\beta - 1)} + O\left(\frac{1}{R}\right) \quad \text{as} \quad R \longrightarrow \infty.$$

In Table 1, we present some numerical computations of the contact angle $\beta$.

 

| $R$ | $b$ | $\delta = \tanh \beta$ | lower estimate | height $h$ | upper estimate |
|---|---|---|---|---|---|
| 1 | 1 | -0.872325 | 0.512319 | 1 | 1.44589 |
| | 2 | -0.98314 | 0.775264 | 2 | 2.98958 |
| | 3 | -0.996532 | 0.887125 | 3 | 4.69422 |
| | 4 | -0.999024 | 0.937328 | 4 | 6.57797 |
| 2 | 1 | -0.82471 | 0.708922 | 1 | 1.23949 |
| | 2 | -0.971606 | 1.21306 | 2 | 2.54026 |
| | 3 | -0.993449 | 1.51804 | 3 | 3.93715 |
| | 4 | -0.997997 | 1.69512 | 4 | 5.44219 |
| 3 | 1 | -0.802982 | 0.798854 | 1 | 1.16434 |
| | 2 | -0.965046 | 1.44425 | 2 | 2.37303 |
| | 3 | -0.991453 | 1.91056 | 3 | 3.65095 |
| | 4 | -0.997268 | 2.23231 | 4 | 5.00739 |

**Table 1** Values of the angle of contact $\beta$ for an unbounded stationary surface and comparison of the height estimates obtained in Theorem 4.5. Here $\kappa = 1$.

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