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ON LINEAR WEINGARTEN SURFACES

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In this paper, we study surfaces in Euclidean 3-space that satisfy a Weingarten condition of linear type as $\kappa_1 = m\kappa_2 + n$, where m and n are real numbers and κ_1 and κ_2 denote the principal curvatures at each point of the surface. We investigate the existence of such surfaces parametrized by a uniparametric family of circles. We prove that the only surfaces that exist are surfaces of revolution and the classical examples of minimal surfaces discovered by Riemann. The latter situation only occurs in the case $(m, n) = (-1, 0)$.

Keywords: Weingarten surface; cyclic surface; Riemann type.

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1. Introduction

A surface S in Euclidean 3-space $\mathbb{R}^3$ is called a *Weingarten surface* if there is some relation between its two principal curvatures κ_1 and κ_2, that is to say, there is a smooth function W of two variables such that $W(\kappa_1, \kappa_2) = 0$. The classification of the Weingarten surfaces in Euclidean space is almost completely open today. These surfaces were introduced by the very Weingarten [18, 19] in the context of the problem of finding all surfaces isometric to a given surface of revolution. Along the history they have been of interest for geometers: for example, [2, 7, 8, 17] and more recently, [6, 9, 16]. Applications of Weingarten surfaces on computer aided design and shape investigation can seen in [1]. In this work, we study Weingarten surfaces that satisfy the simplest case for W, that is, that W is linear:

$$\kappa_1 = m\ \kappa_2 + n, \tag{1.1}$$

where m and n are two constant real numbers. We say then that S is a *linear Weingarten surface* and we abbreviate by an LW-surface. In particular, umbilical surfaces $(m, n) = (1, 0)$ or constant mean curvature surfaces $(m = -1)$ are LW-surfaces. Throughout this work, we exclude the case that one of the principal curvatures is zero, that is, we shall assume that $m \neq 0$.

It is well-known that surfaces with constant mean curvature are motivated by physics and arise from the variational principle of minimizing area keeping the volume fixed (when the constraint on the volume is omitted, then we get minimal surfaces). In a more general sense, linear Weingarten surfaces satisfying the restriction $\kappa_1 = m\kappa_2$ are again solutions of a variational problem. Physically, and for the case that the surfaces are rotational, they are mathematical models in the problem of finding the equilibrium shape of strained balloons. See [12, 13].

Among all LW-surfaces, the class of surfaces of revolution are particularly studied because in such case, Eq. (1.1) leads to an ordinary differential equation. Its study is then simplified to find the profile curve that defines the surface [8]. On the other hand, if S is a closed LW-surface of genus zero, it must be a surface of revolution [17]. See generalizations in [9].

The aim of this paper is the search of new LW-surfaces that generalize the surfaces of revolution. In this sense, we give the following

Definition 1.1. A cyclic surface in Euclidean space $\mathbb{R}^3$ is a surface determined by a smooth uniparametric family of pieces of circles.

See [4, 5]. In particular, surfaces of revolution and tubes are cyclic surfaces. The motivation of the present work comes from what happens for the family of surfaces with constant mean curvature. When the mean curvature vanishes on the surface, that is, if the surface is minimal, $(m, n) = (-1, 0)$, the only rotational minimal surface is the catenoid. Riemann found all non-rotational minimal surfaces foliated by circles in parallel planes [15]. Enneper proved that for a cyclic minimal surface, the planes containing the circles must be parallel [4, 5] and then, it is one of the examples obtained by Riemann. When the mean curvature is a non-zero constant, $(m, n) = (-1, n)$ and $n \neq 0$, Nitsche proved that a cyclic surface must be a surface of revolution [14], whose classification is well-known [3].

In this paper, we study cyclic LW-surfaces. We call that a cyclic surface is of Riemann-type if the planes containing the circles of the foliation are parallel. Our interest in this work is twofold. First, we want to know if a cyclic LW-surface must be of Riemann-type. In this sense, we prove,

S.1: *A cyclic LW-surface with $(m, n) = (m, 0)$ must be of Riemann-type.*

The restriction $n = 0$ is merely technical since, as we will see, the proof involves long computations that in the case $n \neq 0$ become very difficult to manage. However, we hope that the same result holds for the general case of $n \neq 0$. On the other hand, and assuming now that the planes are parallel, we look for new LW-surfaces. However, we conclude then that

S.2: *Besides the surfaces of revolution, the only LW-surfaces of Riemann-type with arbitrary pair (m, n) are the classical Riemann examples of minimal surfaces, that is, if $(m, n) = (-1, 0)$.*

Recall that the Riemann examples play a major role in the theory of minimal surfaces and have been extensively characterized in the literature (see, for example, [10, 11]). Therefore, the statement S.2 gives a new particularity of the Riemann examples in the family of LW-surfaces. See Corollary 4.1.

Remark 1.1. Whenever we talk of an LW-surface, we exclude the umbilical case, that is, $(m, n) = (1, 0)$. Moreover, we point out that any uniparametric family of (non-necessary parallel) planes intersects a sphere into circles.

2. Preliminaries

In this section, we fix some notation on local classical differential geometry of surfaces. Let S be a surface in $\mathbb{R}^3$ and consider $\mathbf{X} = \mathbf{X}(u, v)$ a local parametrization of S defined in the (u, v)-domain. Let $\mathbf{N}$ denote the unit normal vector field on S given by

$$\mathbf{N} = \frac{\mathbf{X}_u \wedge \mathbf{X}_v}{|\mathbf{X}_u \wedge \mathbf{X}_v|}, \quad \mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}, \quad \mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v},$$

where $\wedge$ stands the cross product of $\mathbb{R}^3$. In each tangent plane, the induced metric $\langle , \rangle$ is determined by the first fundamental form

$$I = \langle d\mathbf{X}, d\mathbf{X} \rangle = E du^2 + 2F du dv + G dv^2,$$

with differentiable coefficients

$$E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle, \quad F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle, \quad G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle.$$

The shape operator of the immersion is represented by the second fundamental form

$$II = -\langle d\mathbf{N}, d\mathbf{X} \rangle = e\ du^2 + 2f\ du dv + g\ dv^2,$$

with

$$e = \langle \mathbf{N}, \mathbf{X}_{uu} \rangle, \quad f = \langle \mathbf{N}, \mathbf{X}_{uv} \rangle, \quad g = \langle \mathbf{N}, \mathbf{X}_{vv} \rangle.$$

Under this parametrization $\mathbf{X}$, the mean curvature H and the Gauss curvature K have the classical expressions

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}, \quad K = \frac{eg - f^2}{EG - F^2}.$$

Let us denote by $[,,]$ the determinant in $\mathbb{R}^3$ and put $W = EG - F^2$. Then H and K write as

$$H = \frac{G[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}] - 2F[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}] + E[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}]}{2W^{3/2}} := \frac{H_1}{2W^{3/2}}, \tag{2.1}$$

$$K = \frac{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}][\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}] - [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}]^2}{W^2} := \frac{K_1}{W^2}, \tag{2.2}$$

where H_1 and K_1 denote the numerators in (2.1) and (2.2) respectively. The principal curvatures κ_1 and κ_2 are given then by

$$\kappa_1 = H + \sqrt{H^2 - K}, \quad \kappa_2 = H - \sqrt{H^2 - K}.$$

Then the condition (1.1) writes now as

$$(1-m)H_1 - 2W^{3/2}n = -(1+m)\sqrt{H_1^2 - 4WK_1}. \tag{2.3}$$

After some manipulations, and squaring twice (2.3), we obtain

$$\left(-mH_1^2 + (1+m)^2 WK_1 + n^2 W^3\right)^2 - n^2(1-m)^2 H_1^2 W^3 = 0. \tag{2.4}$$

3. Cyclic LW-Surfaces

In this section, we prove the first statement S.1 of Sec. 1, that is,

Theorem 3.1. *Let S be a cyclic LW-surface with $(m,n) = (m,0)$. Then the planes of the foliation are parallel.*

The methods that we apply in our proofs are based on [14]. For this, we wish to construct an appropriate coordinate system to our foliation of the surface S. Let us denote by Π_u these planes in such way $S \cap \Pi_u$ is each piece of the circles of the foliation. Consider a smooth unit vector field Z that is normal to the planes Π_u. Next, we take a particular integral curve $\Gamma = \Gamma(u)$ of Z parametrized by arclength, that is, $\mathbf{t}(u) := \Gamma'(u) = Z(\Gamma(u))$, where $\mathbf{t}$ is the unit tangent vector to Γ. Consider the Frenet frame of the curve Γ, $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, where $\mathbf{n}$ and $\mathbf{b}$ denote the normal and binormal vectors respectively. Locally we parametrize S by

$$\mathbf{X}(u,v) = \mathbf{c}(u) + r(u)(\cos v\ \mathbf{n}(u) + \sin v\ \mathbf{b}(u)),$$

where $r = r(u) > 0$ and $\mathbf{c} = \mathbf{c}(u)$ denote the radius and centre of each u-circle of the foliation. Consider the Frenet equations of the curve Γ:

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n}, \\ \mathbf{n}' &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \mathbf{b}' &= -\tau \mathbf{n}, \end{aligned}$$

where the prime $'$ denotes the derivative with respect to the u-parameter and κ and τ are the curvature and torsion of Γ, respectively. Observe that $\kappa \neq 0$ because Γ is not a straight-line. Also, set

$$\mathbf{c}' = \alpha \mathbf{t} + \beta \mathbf{n} + \gamma \mathbf{b}, \tag{3.1}$$

where α, β, γ are smooth functions on u.

By using the Frenet equations and (3.1), Eq. (2.4) is an expression of powers of the trigonometric functions $\cos(v)$ and $\sin(v)$, where the highest exponent is 6. Then a straightforward computation allows to write (2.4) as a trigonometric polynomial on $\cos(jv)$, $\sin(jv)$, with $0 \leq j \leq 6$. Exactly, there exist smooth functions on u, namely A_j and B_j, such that (2.4) writes as

$$A_0 + \sum_{j=1}^{6}(A_j \cos(jv) + B_j \sin(jv)) = 0. \tag{3.2}$$

Since this is an expression on the independent trigonometric terms $\cos(jv)$ and $\sin(jv)$, all coefficients A_i, B_i must vanish.

The proof of Theorem 3.1 involves then explicit computations of these coefficients and subsequent manipulations. These calculations are very lengthy and the reader can follow them with a software on symbolic computations. The author was able to obtain the results using the Mathematica program to check his work. The computer was used in each calculation several times, giving understandable expressions of the coefficients A_j and B_j. The first coefficients are A_6 and B_6, whose values are:

$$A_6 = -\frac{1}{32}(m-1)^2\kappa^2 r^6\left(\beta^4 + (\gamma^2 - \kappa^2 r^2)^2 + \beta^2(2\kappa^2 r^2 - 6\gamma^2)\right) = 0,$$
$$B_6 = -\frac{1}{8}(m-1)^2\beta\gamma\kappa^2 r^6(\beta^2 - \gamma^2 + \kappa^2 r^2) = 0.$$

Recall that in the next reasoning, $\kappa \neq 0$. From B_6, we consider three possibilities.

(1) Case $\beta\gamma \neq 0$. Then $\beta^2 = \gamma^2 - \kappa^2 r^2$. From $A_6 = 0$, we obtain $-4\gamma^2(\gamma^2 - \kappa^2 r^2) = 0$. Since $\gamma \neq 0$, then $\gamma^2 = \kappa^2 r^2$. But then, $\beta = 0$. As conclusion, this case is impossible.

(2) Case $\gamma = 0$. Then

$$A_6 = -\frac{1}{32}(m-1)^2\kappa^2 r^6(\beta^2 + \kappa^2 r^2)^2,$$

which yields a direct contradiction.

(3) Case $\beta = 0$. Now

$$A_6 = -\frac{1}{32}(m-1)^2\kappa^2 r^6(\gamma^2 - \kappa^2 r^2)^2.$$

Hence $\gamma^2 = \kappa^2 r^2$. Then

$$A_4 = -\frac{1}{8}(6 + m(6m-13))\kappa^4 r^8(\alpha^2 - r'^2) = 0,$$
$$B_4 = \frac{1}{4}(6 + m(6m-13))\alpha\kappa^4 r^8 r' = 0.$$

(a) If $(6 + m(6m-13)) \neq 0$, then $\alpha^2 = r'^2$ and $\alpha r' = 0$. Thus $\alpha = 0$ and r is a constant function. Then $A_2 = -\frac{1}{2}(2m^2 - 5m + 2)r^{10}\kappa^6 = 0$. Then $m = 1/2$ and $m = 2$. In both cases, the computations of A_1 gives $\tau = 0$ and then (2.4) implies $\frac{9}{4}r^{10}\kappa^6$ and $9r^{10}\kappa^6$ respectively. Anyway, we conclude a contradiction.

(b) Therefore, it suffices to study the case that $(6 + m(6m - 13)) = 0$, that is, $m = 2/3$ and $m = 3/2$. For simplicity, we do the proof in the former case (the case $m = 3/2$ is obtained interchanging the roles of κ_1 and κ_2 in the linear relation $\kappa_1 = m\kappa_2$).

Before to follow, we point out that the case $\alpha = 0$ is impossible, because

$$A_3 = -\frac{5}{18}\kappa^3 r^8 r'^2 \tau = 0.$$

Then this means that $r' = 0$ or $\tau = 0$. If r is a constant function, $A_2 = 2/9r^{10}\kappa^6 = 0$, which it is false. If $\tau = 0$, then B_2 and B_1 give respectively,

$$\kappa^3 r^2 + 2rr'\kappa' - \kappa(9r'^2 + 2rr'') = 0,$$
$$81\kappa^3 r^2 + 2rr'\kappa' + \kappa(71r'^2 - 2rr'') = 0.$$

By combining both equations, we have $80\kappa(\kappa^2 r^2 + r'^2) = 0$, which it is a contradiction.

From now, we assume $\alpha \neq 0$. Then the computation of A_3 and B_3 imply:

$$3\alpha^3\kappa - 2\alpha^2\kappa r\tau - 2\kappa rr'(\alpha' - \tau r') + \alpha(\kappa^3 r^2 + 4r\kappa' r' - \kappa(21r'^2 + 2rr''))$$
$$= 0, \tag{3.3}$$
$$\alpha^2(-2r\kappa' + 15\kappa r') + 2\alpha\kappa r(\alpha' - 2\tau r') + r'(\kappa^3 r^2 + 2r\kappa' r' - \kappa(9r'^2 + 2rr''))$$
$$= 0. \tag{3.4}$$

Denote x_1 and x_2 the left-hand sides of (3.3) and (3.4) respectively. Then $\alpha x_1 + r' x_2 = 0$ yields $(\alpha^2 + r'^2)^2 x_3 = 0$, where

$$x_3 = 3\alpha^2\kappa + \kappa^3 r^2 - 2\alpha\kappa r\tau + 2r\kappa' r' - \kappa(9r'^2 + 2rr'').$$

Since $\alpha \neq 0$, then $x_3 = 0$. Now, $r' x_3 - x_2 = 0$ implies

$$\alpha r\kappa' + \kappa(-6\alpha r' + r(-\alpha' + \tau r')) = 0.$$

In this expression, we obtain κ':

$$\kappa' = \frac{\kappa(r\alpha' + 6\alpha r' - r\tau r')}{\alpha r},$$

and substituting into the value of x_1, we get

$$3\alpha^3 - 2\alpha^2 r\tau + 2rr'(\alpha' - \tau r') + \alpha(\kappa^2 r^2 + 3r'^2 - 2rr'') = 0.$$

Hence we obtain the value of r'', which putting it into $A_2 = 0$ and $B_2 = 0$ give respectively

$$-7\alpha^4 - 8\alpha^2\kappa^2 r^2 - \alpha^3 r\tau + \alpha\kappa^2 r^3\tau + 78\alpha^2 r'^2$$
$$+ 10\kappa^2 r^2 r'^2 - \alpha r\tau r'^2 - 15r'^4 = 0, \tag{3.5}$$
$$-42\alpha^3 - 18\alpha\kappa^2 r^2 - \alpha^2 r\tau + \kappa^2 r^3\tau + 58\alpha r'^2 - r\tau r'^2 = 0, \tag{3.6}$$

where y_1 and y_2 denote the left-hand sides of (3.5) and (3.6) respectively. Now $y_1 - \alpha y_2 = 0$ gives

$$7\alpha^2 + 2\kappa^2 r^2 - 3r'^2 = 0.$$

From this equation, we obtain r'^2,

$$r'^2 = \frac{1}{3}\left(7\alpha^2 + 2\kappa^2 r^2\right) \tag{3.7}$$

and we introduce it into $y_1 = 0$ concluding

$$280\alpha^3 + 62\alpha\kappa^2 r^2 - 10\alpha^2 r\tau + \kappa^2 r^3 \tau = 0.$$

If $\kappa^2 r^2 \neq 10\alpha^2$, then

$$\tau = -\frac{2(140\alpha^3 + 31\alpha\kappa^2 r^2)}{r(\kappa^2 r^2 - 10\alpha^2)}.$$

By substituting in A_1 and using (3.7), we have

$$45\alpha^4 + 31\alpha^2\kappa^2 r^2 + 5\kappa^4 r^4 = 0,$$

which it is a contradiction. As conclusion, $\kappa^2 r^2 = 10\alpha^2$ and $A_3 = 0$ gives

$$\tau = \frac{3\kappa^2 r^2 + 10r'^2}{20rr'}.$$

From (3.7), we have $r'^2 = 9\alpha^2 = 9/10\kappa^2 r^2$. Returning with the computations, the coefficient A_2 (or B_2) gives $\kappa^6 r^{10} = 0$, obtaining the desired contradiction.

4. LW-Surfaces of Riemann-Type

We consider a cyclic surface S of Riemann type, that is, a cyclic surface where the pieces of circles of the foliation lie in parallel planes, for example, parallel to the xy-plane. Because our reasoning is local, we can assume that S writes as

$$X(u,v) = (a(u), b(u), u) + r(u)(\cos v, \sin v, 0),$$

where a, b and r are smooth function in some u-interval I and $r > 0$ denotes the radius of each circle of the foliation. Moreover, S is a surface of revolution if and only if a and b are constant functions. If we compute (2.4), we obtain an expression

$$\sum_{j=0}^{12} A_j(u)\cos(jv) + B_j(u)\sin(jv) = 0. \tag{4.1}$$

Again, the functions A_j and B_j on u vanish on I. We distinguish two cases according to the value of n.

(1) Case $n \neq 0$.
The computation of A_{12} and B_{12} give respectively:
$$A_{12} = \frac{1}{2048} n^4 r^{12} A \qquad B_{12} = \frac{512}{n^4} r^{12} B,$$
where
$$A = a'^{12} - 66a'^{10}b'^2 + 495a'^8b'^4 - 924a'^6b'^6 + 495a'^4b'^8 - 66a'^2b'^{10} + b'^{12},$$
$$B = a'b'(3a'^{10} - 55a'^8b'^2 + 198a'^6b'^4 - 198a'^4b'^6 + 55a'^2b'^8 - 3b'^{10}).$$
We assume now that S is not a surface of revolution and we will arrive to a contradiction. As $r > 0$, $A = B = 0$. Because the expressions of A and B do not depend on r, we do a change of variables. Since the planar curve $\alpha(u) = (a(u), b(u))$ is not constant, we reparametrize it by the length-arc, that is, $(a(u), b(u)) = (x(\phi(u), y(\phi(u)),$ where
$$a'(u) = \phi'(u)\cos(\phi(u)), \qquad b'(u) = \phi'(u)\sin(\phi(u)), \quad \phi'^2 = a'^2 + b'^2. \tag{4.2}$$
With this change, A and B write now as:
$$A = \phi'(u)^{12}\cos(12\phi(u)), \quad B = \phi'(u)^{12}\sin(12\phi(u)).$$
Therefore, $\phi' = 0$, that is, α is a constant curve: contradiction.

(2) Case $n = 0$.
Now (2.4) is simply $-mH_1^2 + (1+m)^2WK_1 = 0$ and Eq. (4.1) is then a sum until $j = 3$, with
$$A_3 = -\frac{1}{4}(1+m)^2 r^5 \left(a''(a'^2 - b'^2) - 2a'b'b''\right),$$
$$B_3 = -\frac{1}{4}(1+m)^2 r^5 \left(b''(a'^2 - b'^2) + 2a'b'a''\right).$$
We assume that S is not a surface with constant mean curvature, that is, $m \neq -1$. As in the case $n \neq 0$, we assume that S is not a surface of revolution and we will obtain a contradiction. As above, we reparametrize the curve $\alpha(u) = (a(u), b(u))$ as in (4.2). Then $A_3 = B_3 = 0$ lead to respectively:
$$\phi'(u)^2\left(-\phi''(u)\cos(3\phi(u)) + \phi'(u)^2\sin(3\phi(u))\right) = 0,$$
$$\phi'(u)^2\left(\phi'(u)^2\cos(3\phi(u)) + \phi''(u)\sin(3\phi(u))\right) = 0.$$
By combining both equations, we obtain $\phi'(u) = 0$ on I, obtaining the desired contradiction.

In the case $m = -1$ and $n = 0$, S is a minimal surface. Then the degree of (4.1) is 2. Here, $A_2 = B_2 = 0$ imply
$$a' = \lambda r^2 \quad b' = \mu r^2 \tag{4.3}$$
for some constants $\lambda, \mu \geq 0$. Hence that (4.1) gives
$$1 + (\lambda^2 + \mu^2)r^4 + r'^2 - rr'' = 0. \tag{4.4}$$
Equations (4.3) and (4.4) define the Riemann examples ($\lambda^2 + \mu^2 \neq 0$) and the catenoid ($\lambda^2 + \mu^2 = 0$).

As conclusion,

Theorem 4.1. *The only LW-surfaces of Riemann-type are*:

(1) *The surfaces of revolution.*
(2) *The classical Riemann examples of minimal surfaces.*

Moreover, the Riemann examples can be viewed as an exceptional case in the family of cyclic LW-surfaces, at least with $n = 0$, that is,

Corollary 4.1. *Riemann examples of minimal surfaces are the only non-rotational cyclic surfaces that satisfy a linear Weingarten relation of type* $\kappa_1 = m\kappa_2$, $m \neq 0$.

Theorem 4.1 and Corollary 4.1 show the statement S.2 of Sec. 1.

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