Parabolic surfaces in hyperbolic space with constant curvature

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Abstract

We study linear Weingarten parabolic surfaces in hyperbolic space $\mathbb{H}^3$. In particular, we classify two family of parabolic surfaces: surfaces with constant Gaussian curvature and surfaces that satisfy the relation $a\kappa_1 + b\kappa_2 = c$, where κ_i are the principal curvatures, and a, b and c are constant.

1 Introduction

Let $\mathbb{H}^3$ be the three-dimensional hyperbolic space. A parabolic group of isometries of $\mathbb{H}^3$ is formed by isometries that leave fix one double point of the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$. We say that a surface S is a *parabolic surface* of $\mathbb{H}^3$ if it is invariant by a group of parabolic isometries. A surface S in $\mathbb{H}^3$ is called a *Weingarten surface* if there is some (smooth) relation $W(\kappa_1, \kappa_2) = 0$ between its two principal curvatures κ_1 and κ_2. In particular, if K and H denote respectively the Gauss curvature and the mean curvature of S, we have a relation $U(K, H) = 0$. In this note we study parabolic Weingarten surfaces that satisfy the simplest case for W and U, that is, of linear type:

$$a\ H + b\ K = c, \tag{1}$$

and

$$a\ \kappa_1 + b\ \kappa_2 = c \tag{2}$$

where $a, b, c \in \mathbb{R}$. We say in both cases that S is a *linear* Weingarten surface. In the set of linear Weingarten surfaces, we mention three families of surfaces that correspond with trivial choices of the constants a, b and c: surfaces with constant Gauss curvature ($a = 0$ in (1)), surfaces with constant mean curvature ($b = 0$ in (1) or $a = b$ in (2)) and umbilical surfaces ($a = -b$ and $c = 0$ in (2)). Although these three kinds of surfaces have been studied in the literature, the classification of linear Weingarten surfaces in the general case is almost completely open today. We find linear Weingarten surfaces in the family of rotational surfaces because in such case, equations (1) and (2) reduce into an ordinary differential equation. In hyperbolic ambient, rotational linear Weingarten surfaces have been studied when the mean curvature is constant [1], in arbitrary dimension [2, 5, 6] or in the spherical case [7, 8].

In this note we give a complete description and classification of parabolic surfaces in $\mathbb{H}^3$ that satisfy equation (1) when $a = 0$ (constant Gaussian curvature) and equation (2). A more detailed study can see in [3] and [4]. Among the facts of our interest, we ask whether the surface can be extended to be complete and if the surface is embedded.

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2 Preliminaries

Let us consider the upper half-space model of the hyperbolic three-space $\mathbb{H}^3$, namely,

$$\mathbb{H}^3 =: \mathbb{R}^3_+ = \{(x,y,z)\in\mathbb{R}^3; z>0\}$$

equipped with the metric

$$\langle,\rangle = \frac{dx^2+dy^2+dz^2}{z^2}.$$

In what follows, we will use the words "vertical" or "horizontal" in the usual affine sense of $\mathbb{R}^3_+$. The ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ is $\mathbb{S}^2_\infty=\{z=0\}\cup\{\infty\}$, the one-compactification of the plane $\{z=0\}$. The asymptotic boundary of a set $\Sigma\subset\mathbb{H}^3$ is defined as $\partial_\infty\Sigma=\overline{\Sigma}\cap\mathbb{S}^2_\infty$, where $\overline{\Sigma}$ is the closure of Σ in $\{z\geq 0\}\cup\{\infty\}$.
Let G be a parabolic group of isometries of $\mathbb{H}^3$. Without loss of generality, we take the point ∞ of $\mathbb{S}^2_\infty$ as the point that fixes G. Then the group G is defined by the horizontal (Euclidean) translations in the direction of a vector $\xi\in\{z=0\}$. The space of orbits is represented in any geodesic plane orthogonal to ξ. Throughout this note, we assume that $\xi=(0,1,0)$.
A surface S invariant by G intersects $P=\{(x,0,z); z>0\}$ in a curve α called the *generating curve* of S. Consider $\alpha(s)=(x(s),0,z(s))$ parametrized by the Euclidean arc-length, $s\in I$ and I an open interval including zero. Then $x'(s)=\cos\theta(s)$ and $z'(s)=\sin\theta(s)$ for a certain differentiable function θ, where the derivative $\theta'(s)$ of the function $\theta(s)$ is the Euclidean curvature of α. A parametrization of S is $X(s,t)=(x(s),t,z(s))$, $t\in\mathbb{R}$. The principal curvatures κ_i of S are

$$\kappa_1(s,t)=z(s)\theta'(s)+\cos\theta(s),\qquad \kappa_2(s,t)=\cos\theta(s), \tag{3}$$

and the Gauss curvature K is $K=\kappa_1\kappa_2-1$. Exactly κ_1 is the hyperbolic curvature of the curve α. Thus a parabolic surface S in $\mathbb{H}^3$ is given by a curve $\alpha=(x(s),0,z(s))$ whose coordinate functions satisfy

$$\left\{\begin{array}{lcl} x'(s) &=& \cos\theta(s)\\ z'(s) &=& \sin\theta(s)\end{array}\right. \tag{4}$$

and the equation

$$K=z(s)\cos\theta(s)\theta'(s)-\sin\theta(s)^2. \tag{5}$$

if the Gaussian curvature K is constant or

$$az(s)\theta'(s)+(a+b)\cos\theta(s)=c \tag{6}$$

if S satisfies the Weingarten relation (2). After an isometry of the ambient space formed by a horizontal translation orthogonal to ξ followed by a dilatation, we consider the initial conditions

$$x(0)=0,\qquad z(0)=1,\qquad \theta(0)=\theta_0. \tag{7}$$

As a consequence of the uniqueness of solutions of an ordinary differential equation, we have

Lemma 2.1. *Let α be a solution of the initial value problem (4)-(5) or (4)-(6). Let $s_0\in I$.*

1. *If $z'(s_0)=0$, then α is symmetric with respect to the vertical line $x=x(s_0)$ of the xz-plane.*
2. *If $\theta'(s_0)=0$, then α is a straight-line.*

3 Parabolic surfaces with constant Gaussian curvature

Let us assume that S is a parabolic surface in $\mathbb{H}^3$ with constant Gauss curvature K. Then the generating curve α satisfies (4)-(5). Consider $z'(s)$ as a function of the new variable $z(s)$. If we put $p = z'$ and $x = z$, we have $xp(x)p'(x) = K + p(x)^2$. Setting $y = p^2$, we write $xy'(x) = 2K + 2y(x)$. The solutions of this equation are $y(x) = Kx^2 - K$, that is,

$$z'(s)^2 = K(z(s)^2 - 1). \tag{8}$$

A new differentiation in (8) gives $z''(s) = Kz(s)$, whose solutions are well known. Next, we express $x(s)$ in terms of an elliptic integral as $x(s) = \int_0^s \sqrt{1 - z'(t)^2}\, dt$.

1. Case $K > 0$. The solution is $z(s) = \cosh(\sqrt{K}s)$ whose domain is $(-s_1, s_1)$ with

   $$s_1 = \frac{1}{\sqrt{K}}\operatorname{arcsinh}(\frac{1}{\sqrt{K}}).$$

   Moreover, the behaviour of α at the ends points of $(-s_1, s_1)$ is

   $$\lim_{s\to s_1} z(s_1) = \sqrt{\frac{1+K}{K}} \qquad \lim_{s\to s_1} z'(s_1) = 1.$$

   The height of S, that is, the hyperbolic distance between the horospheres at heights $z = z(s_1)$ and $z_0 = 1$ is

   $$\frac{1}{2}\log\left(\frac{K+1}{K}\right).$$

2. Case $K = 0$. The solution is $\alpha(s) = (s, 0, 1)$, that is, α is a horizontal straight-line and the surface is a horosphere.

3. Case $K < 0$. The solution is $z(s) = \cos\left(\sqrt{-K}s\right)$. Depending on the value of K, the generating curve α meets $\mathbb{S}^2_\infty$. If $-1 \leq K < 0$, α intersects $\mathbb{S}^2_\infty$ making an angle such that $\sin\theta_1 = \sqrt{-K}$. The domain of α is $(-\pi/2, \pi/2)$. In the particular case that $K = -1$, α is a halfcircle that orthogonally meets $\mathbb{S}^2_\infty$. If $K < -1$, S is not complete and the curve α is a graph on an interval of $\mathbb{S}^2_\infty$. The parameter s goes in the range $(-\frac{1}{\sqrt{-K}}\arcsin(\frac{1}{\sqrt{-K}}), \frac{1}{\sqrt{-K}}\arcsin(\frac{1}{\sqrt{-K}}))$. Analogously as in the case $K > 0$, the height of the surface is

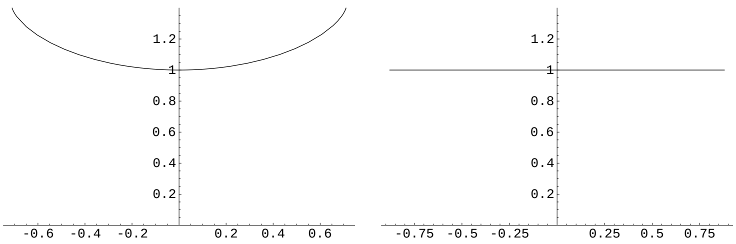
   $$\frac{1}{2}\log\left(\frac{K-1}{K}\right).$$

Theorem 3.1. *Let α be the generating curve of a parabolic surface S in hyperbolic space $\mathbb{H}^3$ with constant Gauss curvature K, where α is the solution of (4)-(5). Assume that the initial velocity of α is a horizontal vector. Then we have:*

1. *Case $K > 0$. The curve α is convex with exactly one minimum and it is a graph on $\mathbb{S}^2_\infty$ defined in some bounded interval $I = (-x_1, x_1)$. See Figure 1, (a).*

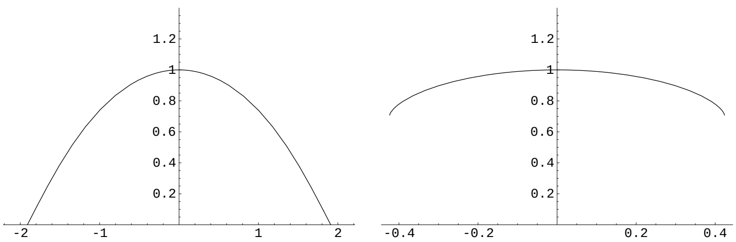
2. *Case $K = 0$. The curve α is a horizontal straight-line and S is a horosphere. See Figure 1, (b).*

3. *Case $K < 0$. The curve α is concave with exactly one maximum and it is a graph on $\mathbb{S}^2_\infty$ defined in some bounded interval $I = (-x_1, x_1)$. If $-1 \leq K < 0$, the curve α meets $\mathbb{S}^2_\infty$ making an angle θ_1 with $\sin\theta_1 = \sqrt{-K}$. See Figure 2 (a). If $K < -1$, α does not intersect $\mathbb{S}^2_\infty$ and at the end points, the curve is vertical. See Figure 2 (b).*



(a) (b)

Figure 1: The generating curves of parabolic surfaces with constant Gaussian curvature K. The initial angle is $\theta(0) = 0$. Case (a): $K = 1$; Case (b): $K = 0$.



(a) (b)

Figure 2: The generating curves of parabolic surfaces with constant Gaussian curvature K. The initial angle is $\theta(0) = 0$. Case (a): $K = -0.5$; Case (b): $K = -2$.

In cases 1) and 3), the height of S is $\frac{1}{2}\log\left(\frac{K+1}{K}\right)$ and $\frac{1}{2}\log\left(\frac{K-1}{K}\right)$ respectively.

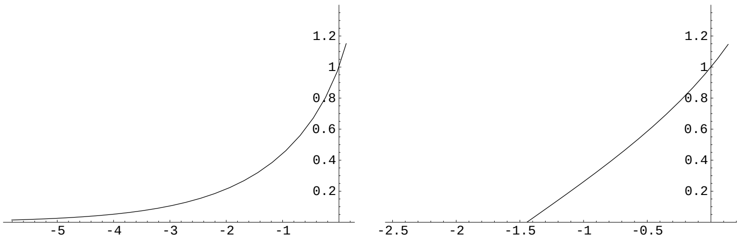
Corollary 3.1. *For each number K with $-1 \leq K < 0$, there exists a non-umbilical complete parabolic surface in $\mathbb{H}^3$ with constant Gauss curvature K. For these surfaces, the asymptotic boundary is formed by two circles tangent at the point fixed by the group of parabolic isometries.*

Theorem 3.2. *Any non-umbilical parabolic surface in $\mathbb{H}^3$ with constant Gaussian curvature K with $K < -1$ or $K \geq 0$ and with a horizontal tangent plane is not complete. Moreover, its asymptotic boundary is the point fixed by the group of parabolic isometries.*

Part of the above results appeared in [5]. Finally, we remark that if we want to have the complete classification of parabolic surfaces with constant Gaussian curvature, we must change the starting angle θ_0 in (7) in order to obtain all such surfaces. See [3]. In the range of value K, with $K \in (1, 0)$, there exist non complete parabolic surfaces and the asymptotic boundary of each such surface is a circle of $\mathbb{S}^2_\infty$. In Figure 3, we show two such parabolic surfaces with $\theta_0 = \pi/4$. As conclusion of our study, we have

Theorem 3.3. *Any non-umbilical parabolic surface in $\mathbb{H}^3$ with constant Gaussian curvature K with $K < -1$ or $K \geq 0$ is not complete. Moreover, its asymptotic boundary is the point fixed by the group of parabolic isometries.*

Corollary 3.2. *Any parabolic surface immersed in hyperbolic space $\mathbb{H}^3$ with constant Gaussian curvature is a graph on $\mathbb{S}^2_\infty$. In particular, it is embedded.*



(a) (b)

Figure 3: The generating curves of parabolic surfaces with constant Gaussian curvature K. The initial angle is $\theta(0) = \pi/4$. Case (a): $K = 0$; Case (b): $K = -1/4$.

4 Linear Weingarten parabolic surfaces

In this section we shall consider parabolic surfaces that satisfy the relation $a\kappa_1 + b\kappa_2 = c$. In the case that a or b is zero, that is, that one of the principal curvatures κ_i is constant, we have

Theorem 4.1. *The only parabolic surfaces in $\mathbb{H}^3$ with one constant principal curvature are totally geodesic planes, equidistant surfaces, horospheres and Euclidean horizontal right-cylinders.*

Proof. We use (3). If $\kappa_1 = c$, then $\theta'(s)z(s) = c - \cos\theta(s)$. By differentiation of this expression and using (4) we obtain $\theta''(s) = 0$ for all s. Then θ' is constant and hence that from the Euclidean viewpoint, the curve is a piece of a straight-line or a circle. If κ_2 is constant, then $\cos\theta(s) = c$ and this means that θ is constant and α is a straight-line. $\square$

We write the general case (2) as

$$\kappa_1 = m\kappa_2 + n \tag{9}$$

where $m, n \in \mathbb{R}$, $m \neq 0$. By using (3), Equation (9) writes as

$$\theta'(s) = \frac{(m-1)\cos\theta(s) + n}{z(s)}. \tag{10}$$

After a change of orientation on the surface, we suppose in our study that $n \geq 0$. We discard the trivial cases of Weingarten surfaces, that is, $(m, n) = (1, 0)$ and $m = -1$. We consider that the starting angle θ_0 in (7) is $\theta_0 = 0$. Equation (10) yields at $s = 0$, $\theta'(0) = n + m - 1$. By Lemma 2.1, if $\theta'(0) \neq 0$, then $\theta(s)$ is a monotonic function on s. Let $(-\bar{s}, \bar{s})$ be the maximal domain of solutions of (4)-(10) under the initial conditions (7) and denote $\bar{\theta} = \lim_{s\to\bar{s}} \theta(s)$. Depending on the sign of $\theta'(0)$, we consider three cases.

4.1 Case $n + m - 1 > 0$

As $\theta'(0) > 0$, $\theta(s)$ is a strictly increasing function.

1. Subcase $m < n + 1$. In particular, $n > 0$. We prove that θ attains the value $\pi/2$. Assume on the contrary that $\bar{\theta} \leq \pi/2$ and we will arrive to a contradiction. As $z'(s) = \sin\theta(s) > 0$, $z(s)$ is strictly increasing in $(0, \bar{s})$. Then $z(s) \geq z_0$ and the derivatives of $\{x(s), z(s), \theta(s)\}$ in equations (4)-(10) are bounded. This means that $\bar{s} = \infty$. As $\lim_{s\to\infty} z'(s) = \sin\bar{\theta} > 0$, then $\lim_{s\to\infty} z(s) = \infty$. Multiplying in (10) by $\sin\theta$ and integrating, we obtain

$$n + \cos\theta(s) = \frac{2-m}{z(s)} \int_0^s (\sin\theta(t)\cos\theta(t))\ dt + \frac{n+1}{z(s)}. \tag{11}$$

Let $s \to \infty$ in (11). If the integral that appears in (11) is bounded, then $n + \cos\bar{\theta} = 0$, that is, $\cos\bar{\theta} = n = 0$: contradiction. If the integral is not bounded, and using the L'Hôpital's rule, $n + \cos\bar{\theta} = (2-m)\cos\bar{\theta}$, that is, $(m-1)\cos\bar{\theta} + n = 0$. Then $m - 1 \leq 0$ and the hypothesis $n + m - 1 > 0$ yields $\cos\bar{\theta} = n/(1-m) > 1$: contradiction.

Therefore, there exists a first value s_1 such that $\theta(s_1) = \pi/2$. We prove that $\theta(s)$ attains the value π. By contradiction, we assume $\bar{\theta} \leq \pi$ and $z(s)$ is strictly increasing again. We then have $\bar{s} = \infty$ again and $\theta'(s) \to 0$ as $s \to \infty$. If $z(s)$ is bounded, then (11) implies $(m-1)\cos\bar{\theta} + n = 0$. As $m - 1 = n = 0$ is impossible, then $m - 1 > 0$ since $\cos\bar{\theta} < 0$. But the hypothesis $m < n + 1$ implies that $\cos\bar{\theta} = -n/(m-1) < -1$, which it is a contradiction. Thus $z(s) \to \infty$ as $s \to \infty$. By using (11) again, and letting $s \to \infty$, we have $n + \cos\bar{\theta} = 0$. In particular, $0 < m < 2$. We obtain a second integral from (10) multiplying by $\cos\theta(s)$:

$$\sin\theta(s) = \frac{s}{z(s)} + \frac{1}{z(s)} \int_0^s \Big(n\cos\theta(t) + (m-2)\cos^2\theta(t) \Big)\, dt.$$

If the integral is bounded, then $\sin^2\bar{\theta} = 1$: contradiction. Thus, the integral is not bounded and L'Hôpital rule implies $\sin^2\bar{\theta} = 1 + n\cos\bar{\theta} + (m-2)\cos^2\bar{\theta}$. This equation, together $n + \cos\bar{\theta} = 0$ yields $(m-2)\cos^2\bar{\theta} = 0$: contradiction.

As conclusion, there exists a first value s_2 such that $\theta(s_2) = \pi$. By Lemma 2.1, the curve α is symmetric with respect to the line $x = x(s_2)$. By symmetry, α is invariant by a group of horizontal translations orthogonal to the orbits of the parabolic group.

2. Subcase $m \geq n + 1$. Under this hypothesis and as $\theta'(s) > 0$, Equation (10) implies that $\cos\theta(s) \neq -1$ for any s. Thus $-\pi < \theta(s) < \pi$. For $s > 0$, $z'(s) = \sin\theta(s) > 0$ and then $z(s)$ is increasing on s and so, $\theta'(s)$ is a bounded function. This implies $\bar{s} = \infty$. We show that either there exists $s_0 > 0$ such $\theta(s_0) = \pi/2$ or $\lim_{s\to\infty}\theta(s) = \pi/2$.

   As in the above subcase, and with the same notation, if $\theta(s) < \pi/2$ for any s, then $n + \cos\bar{\theta} = 0$ or $(m-1)\cos\bar{\theta} + n = 0$. As $\cos\bar{\theta} \geq 0$ and since $m - 1 \geq n$, it implies that this occurs if and only if $n = 0$ and $\bar{\theta} = \pi/2$. In such case, $z''(s) = \theta'(s)\cos\theta(s) > 0$, that is, $z(s)$ is a convex function. As conclusion, if $n > 0$, there exists a value s_0 such that $\theta(s_0) = \pi/2$, and there exists $\bar{\theta} \in (\pi/2, \pi]$ such that $\lim_{s\to\infty}\theta(s) = \bar{\theta}$.

Theorem 4.2. *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface S in $\mathbb{H}^3$ whose principal curvatures satisfy the relation $\kappa_1 = m\kappa_2 + n$. Consider $n \geq 0$ and that $\theta(0) = 0$ in the initial condition (7). Assume $n + m - 1 > 0$.*

1. *Case $m < n + 1$. Then α is invariant by a group of translations in the x-direction. Moreover, α has self-intersections and it presents one maximum and one minimum in each period, with vertical points between maximum and minimum. See Figure 4 (a).*

2. *Case $m \geq n + 1$. If $n > 0$, then α has a minimum with self-intersections. See Figure 4 (b). If $n = 0$, then α is a convex graph on $\mathbb{S}^2_\infty$, with a minimum. See Figure 5 (a).*

4.2 Case $n + m - 1 = 0$

In the case that $n + m - 1 = 0$ where $\theta'(0) = 0$, and by Lemma 2.1, $\theta(s) = 0$ for any s.

Theorem 4.3. *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface S in $\mathbb{H}^3$. Assume that the principal curvatures of S satisfy the relation $\kappa_1 = m\kappa_2 + n$ with $n + m - 1 = 0$ and $n \geq 0$. If $\theta(0) = 0$ in the initial condition (7), then S is a horosphere.*

4.3 Case $n+m-1<0$

As $\theta'(0)<0$, $\theta(s)$ is a decreasing function. Since $n\geq 0$ and from (10), $\cos\theta(s)\neq 0$. This implies that $\theta(s)$ is a bounded function with $-\pi/2<\theta(s)<\pi/2$. If $\bar{s}=\infty$ and as $z(s)>0$, then both functions $\theta'(s)$ and $z'(s)$ go to 0 as $s\to\infty$. By (7) and (10), we have $(m-1)\cos\bar{\theta}+n=0$ and $\sin\bar{\theta}=0$: contradiction. This proves that $\bar{s}<\infty$.
As consequence, $z(s)\to 0$ since on the contrary, $\theta'(s)$ would be bounded and $\bar{s}=\infty$. We now use (11). Letting $s\to\bar{s}$ and by L'Hôpital rule again, we obtain $(m-1)\cos\bar{\theta}+n=0$, that is, $\cos\bar{\theta}\geq -n/(m-1)$. Finally, $z''(s)=\theta'(s)\cos\theta(s)<0$, that is, α is concave.

Theorem 4.4. *Let $\alpha(s)=(x(s),0,z(s))$ be the generating curve of a parabolic surface S in $\mathbb{H}^3$ whose principal curvatures satisfy the relation $\kappa_1=m\kappa_2+n$. Consider $n\geq 0$ and that $\theta(0)=0$ in the initial condition (7). Assume $n+m-1<0$. Then α is a concave graph on $\mathbb{S}^2_\infty$ with one maximum and it intersects $\mathbb{S}^2_\infty$ with a contact angle $\bar{\theta}$, $\cos\bar{\theta}=-n/(m-1)$. See Figure 5 (b).*

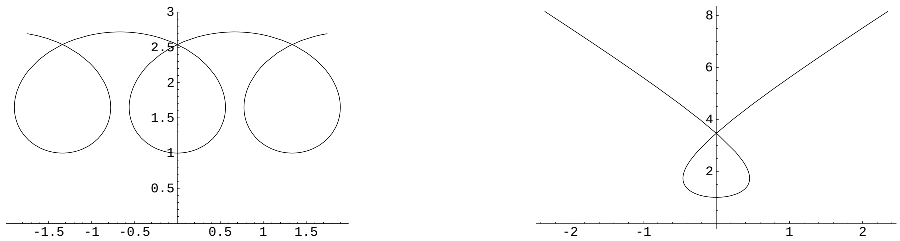
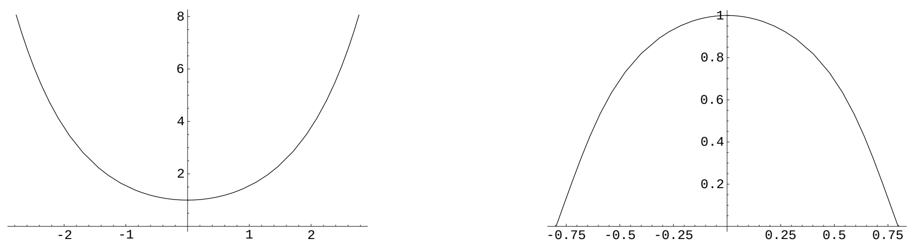


Figure 4: The generating curves of a parabolic surfaces with $\kappa_1=m\kappa_2+n$ and $n+m-1>0$. Let $\theta(0)=0$. We consider in (a) the subcase $m<n+1$, with $m=1$ and $n=2$. In (b) we show the subcase $m\geq n+1$ with $m=3$ and $n=1$.



(a) (b)

Figure 5: The generating curves of a parabolic surfaces with $\kappa_1=m\kappa_2+n$. Let $\theta(0)=0$. We consider in (a) the case $n+m-1>0$ and subcase $m\geq n+1$, with $m=2$ and $n=0$. In (b), we show the case $n+m-1<0$ with $m=-2$ and $n=1$.

As it as pointed out in the above Section 3, the classification is complete when we change the initial angle θ_0 in (7) in the range $0\leq\theta_0\leq 2\pi$. For example, in the case studied in subsection 4.1, that is, $n+m-1>0$, and subcase $m<n+1$, the velocity vector $\alpha'(s)$ takes all values of the interval $[0,2\pi]$. Thus, and using the uniqueness of solutions of an ordinary differential equation, the case $\theta_0=0$ covers all possibilities. As an example of the problem, we focus in the case of subsection 4.2. We omit the proof.

Theorem 4.5. *Let $\alpha(s)=(x(s),0,z(s))$ be the generating curve of a parabolic surface S in hyperbolic space $\mathbb{H}^3$. Assume that the principal curvatures of S satisfy the relation $\kappa_1=m\kappa_2+n$*

with $n+m-1=0$. If $\theta(0)\in(0,2\pi)$ in the initial condition (7), then α is a curve with self-intersections, with one maximum and asymptotic to $\mathbb{S}^2_\infty$ at infinity, that is, $\lim_{s\to\pm\infty} z(s)=0$. *See Fig. 6.*

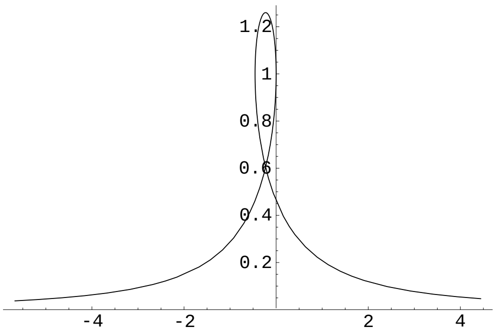


Figure 6: The generating curve of a parabolic surface with $\kappa_1=m\kappa_2+n$ and $n+m-1=0$. Here $m=-2$ and $n=3$. The starting angle θ_0 is $\theta_0=\pi/2$.

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