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Capillary channels in a gravitational field

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Abstract

We investigate the shape of the liquid–air free interface in the next two settings. First, we insert two parallel vertical plates sufficiently close in an infinite reservoir of liquid. Due to capillary and gravity forces and when the equilibrium is achieved, the liquid rises to a certain height. Then the liquid–air interface meets the vertical walls at prescribed angles and its mean curvature is a linear function of the height, that is, of the coordinate function that defines the gravity field. We study the shapes of these interfaces and their qualitative properties assuming natural hypothesis on symmetry. One matter of interest is to obtain estimates of the size of the capillary meniscus, such as its height, in terms of the boundary data. In the second setting, we consider a horizontal hydrophilic strip surrounded by a solid region of hydrophobic character. We spread the liquid over the strip driven by wettability in such a way that the liquid remains confined up to the boundary of the strip. In a state of equilibrium and assuming that the liquid is invariant in the direction of the non-bounded coordinate of the strip, we prove results on the existence and the uniqueness and we analyse the behaviour of the interface, specially related to the estimates of volume enclosed by the surface. Both settings are particular situations of the one-dimensional case of the capillarity problem, which has been studied in the literature to describe the shape of a liquid that faces a vertical plate.

Mathematics Subject Classification: 35Q35, 76B45, 35J65, 53A10

1. Introduction

Consider an infinite horizontal reservoir of fluid and let us introduce two parallel vertical plates. The action of capillarity causes the liquid to rise between both plates until a state of mechanical equilibrium. Denote by S the liquid–air interface formed by the fluid between the two plates and whose shape we would like to determine. See figure 1(a). The shape of the interface is determined by the equilibrium between the capillary and the gravity forces. The fluid surface

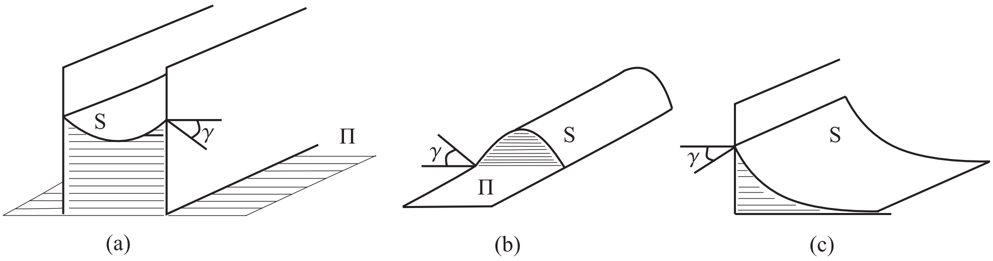

Figure 1. (*a*) A capillary surface between two parallel vertical planes, (*b*) adhesion of a liquid channel on a long striped domain and (*c*) meniscus of a liquid facing a vertical plate.

level at a large distance from the plates provides a reference level Π for atmospheric pressure that does not change with perturbations of the fluid surface between the plates. According to the principle of virtual work, the configurations that the capillary meniscus adopts between the two plates are characterized by two facts [7].

(i) The mean curvature of S is proportional to the height function with respect to Π (Laplace equation).
(ii) The angles γ_i with which S intersects the plates are constant (Young condition). These constants depend only on the physical properties of the liquid and the plates.

We also study the following wetting phenomenon. We consider a long, straight and completely wetting strip embedded in a hydrophobic solid substrate, i.e. of non-wetting type. The liquid that covers the strip wants to wet the hydrophilic strip, but wants to dewet the hydrophobic support. As a result, the liquid is confined to the strip by wettability up to its boundary, namely, two parallel straight lines, and does not contact the exterior. See figure 1(*b*). Then the liquid–air interface satisfies Laplace and Young equations again and we then want to know the shape of such an interface, specially when it ceases to be a graph and the contact angle γ with the horizontal substrate lies in the range $\pi/2 \leqslant \gamma \leqslant \pi$.

Consider (x, y, z) the usual coordinates in Euclidean space $\mathbb{R}^3$. Let $P_1 = \{x = -a\}$ and $P_2 = \{x = a\}$ be two vertical planes which represent the vertical plates and let $\Pi = \{z = 0\}$ be the horizontal plane. Set $L_i = \Pi \cap P_i$. Denote $\Omega = \{(x, y) \in \mathbb{R}^2; |x| < a\}$ the strip in Π determined by the two planes, identifying $\mathbb{R}^2$ with Π as usual. Let the height of this capillary free surface S with respect to Π, assumed nonparametric over Ω, be given by the scalar function $u = u(x, y)$, $(x, y) \in \Omega$. When the capillary and the gravity forces are in equilibrium, u satisfies the partial differential equation

$$\operatorname{div} Tu = \kappa u, \qquad Tu = \frac{Du}{\sqrt{1 + |Du|^2}} \tag{1}$$

in Ω. Here $\kappa = \rho g/\sigma$ is the capillarity constant with ρ the difference in densities across the interface S, g the gravitational acceleration, with a positive and a negative sign in the sessile and the pendant case, respectively, and σ the surface tension. We assume that the gravity field acts in the direction of the z-coordinate of $\mathbb{R}^3$. Equation (1) can be interpreted as that the mean curvature H of the surface $z = u(x, y)$ is $\kappa u/2$. From a mathematical point of view, there exists a considerable body of literature on the subject of capillarity (for an extensive list of references, see [7]).

The Young condition is written as

$$\nu_i \cdot Tu = \cos \gamma_i \qquad \text{along } L_i, \tag{2}$$

where ν_i is the unit exterior normal on L_i. Here γ_i are the contact angles with which S meets P_i, $i = 1, 2$. The orientation on S points in the z-positive direction. If the two plates are made of the same materials, $\gamma = \gamma_1 = \gamma_2$. We may normalize so that $0 \leqslant \gamma \leqslant \pi$. The range $0 \leqslant \gamma \leqslant \pi/2$ indicates a capillary rise; $\pi/2 < \gamma \leqslant \pi$ yields a capillary fall.

When the effect of gravity is ignored, the liquid–air interface describes a surface with constant mean curvature. A first example of a graph with constant mean curvature on a band is any section of an infinite round cylinder positioned with its axis parallel to P_i. In our both settings, it is natural to assume that the interface is homogeneous in the direction of the y-coordinate. This motivates us to consider that the surface S is invariant by the reflection with respect to any vertical plane $\{y = t\}$ and S will be determined by its intersection curve with any such plane. This means that S is a cylindrical ruled surface. As a consequence, the Young condition (2) on the contact angle is replaced by a Dirichlet condition on the boundary $\partial\Omega$ of the strip of Ω.

Classically, the capillarity and the wetting problems were studied when the liquid rises in a tube with a circular section or when one deposits liquid on a circular domain, respectively. Thus, our settings then reduce to consider one of the curvature radii is infinite. For this reason, our context is within the so-called one-dimensional case of the capillarity problem. In the literature, and within this framework, the study of the shape of a liquid in contact with a vertical plate as is shown in figure 1(c) has been considered. As we will see in section 2, a first integration of (1) is obtained in such a way that the solutions can be expressed in terms of elliptic integrals and some estimates of the height of the meniscus have been obtained from these integrals or as a limit case of the two-dimensional case [1, 2, 9, 10, 14–16, 18].

The aim of this paper is to give a qualitative and a quantitative description of solutions of (1) when the function u is defined in a strip Ω and assuming the invariance of the shape of the interface with respect to the y-coordinate. In this sense, our point of view will be twofold: first, we intend to analyse the symmetries of the surface and the shapes adopted depending on the sign of κ. Second, we are interesting in obtaining, if possible, successive estimates of the height of the meniscus, as well as of the volume per unit of length enclosed by the surface. In this sense, we follow the same spirit as in [3–6].

This paper is organized as follows. In section 2, we consider the capillarity equation in the one-dimensional problem, doing a first integral of this equation and describing each one of the above settings, including the case that appears in figure 1(c). In section 3 we make a study of the symmetries of the solutions of such an equation. In section 4, we obtain estimates of the height of the meniscus in the capillary problem. In sections 5 and 6, we consider the setting of the spreading of liquid confined in a strip by wettability. In section 5 we consider sessile liquid channels with results on the existence with respect to the volume enclosed by the channel. Finally, in section 6 we study pendant liquid channels.

2. The capillarity equation in the one-dimensional problem

In the three settings described in the introduction, the shape of the liquid–air interface is invariant in one horizontal coordinate. Then the interface S is a cylindrical ruled surface in Euclidean space $\mathbb{R}^3$ and parametrized as

$$\boldsymbol{x}(s, t) = \alpha(s) + t\vec{w}, \qquad s \in I, t \in \mathbb{R},$$

where α is a regular planar curve of $\mathbb{R}^3$ and $\vec{w} \in \mathbb{R}^3$, $|\vec{w}| = 1$ with $\langle \alpha'(s), \vec{w} \rangle = 0$. The curve α is called the *directrix* of S and $\vec{w}$ gives the directions of the *rulings* $t \longmapsto \alpha(s) + t\vec{w}$. A simple computation of the mean curvature H of S at each point $\boldsymbol{x}(s, t)$ gives $H(\boldsymbol{x}(s, t)) = C_\alpha(s)/2$, where C_α is the curvature of α. Thus, S satisfies the capillary equation (1) if and only if

$$C_\alpha(s) = \kappa(z \circ \alpha)(s) \tag{3}$$

for all $s \in I$.

Definition 2.1. *Let $\kappa \neq 0$. A κ-cylindrical surface is a cylindrical ruled surface that locally satisfies the capillary equation (3).*

Let us consider that the curve $\alpha(s)$ lies in the xz-plane and that it is a graph of a function u. Then S is parametrized by $\boldsymbol{x}(r, t) = (r, t, u(r))$, $r \in I$, $t \in \mathbb{R}$. By computing C_α, the capillary equation (3) in its one-dimensional form gives

$$\frac{u''(r)}{(1 + u'(r)^2)^{3/2}} = \kappa u(r). \tag{4}$$

Equation (4) is a second-order differential equation for the profile $u(r)$ of the meniscus subject to the appropriate boundary conditions. The next calculations are classical and we refer to the encyclopedic article [15, pp 1130–40] (see also [1, pp 71–87]). We have included them here for the sake of completeness. Multiplying by u' in equation (4), we have a first integration:

$$\frac{\kappa}{2} u(r)^2 = m - \frac{1}{\sqrt{1 + u'(r)^2}}, \tag{5}$$

for some constant $m \in \mathbb{R}$. We now consider the settings that appear in figure 1. In section 2.1, we analyse the cases that we will study in this work, that is, the capillarity between two parallel vertical plates and the liquid covering an infinite strip, figures 1(*a*) and (*b*), respectively. In order to complete with the analysis of (5), we briefly treat the case described in figure 1(*c*) in section 2.2 as a further example of the one-dimensional case. In both cases, we assume that $\kappa > 0$.

2.1. Capillarity between two parallel vertical plates; adhesion on a long strip

We consider the settings of figures 1(*a*) and (*b*). Here we assume that at $r = 0$, $u'(0) = 0$ a certain height $u(0) = u_0$. Under these boundary conditions, the calculation of the constant m in (5) gives $m = 1 + \kappa u_0^2/2$, that is,

$$\frac{1}{\sqrt{1 + u'(r)^2}} = -\frac{\kappa}{2}(u^2(r) - u_0^2) + 1$$

or,

$$u(r)^2 = u_0^2 + \frac{2}{\kappa}(1 - \cos\psi(r)) = u_0^2 + \frac{4}{\kappa} - \frac{4}{\kappa}\cos^2\left(\frac{\psi(r)}{2}\right).$$

By doing the change of variables $v = \cos(\psi(r)/2)$ and putting into (4), it allows integration to obtain the formula

$$\frac{\sqrt{\kappa}}{\lambda} r = \left(\frac{2}{\lambda^2} - 1\right)\left(K(\lambda) - F\left(\frac{\pi - \psi}{2}, \lambda\right)\right) - \frac{2}{\lambda^2}\left(E(\lambda) - E\left(\frac{\pi - \psi}{2}, \lambda\right)\right),$$

where, as usual, here we denote by $K(\lambda)$ and $E(\lambda)$ the complete elliptic integrals $K(\lambda) = F(\pi/2, \lambda)$ and $E(\lambda) = E(\pi/2, \lambda)$.

2.2. *Capillarity of a liquid surface in contact with a vertical plate*

We study the setting of figure 1(*c*), that is, the meniscus of a liquid facing a vertical plate. The function u is physically the height of a capillary surface on one side of an infinite vertical plate. Assume that $u(0) > 0$ and denote γ as the angle of contact of the liquid with the plate, $0 \leqslant \gamma < \pi/2$. In this case, the boundary conditions are different from the case of the parallel vertical plates. We consider the natural requirement that the directrix is asymptotic to the x-axis at infinity. Then u satisfies (4) together with the boundary conditions

$$nu'(0) = -\cot\gamma \qquad \lim_{r\to\infty} u'(r) = 0.$$

Thus, $u'(r) \to 0$ as $u(r) \to 0$. This mean that the constant m in (5) is $m = 1$ and equation (4) written as

$$u'(r) = \pm\frac{u(r)\sqrt{4\kappa - \kappa^2 u(r)^2}}{2 - \kappa u(r)^2}.$$

Then it is possible to obtain an integration as follows:

$$-\frac{\sqrt{4-\kappa u(r)^2}}{\sqrt{\kappa}} + \frac{1}{\sqrt{\kappa}}\log\left(\frac{2+\sqrt{4-\kappa u(r)^2}}{u(r)}\right) = r + c,$$

where c is chosen so that $u'(0) = -\cot\gamma$. We can change the logarithm by a hyperbolic function:

$$-\frac{\sqrt{4-\kappa u(r)^2}}{\sqrt{\kappa}} + \frac{1}{\sqrt{\kappa}}\,\mathrm{arc}\cosh\left(\frac{2}{\sqrt{\kappa}u(r)}\right) = r + C,$$

with $C = c - (\log\sqrt{\kappa})/\sqrt{\kappa}$.

3. Symmetries of capillary strips

Our main objective in this section is the study of the shapes of κ-cylindrical surfaces focusing on their symmetries and distinguishing the cases in which κ is a positive or a negative constant. We note that each vertical plane orthogonal to the rulings is a plane of symmetry of S. Without loss of generality, we assume $\alpha(s) = (x(s), 0, z(s))$ is parametrized by an arc length:

$$x'(s)^2 + z'(s)^2 = 1, \qquad s \in I. \tag{6}$$

Let $\theta(s)$ be the angle between the vectors $\partial/\partial x$ and $\alpha'(s)$. It is well known that the curvature C_α of the curve α at the point s is exactly $\theta'(s)$, the derivative of $\theta(s)$. Using (6), equation (3) converts into the following three-dimensional autonomous differential equation $\mathcal{P}$:

$$\mathcal{P}: \begin{cases} x'(s) = \cos\theta(s), \\ z'(s) = \sin\theta(s), \\ \theta'(s) = \kappa z(s). \end{cases} \tag{7}$$

Theorem 3.1. *The system of ordinary differential equations $\mathcal{P}$ has a unique solution for each initial condition. Moreover, the maximal interval of the solution is $\mathbb{R}$.*

Proof. Classical theory yields the existence of solutions for each initial datum $x(0) = x_0$, $z(0) = z_0$, $\theta(0) = \theta_0$. If $z_0 = 0$, then the solution of $\mathcal{P}$ is $(x(s), z(s), \theta(s)) = (s, 0, 0)$. Assume now that $z_0 \neq 0$. From (7),

$$x''(s) = -\theta'(s)z'(s) = -\kappa z(s)z'(s) = -\frac{\kappa}{2}(z(s)^2)'.$$

Then there exists a constant $m \in \mathbb{R}$ such that

$$\cos\theta(s) = x'(s) = -\frac{\kappa}{2}z(s)^2 + m.$$

At $s = 0$, we have $m = \cos\theta_0 + \kappa z_0^2/2$. Then

$$z(s)^2 = z_0^2 + \frac{2}{\kappa}(\cos\theta_0 - \cos\theta(s)). \tag{8}$$

Therefore $z(s)$ is a bounded function. As a consequence of (8), the first derivatives of x, z and θ in (7) are bounded functions and this yields that the solutions can be continued indefinitely. This proves the result. q.e.d.

We recall that in the problems which we are interested in, figures 1(*a*) and (*b*), the initial conditions are

$$x(0) = 0, \qquad z(0) = z_0, \qquad \theta(0) = 0. \tag{9}$$

In view of our notation, this means that the starting point of α is $\alpha(0) = (0, 0, z_0)$ and the initial velocity is the horizontal vector $\alpha'(s) = (1, 0, 0)$. When we consider the directrix α of a κ-cylindrical surface, we assume that α satisfies $\mathcal{P}$ under the initial conditions (9). We prove that our κ-cylindrical surfaces have a rich symmetry.

Theorem 3.2 (Symmetry I). *Let $\alpha(s) = (x(s), 0, z(s))$ be the directrix of a κ-cylindrical surface. If $\sin\theta(s_0) = 0$, then α is symmetric with respect to the vertical line $x = x(s_0)$.*

Proof. Let $m \in \mathbb{Z}$ be such that $\theta(s_0) = m\pi$. The theorem is proved if for $s \in \mathbb{R}$,

$$x(s + s_0) - x(s_0) = x(s_0) - x(s_0 - s),$$

$$z(s + s_0) = z(s_0 - s),$$

$$\theta(s + s_0) = 2m\pi - \theta(s_0 - s).$$

However, these two sets of functions satisfy $\mathcal{P}$ with the same initial conditions at $s = 0$. The uniqueness of solutions of an O.D.E. concludes the proof. q.e.d.

In a similar way, one can show the following theorem.

Theorem 3.3 (Symmetry II). *Let $\alpha(s) = (x(s), 0, z(s))$ be the directrix of a κ-cylindrical surface. Assume that $z(s_0) = 0$. Then α is symmetric with respect to the point $\alpha(s_0)$.*

We end this section by studying the symmetries of our surfaces depending on the sign of κ. We point out that if (x, z, θ) is a solution of $\mathcal{P}$ and (9), then $(x, -z, -\theta)$ is a solution of $\mathcal{P}$ but with the initial condition $z(0) = -z_0$. This allows us to suppose that the signs of κ and z_0 agree.

Theorem 3.4 (Sessile case). *Let S be a κ-cylindrical surface with $\kappa > 0$ and let $\alpha(s)$ be its directrix curve. Then there exists a horizontal vector $\vec{v}$ orthogonal to the rulings such that S is invariant by the group of translations generated by $\vec{v}$. Moreover, the function $z = z(s)$ is periodic.*

Proof. From equation (8), $z \geqslant z_0$. On the other hand, equations (7) imply that θ is strictly increasing and its limit is ∞. Set $T > 0$ the first number such that $\theta(T) = 2\pi$. Again, the uniqueness of solutions in a ODE gives $\alpha(s + T) = \alpha(s) + (x(T), 0, 0)$. See figure 2. This means that the surface is invariant by the group of translations G, where the translation vector is $\vec{v} = (x(T), 0, 0)$, and that $z(s + T) = z(s)$ for any s. q.e.d.

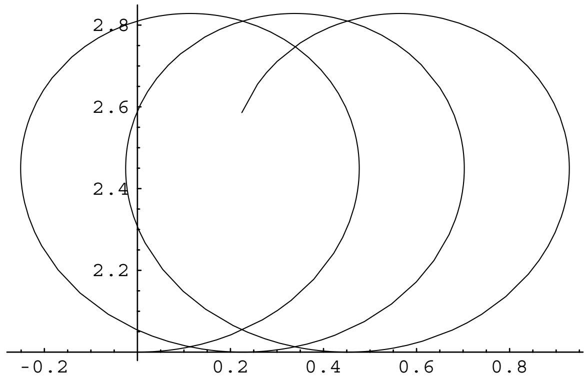

Figure 2. Directrix of a κ-cylindrical surface. Here $\kappa = 1$ and $z_0 = 2$.

Remark 3.1. As a consequence of theorem 3.4, and since $\theta(s)$ increases to infinity, the velocity vector rotates infinite times around the origin.

From (8) and because $\cos\theta(s)$ takes all the values into the interval $[-1, 1]$, we estimate the function z in terms of the lowest height z_0.

Corollary 3.1. *Let S be a κ-cylindrical surface with $\kappa > 0$ and denote z the height with respect to the plane Π. Then z satisfies*

$$z_0 \leqslant z(p) \leqslant \sqrt{\frac{4}{\kappa} + z_0^2}, \qquad p \in S,$$

where the upper and lower bounds are achieved on S.

Theorem 3.5 (Pendant case). *Let S be a κ-cylindrical surface with $\kappa < 0$ and let α be its directrix curve. Assume $z_0 < 0$.*

(i) If $z_0 < -2/\sqrt{-\kappa}$, there exists a horizontal vector $\vec{v}$ orthogonal to the rulings such that S is invariant by the group of translations generated by $\vec{v}$. Moreover $z(s)$ is a periodic function and

$$z_0 \leqslant z(s) \leqslant -\sqrt{z_0^2 + \frac{4}{\kappa}}. \tag{10}$$

(ii) If $z_0 = -2/\sqrt{-\kappa}$, then $z_0 \leqslant z < 0$, z is strictly increasing and $\lim_{s\to\infty} z(s) = 0$.

(iii) If $-2/\sqrt{-\kappa} < z_0 < 0$, then there exists a horizontal vector $\vec{v}$ orthogonal to the rulings such that S is invariant by the group of translations generated by $\vec{v}$. Moreover, $z(s)$ is a periodic function that vanishes in a discrete set of points, $z_0 \leqslant z(s) \leqslant -z_0$, where both extrema are achieved.

Proof.

(i) The hypothesis on z_0 together with (8) implies that z does not vanish and the inequalities (10). From (7), θ is a function strictly increasing with

$$\theta' \geqslant -\kappa\sqrt{z_0^2 + \frac{4}{\kappa}}.$$

This means that θ increases until infinity. Again, let $T > 0$ be the first number where $\theta(T) = 2\pi$. The same reasoning as in theorem 3.4 proves statement (i). In particular, $\cos\theta(s)$ takes all values in $[-1, 1]$ and the bounds in (10) are achieved. See figure 3.

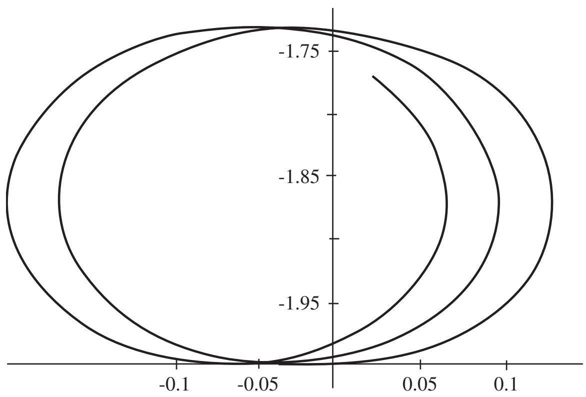

Figure 3. Directrix of a κ-cylindrical surface. Here $\kappa = -4$ and $z_0 = -2$.

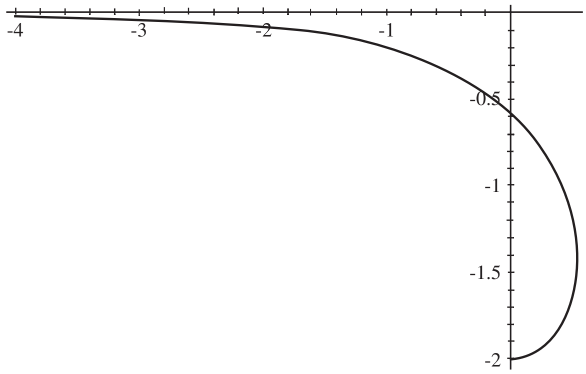

Figure 4. Directrix of a κ-cylindrical surface. Here $\kappa = -4$ and $z_0 = -1$.

(ii) Using (8) again, the only zeros of z occur when $\cos\theta(s) = -1$. If z vanishes at some point, the uniqueness of solutions would imply that $z = 0$, which is a contradiction. Thus $z < 0$ and $\cos\theta > -1$. Near $s = 0$, θ increases and the same occurs for the function z, with $0 \leqslant \theta(s) < \pi$. Moreover, $z(s) < 0$, $z'(s) > 0$ for $s \in \mathbb{R}$ and

$$\lim_{s\to\infty} z(s) = z_1 \qquad \lim_{s\to\infty} z'(s) = 0,$$

for some number $z_1 \leqslant 0$. If $z_1 < 0$, by (7) $\theta' \geqslant kz_1 \gg 0$, and this implies that θ increases indefinitely and reaches the value π. This contradiction yields $z_1 = 0$. See figure 4.

(iii) We first prove that z vanishes. Because $z_0 < 0$, the functions z and θ increase near $s = 0$. If $z(s) \leqslant 0$, then

$$\lim_{s\to\infty} z(s) = \delta \qquad \lim_{s\to\infty} z'(s) = 0,$$

for some number $\delta \leqslant 0$. If $\delta < 0$, the third equation in (7) implies that θ increases until ∞ and $\cos\theta(s)$ takes all possible values. Equation (8) together with $-2/\sqrt{-\kappa} < z_0$ implies that $z = 0$ at some point. If $\delta = 0$, equations (7) give again that either $\theta \to \infty$, which is a contradiction, or $\theta(s) \to \theta_0$, for some $\theta_0 < \infty$. Letting $s \to \infty$ in (7), we conclude that $\theta = \pi$, in contradiction to (8) and $-2/\sqrt{-\kappa} < z_0$.

Therefore, $z(s)$ vanishes at some point $s = s_0$. As $z(s)$ has a minimum at $s = 0$ ($z''(0) = \kappa z_0 > 0$), it follows by the symmetry of α with respect to that zero that

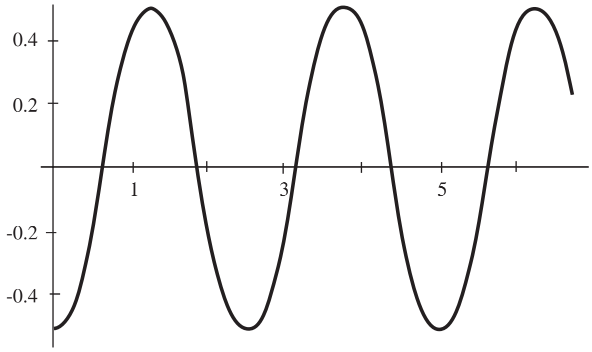

Figure 5. Directrix of a κ-cylindrical surface. Here $\kappa = -4$ and $z_0 = -0.5$.

$z_0 \leqslant z \leqslant -z_0$ (theorem 3.3). Using the same theorem, we have $z(2s_0) = -z_0$. The proof finishes using the symmetries of α given in theorem 3.2. See figure 5. Exactly, it follows that $T = 4s_0$ is the period of the function $z(s)$ and

$$x(s + 4s_0) = x(s) + x(4s_0), \tag{11}$$

$$z(s + 4s_0) = z(s), \tag{12}$$

$$\theta(s + 4s_0) = \theta(s). \tag{13}$$

q.e.d.

4. Estimates of capillary strips: case $\kappa > 0$

In this section, we consider κ-cylindrical surfaces S that are graphs of a function u over the strip Ω, that is, S is the surface $z = u(x, y)$ that projects simply onto Ω, where $u(x) = u(x, y)$, $-a < x < a$. The aim of this section is to derive estimates for the capillary rise, as for example, the centre height $u(0) = u_0$ and the outer height $u(a)$. See figure 6. For the two-dimensional problem, that is, the capillary problem when we dip a tube of circular section, we refer to [5–7]. Although a part of the following results holds independently of the sign of κ, in this section we restrict to the case that κ is a positive constant.

In analogy with the initial conditions (9), we consider

$$u(0) = u_0 > 0, \qquad u'(0) = 0. \tag{14}$$

Denote $u = u(r; u_0)$ the dependence of u with respect to the initial condition u_0. It follows from the uniqueness of solutions of (4)–(14) that

(i) $u(r; 0) = 0$ and $u(-r; u_0) = u(r; u_0)$.
(ii) $u(r; u_0) = -u(r; -u_0)$.

As pointed out in the previous section, the second property allows us to prescribe the sign of u_0 to be the same as the κ one. Thus, we assume that $u_0 > 0$. The boundary condition (2) is written now as $u'(a) = \cot\gamma$. By standard theory, we know that the solution u is defined around $r = 0$. We write

$$\sin\psi(r) = \frac{u'(r)}{\sqrt{1 + u'(r)^2}}, \qquad \cos\psi(r) = \frac{1}{\sqrt{1 + u'(r)^2}},$$

where $\psi(r)$ is the angle that it makes with $u(r)$ in the horizontal direction. Then (4)–(14) take the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\sin\psi(r) = \kappa u(r), \qquad \psi(0) = 0. \tag{15}$$

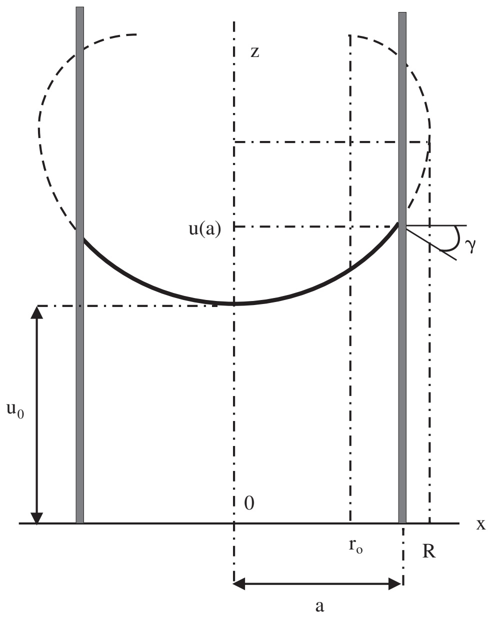

Figure 6. A section of a κ-cylindrical surface, $\kappa > 0$.

For $r > 0$ and close to 0,

$$\sin\psi(r) = \kappa \int_0^r u(t)\,\mathrm{d}t. \tag{16}$$

As $u_0 > 0$, the integrand is positive near to $r = 0$. Then $\sin\psi(r) > 0$, and so $u'(r) > 0$. This means that u increases provided that u is defined in the maximal interval $(0, R)$, $0 < R \leqslant \infty$. Multiplying by u' in (4), we have a first integration

$$\frac{1}{\sqrt{1 + u'(r)^2}} = -\frac{\kappa}{2}(u^2(r) - u_0^2) + 1. \tag{17}$$

Therefore

$$u'(r) = \sqrt{\frac{4}{(2 + \kappa(u_0^2 - u(r)^2))^2} - 1} \tag{18}$$

and

$$u(r) = u_0 + \int_0^r \sqrt{\frac{4}{(2 + \kappa(u_0^2 - u(t)^2)^2} - 1}\,\mathrm{d}t.$$

From (4)–(14), we have $u'' \geqslant \kappa u \geqslant \kappa u_0 > 0$. This implies that u' increases on r and $u'(R) = \infty$. Equation (18) gives

$$u(R) = \sqrt{\frac{2}{\kappa} + u_0^2}.$$

This means that $R < \infty$ and that the maximal distance between the centre and the outer heights of a κ-cylindrical surface is

$$u(R) - u_0 = \frac{2/\kappa}{u_0 + \sqrt{(2/\kappa) + u_0^2}}. \tag{19}$$

This was to be expected according to remark 3.1 and (8). As a consequence, if we fix $\kappa > 0$ and $u_0 > 0$, the angle of contact γ takes all the values in the range $0 \leqslant \gamma \leqslant \pi/2$. In addition, we have from (17) (or (8)) the following corollary.

Corollary 4.1. *Let $u = u(r; u_0)$ be the profile curve of a κ-cylindrical surface S. If γ is the contact angle with the vertical walls at $r = a$, then the difference between the centre and the outer heights $q := u(a) - u(0)$ satisfies*

$$q = \frac{2/\kappa(1-\sin\gamma)}{u_0 + \sqrt{u_0^2 + \frac{2}{\kappa}(1-\sin\gamma)}} < \sqrt{\frac{2}{\kappa}(1-\sin\gamma)}. \tag{20}$$

Fixing γ, the function $q = q(u_0)$ depending on u_0 satisfies

$$\lim_{u_0\to 0} q(u_0) = \sqrt{\frac{2}{\kappa}(1-\sin\gamma)}, \qquad \lim_{u_0\to\infty} q(u_0) = 0.$$

As u increases on r, we bound the integrand in (15) by $u_0 < u(t) < u(r)$, obtaining

$$\kappa u_0 < \frac{\sin\psi(r)}{r} < \kappa u(r). \tag{21}$$

Moreover,

$$\lim_{r\to 0} \frac{\sin\psi(r)}{r} = \kappa u_0.$$

This allows us to give the following result on existence.

Theorem 4.1. *Let $\kappa > 0$ be a constant of capillarity and let Ω be a strip. Given $0 \leqslant \gamma \leqslant \pi/2$, there exists a κ-cylindrical surface on Ω that makes a contact angle γ with the plates $P_1 \cup P_2$.*

Proof. The problem is reduced to finding $u_0 > 0$ such that $u'(a; u_0) = \cot\gamma$ or, in terms of the function ψ, that $\sin\psi(a) = \cos\gamma$, where $0 < \cos\gamma \leqslant 1$. If $u_0 = 0$, we know that $u(r; 0) = 0$. By continuity on the parameter u_0 for the solutions of (4)–(14),

$$\lim_{u_0\to 0} \sin\psi(a; u_0) = \sin\psi(a; 0) = 0.$$

Denote $(-R, R)$ the domain of $u(r; u_0)$, with $R = R(u_0)$. Since $R(0) = \infty$, there exists u_0 close to 0 such that the following holds:

$$R(u_0) > a \qquad \sin\psi(a; u_0) < \cos\gamma.$$

From (21), $u_0 \leqslant 1/(\kappa a)$. Again, the left inequality in (21) leads to

$$\lim_{u_0\to 1/(\kappa a)} \sin\psi(a; u_0) = 1.$$

By continuity, there exists $u_0 \in (0, 1/(\kappa a))$, such that the solution $u(r; u_0)$ satisfies $\sin\psi(a) = \cos\gamma$ and the result follows. q.e.d.

We begin with the purpose to control the centre height u_0 and the outer height $u(a)$ of a κ-cylindrical surface S. See figure 6. Consider two lower circular arcs $\Sigma^{(1)}$ and $\Sigma^{(2)}$ centred at the u-axis and defined, respectively, by two functions $u^{(1)}$ and $u^{(2)}$ as follows. The function $u^{(1)}$ satisfies $u^{(1)}(0) = u_0$ and the radius is $R_1 = 1/(\kappa u_0)$. On the other hand, for $u^{(2)}$ we have $u^{(2)}(0) = u_0$ and such that

$$\frac{\mathrm{d}}{\mathrm{d}r}u^{(2)}(a) = \frac{\mathrm{d}}{\mathrm{d}r}u(a)$$

so that $\Sigma^{(2)}$ meets the vertical plates at the same angle as does the solution surface.

The explicit functions $u^{(1)}$ and $u^{(2)}$ are

$$u^{(1)}(r) = u_0 + R_1 - \sqrt{R_1^2 - r^2}, \qquad R_1 = \frac{1}{\kappa u_0}, \tag{22}$$

$$u^{(2)}(r) = u_0 + R_2 - \sqrt{R_2^2 - r^2}, \qquad R_2 = \frac{a}{\cos\gamma}. \tag{23}$$

Claim. The three functions satisfy $u^{(1)}(r) < u(r) < u^{(2)}(r)$ in the interval $(0, a]$.

Proof (Of the claim). By (4), the curvature of $u(r)$ is

$$C_u(r) = \frac{u''(r)}{(1 + u'(r)^2)^{3/2}} = \kappa u(r).$$

Moreover C_u increases on r since $C_u'(r) = \kappa u'(r)$ and κ and $u'(r)$ are positive. At $r = 0$, $C_u(0) = \kappa u_0 = C_{u^{(1)}}(0)$ and $\Sigma^{(1)}$ has a constant curvature. Because $u(0) = u^{(1)}(0)$, we conclude then

$$\frac{\mathrm{d}}{\mathrm{d}r}u^{(1)}(r) < \frac{\mathrm{d}}{\mathrm{d}r}u(r), \qquad u^{(1)}(r) < u(r), \qquad 0 < r < a.$$

On the other hand, as $u^{(1)}$ and $u^{(2)}$ are circles and $C_{u^{(2)}}(a) > C_{u^{(1)}}(a)$, then $C_{u^{(2)}}(r) > C_{u^{(1)}}(r)$ for any r. At $r = 0$, $C_{u^{(2)}}(0) > C_u(0)$ and $u^{(2)}(0) = u(0)$. Thus, there exists $\delta > 0$ such that $u^{(2)}(r) > u(r)$ for $0 < r < \delta$. We assume that δ is the least upper bound of such values. By contradiction, suppose that $\delta < a$. As $u^{(2)}(\delta) = u(\delta)$ and $u^{(2)\prime}(\delta) \leqslant u'(\delta)$, we have $\psi^{(2)}(\delta) \leqslant \psi(\delta)$ and a consequence,

$$\int_0^\delta (C_u(r) - C_{u^{(2)}}(r))\,\mathrm{d}r = \int_0^\delta \frac{\mathrm{d}}{\mathrm{d}r}(\sin\psi(r) - \sin\psi^{(2)}(r))\,\mathrm{d}r \tag{24}$$

$$= \sin\psi(\delta) - \sin\psi^{(2)}(\delta) := \alpha(\delta) \geqslant 0. \tag{25}$$

Then there exists $\bar{r} \in (0, \delta)$ such that $C_u(\bar{r}) > C_{u^{(2)}}(\bar{r})$. As $C_u(r)$ increases, $C_u(r) > C_{u^{(2)}}(r)$ for $r \in (\bar{r}, a)$. In particular, and using $u'(a) = u^{(2)'}(a)$,

$$0 < \int_\delta^a (C_u(r) - C_{u^{(2)}}(r))\,\mathrm{d}r = \int_\delta^a \frac{\mathrm{d}}{\mathrm{d}r}(\sin\psi(r) - \sin\psi^{(2)}(r))\,\mathrm{d}r = -\alpha(\delta),$$

in contradiction to (24). q.e.d.

As a conclusion of the claim, the circular arcs $\Sigma^{(1)}$, $\Sigma^{(2)}$ lie, respectively, below and above the solution curve. Using the explicit formulae for $u^{(1)}$ and $u^{(2)}$, namely, (22)–(23), the statement of the claim is written, at the value $r = a$, as

$$u_0 + \frac{1}{\kappa u_0} - \sqrt{\frac{1}{\kappa^2 u_0^2} - a^2} < u(a) < u_0 + \frac{a}{\cos\gamma}(1 - \sin\gamma).$$

Together with the fact that $u_0 < 1/(a\kappa)$ and with corollary 4.1, we conclude the following theorem.

Theorem 4.2. *Let S be a κ-cylindrical surface where γ denotes the contact angle with the vertical plates P_i and $0 \leqslant \gamma < \pi/2$. Then the difference value $q = u(a) - u(0)$ satisfies*

$$\frac{1}{\kappa u_0}\left(1 - \sqrt{1 - a^2\kappa^2 u_0^2}\right) < q < \frac{a}{\cos\gamma}(1 - \sin\gamma), \tag{26}$$

$$q > \frac{2a(1 - \sin\gamma)}{1 + \sqrt{1 + 2\kappa a^2(1 - \sin\gamma)}}. \tag{27}$$

In the case that S is vertical at the walls P_i, and also using (20), we obtain

$$\frac{2a}{1+\sqrt{1+2\kappa a^2}} < q < \left\{a, \sqrt{\frac{2}{\kappa}}\right\}.$$

Another consequence of the claim is that it allows us to compare the volume per unit of length enclosed by the capillary surface S and the horizontal round cylinders determined by $\Sigma^{(1)}$ and $\Sigma^{(2)}$ (we refer to section 5 for a further discussion on the volume enclosed by the surface). Using the claim, we have

$$\int_0^a u^{(1)}(r)\,\mathrm{d}r < \int_0^a u(r)\,\mathrm{d}r < \int_0^a u^{(2)}(r)\,\mathrm{d}r. \tag{28}$$

The integral (28) for u can be computed by (4):

$$\int_0^a \kappa u(r)\,\mathrm{d}r = \cos\gamma. \tag{29}$$

For the integrals of $u^{(i)}$, we denote

$$F(u_0; R) = a(R+u_0) - \frac{a}{2}\sqrt{R^2-a^2} - \frac{R^2}{2}\arcsin\left(\frac{a}{R}\right).$$

Then (28) and (29) imply

$$F(u_0; R_1) < \frac{\cos\gamma}{\kappa} < F(u_0; R_2).$$

Thus, each one of the two above inequalities gives

$$a\left(\frac{1}{\kappa u_0}+u_0\right) - \frac{a}{2}\sqrt{\frac{1}{\kappa^2 u_0^2}-a^2} - \frac{\arcsin(a\kappa u_0)}{2\kappa^2 u_0^2} < \frac{\cos\gamma}{\kappa}, \tag{30}$$

$$\frac{\cos\gamma}{\kappa} < \frac{a^2}{\cos\gamma} + au_0 - \frac{a^2\tan\gamma}{2} - \frac{a^2}{2\cos^2\gamma}\left(\frac{\pi}{2}-\gamma\right). \tag{31}$$

From (31), we obtain a lower bound for u_0. On the other hand, and since $\partial F/\partial u_0 > 0$, let $u_0^+ > u_0$ be the unique number such that $F(u_0^+; R_1) = \cos\gamma/\kappa$. As $F(x; R) - ax$ is positive,

$$F\left(\frac{\cos\gamma}{a\kappa}; R_1\right) > \frac{\cos\gamma}{\kappa} = F(u_0^+; R_1),$$

and thus

$$u_0^+ < \frac{\cos\gamma}{a\kappa}.$$

Theorem 4.3. *Let $u = u(r; u_0)$ be the directrix of a κ-cylindrical surface S, $\kappa > 0$. If $0 \leqslant \gamma < \pi/2$ denotes the contact angle with the vertical plates P_i at $r = a$, then*

$$\frac{\cos\gamma}{a\kappa} - \frac{a}{\cos\gamma} + \frac{a\tan\gamma}{2} + \frac{a}{2\cos^2\gamma}\left(\frac{\pi}{2}-\gamma\right) < u_0 < u_0^+ < \frac{\cos\gamma}{a\kappa}. \tag{32}$$

The left inequality in (32) extends the one obtained by Laplace for the circular capillary tube [11]. The right inequality $u_0 < \cos\gamma/(a\kappa)$ is also a consequence by comparing the slopes of $u^{(1)}$ and u at the point $r = a$: $u^{(1)\prime}(a) < u'(a)$. On the other hand, the combination of inequalities (30) and (31) gives an estimate of u_0 that is rather cumbersome, even in the case $\gamma = 0$:

$$a\left(\frac{1}{\kappa u_0}+u_0\right) - \frac{a}{2}\sqrt{\frac{1}{\kappa^2 u_0^2}-a^2} - \frac{\arcsin(a\kappa u_0)}{2\kappa^2 u_0^2} < a^2 + au_0 - \frac{\pi a^2}{4}.$$

Now, we bound the outer height $u(a)$. Let us move down the circular arc $\Sigma^{(2)}$ until it meets the solution curve (tangentially) at $(a, u(a))$.

Claim. At the contact point $(a, u(a))$, the arc $\Sigma^{(2)}$ in its new position lies below the solution curve u.

Proof (Of the claim). The argument is similar as in the above proof. In the new position, we compare the curvatures of u and $\Sigma^{(2)}$: by (21), we have

$$C_u(a) = \kappa u(a) > \frac{\sin\psi(a)}{a} = C_{u^{(2)}}(a).$$

Thus, around the point $r = a$, $u(r) > u^{(2)}(r)$. By contradiction, assume that there is $\delta \in (0, a)$ such that $u^{(2)}(r) < u(r)$ for $r \in (\delta, a)$ and $u^{(2)}(\delta) = u(\delta)$. Since $u'(\delta) \geqslant u^{(2)\prime}(\delta)$, then $\psi^{(2)}(\delta) \leqslant \psi(\delta)$. This implies

$$\int_\delta^a (C_{u^{(2)}}(r) - C_u(r))\,\mathrm{d}r = \sin\psi(\delta) - \sin\psi^{(2)}(\delta) \geqslant 0. \tag{33}$$

Then there would be $\bar{r} \in (\delta, a)$ such that $C_{u^{(2)}}(\bar{r}) - C_u(\bar{r}) > 0$. As $C_u(r)$ increases on r, $C_u(r) < C_{u^{(2)}}(r)$ on $(0, \bar{r})$ and hence also throughout $(0, \delta) \subset (0, \bar{r})$. Using (33),

$$0 < \int_0^\delta (C_{u^{(2)}}(r) - C_u(r))\,\mathrm{d}r = \sin\psi(\delta) - \sin\psi^{(2)}(\delta) \leqslant 0.$$

This contradiction shows the claim. q.e.d.

Let $u^{(3)}$ be the function that defines the displaced arc $\Sigma^{(2)}$. Then the claim allows us to estimate the value $u(a)$ by using

$$\int_0^a u^{(3)}(r)\,\mathrm{d}r < \int_0^a u(r)\,\mathrm{d}r.$$

Recall that the centre of $u^{(3)}$ is $u_0 - (u^{(2)}(a) - u(a))$. Then

$$F(u_0 + u(a) - u^{(2)}(a); R_2) < \frac{\cos\gamma}{\kappa}.$$

As a consequence theorem 4.4 follows.

Theorem 4.4. *With the same notation as in theorem 4.3, for any $0 \leqslant \gamma < \pi/2$, we have*

$$u(a) < \frac{\cos\gamma}{\kappa a} - \frac{a}{2}\tan\gamma + \frac{a}{2\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right). \tag{34}$$

Using again the circular arc $u^{(3)}$, we obtain corollary 4.2.

Corollary 4.2. *For any $r \in (0, a)$ and $0 \leqslant \gamma < \pi/2$, we have*

$$\frac{r^2\kappa u_0}{1 + \sqrt{1 - r^2\kappa^2 u_0^2}} < u(r) - u_0 < \frac{a}{\cos\gamma} - \sqrt{\frac{a^2}{\cos^2\gamma} - r^2} \tag{35}$$

and

$$u(a) + \frac{a\sin\gamma}{\cos\gamma} - \sqrt{\frac{a^2}{\cos^2\gamma} - r^2} < u(r) - u_0. \tag{36}$$

Proof. Inequalities (35) are a consequence of $u^{(1)}(r) < u(r) < u^{(2)}(r)$ and (22)–(23). The lower bound (36) is a consequence of $u^{(3)}(r) < u(r)$ in $(0, a)$. q.e.d.

We finish this section obtaining new estimates of the values u_0 and $q = u(a) - u(0)$ from below. As $u'(r) > 0$ in $(0, a)$, we may introduce the inclination angle $\psi = \arctan u'(r)$ as an independent variable. We have then

$$\frac{\mathrm{d}r}{\mathrm{d}\psi} = \frac{\cos\psi}{\kappa u} \qquad \frac{\mathrm{d}u}{\mathrm{d}\psi} = \frac{\sin\psi}{\kappa u}. \tag{37}$$

Simple quadratures give

$$u(\psi) = \sqrt{u_0^2 + \frac{2}{\kappa}(1 - \cos\psi)} \tag{38}$$

previously obtained in (8). As a consequence, *the difference in squares of the maximum and the minimum heights satisfies*

$$u^2(\psi) - u_0^2 = \frac{2}{\kappa}(1 - \cos\psi), \tag{39}$$

and thus is independent of the width of the strip Ω.

As $r\kappa u_0 < \sin\psi$, we deduce from (38) corollary 4.3.

Corollary 4.3. *In the range* $0 < \psi \leqslant \pi/2$ *there holds*

$$u(\psi) < \sqrt{\left(\frac{\sin\psi}{\kappa r}\right)^2 + \frac{2}{\kappa}(1 - \cos\psi)}.$$

Now, the following computations are similar to the case that Ω is a circular disc [4]. Let

$$m = \cos(\psi/2), \qquad p = \sqrt{1 + \kappa(r/m)^2}.$$

The function r/m increases in ψ. As

$$u < \frac{\sin\psi}{\kappa r} p,$$

it follows from (37) that $p\,\mathrm{d}r > r\cot\psi\,\mathrm{d}\psi$, that is

$$\frac{\sqrt{m^2 + \kappa r^2}}{mr}\,\mathrm{d}r > \cot\psi\,\mathrm{d}\psi. \tag{40}$$

From (21),

$$\lim_{\psi\to 0} \frac{r(\psi)}{\sin\psi} = \frac{1}{\kappa u_0}.$$

An integration in (40) leads to theorem 4.5.

Theorem 4.5. *In the range* $0 < \psi \leqslant \pi/2$*, there holds*

$$u_0 > \frac{\sin\psi}{2\kappa r}\frac{\kappa}{m}(1 + p)\mathrm{e}^{1-p}. \tag{41}$$

Theorem 4.6. *For any* $0 \leqslant \gamma < \pi/2$*, we have*

$$\frac{2(1 - \sin\gamma)}{\kappa f(\gamma)} < u(a) - u_0 < \frac{a(1 - \sin\gamma)}{\cos\gamma},$$

where

$$f(\gamma) = \frac{2\cos\gamma}{\kappa a} - \frac{a\tan\gamma}{2} + \frac{a}{2\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right).$$

Proof. The right inequality is a consequence of (26). For the left one, we know from (39) that

$$u(a) - u_0 = \frac{2(1 - \sin\gamma)}{\kappa(u(a) + u_0)}.$$

Then we bound $u(a)$ and u_0 using (32) and (34). q.e.d.

5. Sessile liquid channels

In next two sections we study the setting of figure 1(b), that is, the spreading of a liquid on the strip $\Omega = \{(x, y); -a < x < a\}$ of the plane $\Pi = \{z = 0\}$, completely wetting the domain Ω and confining the liquid up the boundary $\partial\Omega$. We assume that the liquid–air interface S is invariant in the y-direction. Firstly, in this section we address the case in which the capillarity constant κ is positive. This indicates that the gravity field points towards Π.

Let $u = u(r; u_0)$ be the solution of (4)–(14), with $u_0 > 0$. We write (4) in terms of the inclination angle ψ with respect to the r-axis:

$$\frac{\mathrm{d}r}{\mathrm{d}\psi} = \frac{\cos\psi}{\kappa u} \qquad \frac{\mathrm{d}u}{\mathrm{d}\psi} = \frac{\sin\psi}{\kappa u}. \tag{42}$$

We point out that this angle ψ agrees at the contact between S and Π with the value γ, the angle at which the surface meets Π along the boundary. We know from section 3 that there exists a finite value $R > 0$ where u is vertical, that is, $r(\pi/2) = R$. Theorem 3.4 asserts that ψ takes any real number and the solution $u(\psi)$ can be continued as a solution of (4).

Theorem 5.1. *The functions $r(\psi)$ and $u(\psi)$ can be continued throughout the range $0 < \psi \leqslant \pi$. Moreover, there exists a value $r_o = \lim_{\psi\to\pi} r(\psi)$, being $r_o > 0$. The function $u(\psi)$ monotonically increases in $(0, \pi)$ whereas the function $r(\psi)$ increases to the value $r = R$ at $\psi = \pi/2$ and decreases next in the interval $(\pi/2, \pi)$ until the value r_o.*

Proof. From (21), we know that $r < 1/(\kappa u_0)$. Denote by $(-)$ and $(+)$ the parts of the meniscus defined for $\psi \in (0, \pi/2)$ and $\psi \in (\pi/2, \pi)$, respectively. For values $r < R$ close to R, we have from an integration of (15) from r to R:

$$1 - \sin\psi^-(r) = \kappa\int_r^R u^-(t)\,\mathrm{d}t, \qquad 1 - \sin\psi^+(r) = \kappa\int_r^R u^+(t)\,\mathrm{d}t.$$

Subtracting both expressions, we obtain

$$\sin\psi^-(r) - \sin\psi^+(r) = \kappa\int_r^R (u^+(t) - u^-(t))\,\mathrm{d}t. \tag{43}$$

In particular, $u^+(r) > u^-(r)$ and hence $\sin\psi^-(r) > \sin\psi^+(r)$. From (42), u^+, u^- both increase in ψ as one can continue the solution until $\psi = \pi$. However, it is not possible to arrive at $r = 0$ in (43), since it would imply

$$0 > -\sin\psi^+(0) = \kappa\int_0^R (u^+(t) - u^-(t))\,\mathrm{d}t > 0.$$

Hence, we conclude the existence of the value $r_o = \lim_{r\to\pi} r(\psi) > 0$. The monotonicity of the functions $u(\psi)$ and $r(\psi)$ is a consequence of (42). q.e.d.

Corollary 5.1. *Given $\kappa, u_0 > 0$ and $0 < \gamma \leqslant \pi$, there exists exactly one κ-cylindrical surface given by the profile $u(r; u_0)$ which makes a contact angle γ with the support plane.*

We now study the behaviour of a sessile liquid channel with respect to the volume that encloses the support plane. Although our channels have infinite volume, we can consider the volume per unit of length. For $0 < r \leqslant a$, let $\Omega_b = (-r, r) \times (-b/2, b/2) \subset \Omega$ be a bounded rectangle in Ω. The enclosed volume of S in Ω_b is

$$2b\left(ru(r) - \int_0^r u(t)\,\mathrm{d}t\right).$$

We call the *volume* of S the volume per unit of length, that is,

$$\mathcal{V}(r) = 2\left(ru(r) - \int_0^r u(t)\,\mathrm{d}t\right).$$

If we view $r = r(\psi)$, we also write $\mathcal{V}(\psi)$ to denote the dependence on the angle ψ. Using (15), we have

$$\mathcal{V}(\psi) = 2\left(ru(r) - \frac{\sin\psi}{\kappa}\right). \tag{44}$$

As we have done in this paper, we use γ to denote the value of the angle ψ at which the interface meets the support plane, and let $a = a(\gamma)$. We first establish a result of existence and uniqueness for a given volume; next, we will do estimates of such a volume as a function on the initial data.

Theorem 5.2. *Let $\kappa > 0$. Given $V > 0$ and $0 < \gamma \leqslant \pi$, there is exactly one κ-cylindrical surface with boundary angle γ and volume $\mathcal{V}(\gamma) = V$.*

Proof. The function $\mathcal{V} = \mathcal{V}(u_0)$ is continuously differentiable on the initial condition u_0: this follows from the standard continuous dependence theorem of the ODE theory.

We first prove the existence. Recall the notation $R = r(\pi/2)$. By (21), $R < 1/(\kappa u_0)$. Thus $R \to 0$ as $u_0 \to \infty$. We know from section 4 that the function $u^{(3)}$ defined there lies below u. Hence a circle of radius R contains the function $u(\psi)$, for $0 < \psi < \pi/2$. As we have viewed in the proof of theorem 5.1, $\sin\psi^+ < \sin\psi^-$ and thus the same circle $u^{(3)}$ also contains the upper part of u, that is, $u(\psi)$, for $\pi/2 < \psi \leqslant \pi$. Consequently the function volume $\mathcal{V}(u_0)$ satisfies $\mathcal{V} < \pi R^2$, which goes to 0 as $u_0 \to \infty$.

Now, let $u_0 \to 0$. From (41), $r(\gamma; u_0) \to \infty$ for any fixed $0 \leqslant \gamma \leqslant \pi/2$. By (38), we know that $u(\gamma; u_0) > \sqrt{2/\kappa(1 - \cos\gamma)}$. Since the surface is convex, we have $\mathcal{V} \to \infty$ as $u_0 \to 0$. In the case $\gamma > \pi/2$, we obtain $\mathcal{V}(\gamma; u_0) > \mathcal{V}(\pi/2; u_0)$ and the same conclusion holds.

Now, let us fix γ and V. By letting u_0 be between 0 and ∞, the volume function $\mathcal{V}$ takes all values. The continuity of $\mathcal{V}$ with respect to u_0 gives the existence of a κ-cylindrical surface with the prescribed volume V.

The proof of uniqueness is obtained if, fixing the value of the angle ψ, $0 < \psi \leqslant \pi$, we show

$$\dot{\mathcal{V}}(\psi) = \dot{\mathcal{V}}(\psi; u_0) := \frac{\partial\mathcal{V}(\psi; u_0)}{\partial u_0} < 0, \tag{45}$$

for all $u_0 > 0$ and each fixed ψ in $0 < \psi \leqslant \pi$. For this purpose, we first prove the inequality

$$\frac{\mathrm{d}\dot{\mathcal{V}}}{\mathrm{d}\psi} < 0 \qquad \text{in } (0, \pi]. \tag{46}$$

Once proved, we continue the proof of (45) as follows. The $(\cdot)$ symbol indicates the differentiation with respect to u_0:

$$\dot{r} = \dot{r}(\psi) = \frac{\partial r}{\partial u_0}(\psi; u_0), \qquad \dot{u} = \dot{u}(\psi) = \frac{\partial u}{\partial u_0}(\psi; u_0).$$

Fixing the boundary angle ψ, a differentiation of (44) with respect to u_0 yields

$$\dot{\mathcal{V}} = 2(\dot{r}u + r\dot{u}). \tag{47}$$

It is known that $\dot{r}(0) = 0$ and $\dot{u}(0) = 1$. Then it follows from (47) that $\dot{\mathcal{V}}(0; u_0) = 0$. Then (46) leads to $\dot{\mathcal{V}}(u_0) < 0$ in $(0, \pi)$ for any u_0. This proves (45) and so the uniqueness of the statement of the theorem. Thus, we focus on proving inequality (46). By using (42),

$$\frac{\mathrm{d}\dot{\mathcal{V}}}{\mathrm{d}\psi} = \frac{2\sin\psi}{\kappa u^2}(\dot{r}u - r\dot{u}). \tag{48}$$

We shall prove that $\dot{r} < 0$ and $\dot{u} > 0$. Again, (42) yields

$$\frac{\mathrm{d}\dot{r}}{\mathrm{d}\psi} = -\frac{\cos\psi\,\dot{u}}{\kappa u^2}, \qquad \frac{\mathrm{d}\dot{u}}{\mathrm{d}\psi} = -\frac{\sin\psi\,\dot{u}}{\kappa u^2}. \tag{49}$$

As $u_0 > 0$ and $(\mathrm{d}r/\mathrm{d}\psi)(0) = -1/(\kappa u_0^2) < 0$, we know that $\dot{r} < 0$ in an initial interval $J = (0, \delta)$, with $\delta \leqslant \pi$.

On the other hand, and as $\dot{u}(0) = 1$, then $\dot{u} > 0$ for sufficiently small values of ψ. Then and from (49) we have

$$\frac{\mathrm{d}\dot{u}}{\mathrm{d}\psi} > -\frac{\sin\psi\,\dot{u}}{\kappa u_0^2}.$$

By integrating this expression between the angles 0 and ψ, we obtain

$$\dot{u} > \exp\left\{\frac{\cos\psi - 1}{\kappa u_0}\right\}. \tag{50}$$

It follows that $\dot{u} > 0$ and (50) holds in J. From (48),

$$\frac{\mathrm{d}\dot{\mathcal{V}}}{\mathrm{d}\psi} < 0 \qquad \text{in } J. \tag{51}$$

By contradiction, we suppose there exists a value ψ_0, $0 < \psi_0 < \pi$, such that $\dot{r}(\psi_0) = 0$. Take ψ_0 the first ψ with this property. As $\dot{\mathcal{V}}(0) = 0$, inequality (51) implies that $\dot{\mathcal{V}}(\psi_0) < 0$. However, inequality (50) gives $\dot{u}(\psi_0) > 0$ and from the expression of $\dot{\mathcal{V}}$ in (47), we conclude

$$\dot{\mathcal{V}}(\psi_0) = 2r(\psi_0)\dot{u}(\psi_0) > 0.$$

This contradiction implies then that $\dot{r}$ is negative in $(0, \pi)$ and consequently $\dot{u} > 0$. This proves inequality (46). q.e.d.

After this result, and as was announced in the introduction of the paper, we now give a control of the volume $\mathcal{V}$ enclosed by a liquid channel. Let $0 < \gamma \leqslant \pi$ be the contact angle and $a = r(\gamma)$. We recall the function $u^{(3)}$ defined in section 4, that is, the circular arc $u^{(3)}$ is centred on the z-axis and is tangent to u at $r = a$ when $0 < \gamma \leqslant \pi/2$ and $u^{(3)}(r) < u(r)$ for $0 < r < a$, with $u^{(3)}(a) = u(a)$. We claim that the halfcircle determined by $u^{(3)}$ in the halfplane $r > 0$ is contained inside the solution curve u, with a single point of contact, namely, $(a, u(a))$. For this purpose, we compare the curvatures of the curves u and $u^{(3)}$. By using (15), (21) and (23),

$$C_u(a) = \kappa u(a) > \frac{\cos\gamma}{a} = \frac{1}{R_2} = C_{u^{(3)}}.$$

Using (15) and (42) again, the function C_u increases on ψ since

$$\frac{\mathrm{d}C_u}{\mathrm{d}\psi} = \kappa\frac{\mathrm{d}u}{\mathrm{d}\psi} = \frac{\sin\psi}{u} > 0.$$

This proves the inclusion property. As a consequence, we can compare the volume $\mathcal{V}$ of the liquid channel with respect to the halfcylinder determined by $u^{(3)}$. Using the notation $R = r(\pi/2)$, we have theorem 5.3.

Theorem 5.3. *Let $\kappa > 0$ and let S be a κ-cylindrical surface resting on a horizontal plane. Denote by γ the angle of contact and $\mathcal{V}(\gamma)$ the volume enclosed by S. In the range $0 < \gamma \leqslant \pi/2$, there holds*

$$\mathcal{V}(\gamma) < \frac{a^2}{\sin^2\gamma}(\gamma - \sin\gamma\cos\gamma). \tag{52}$$

If $\pi/2 \leqslant \gamma \leqslant \pi$, we have

$$\mathcal{V}(\gamma) < R^2(\gamma - \sin\gamma\cos\gamma). \tag{53}$$

Proof. For the case $0 < \gamma \leqslant \pi/2$, it suffices to point out that

$$\int_0^a u^{(3)}(r)\,\mathrm{d}r = a^2 \cot\gamma + a\,u(a) - \frac{a^2}{\sin^2\gamma}\left(\frac{\gamma}{2} + \frac{1}{2}\sin\gamma\cos\gamma\right).$$

In order to prove (53), we consider the halfcircle $u^{(3)}$ centred at $(0, u(R))$ of radius R, which is tangent to u at $r = R$. It is known that the lower part of this circle lies below u. We parametrize $u^{(3)}$ by the angle ϕ with the r-axis in each point. We prove that $u^{(3)}(\gamma) > u(\gamma)$. Fixing $r < R$ and $\pi/2 \leqslant \psi(r) \leqslant \pi$, the function $u^{(3)}$ lies above u at r, $u^{(3)}(r) > u(r)$ and $\sin\psi^+(r) < \sin\phi^+(r)$. As $\sin\phi^+$ decreases as $\phi^+ \to \pi$, $u^{(3)}(\gamma) > u(\gamma)$. Then inequality (53) is a consequence of the explicit computation of the volume of $u^{(3)}$ until $\phi = \gamma$. q.e.d.

We now see a new bound of the volume.

Theorem 5.4. *With the same notation as in theorem 5.3, we have*

$$\mathcal{V}(\gamma) > \frac{\gamma - \sin\gamma\cos\gamma}{\kappa^2 u(\gamma)^2} \qquad 0 < \gamma \leqslant \pi/2. \tag{54}$$

Proof. Consider the circle

$$v(r) = u_0 + R - \sqrt{R^2 - r^2}, \qquad R = \frac{1}{\kappa u(\gamma)}.$$

The curve v touches u tangentially at $(0, u_0)$. As $C_u(0) = \kappa u_0 < C_v(0) = \kappa u(\gamma)$, we have $v(r) > u(r)$, for each r where v is defined. If we prove that $v(\gamma) < u(\gamma)$, then the arc v until $\phi = \gamma$ lies above u and this allows us to obtain the lower bound for the volume of u, namely, the volume enclosed by v gives (54). At the point where v attains the inclination angle $\phi = \gamma$,

$$v(\gamma) = u_0 + \frac{1 - \cos\gamma}{\kappa u(\gamma)}.$$

Then $u(\gamma) > v(\gamma)$ if $u(\gamma) - u_0 > v(\gamma) - u_0$. By using (38), we have to prove

$$\frac{2(1-\cos\gamma)/\kappa}{u_0 + \sqrt{u_0^2 + (2/\kappa)(1-\cos\gamma)}} > \frac{1-\cos\gamma}{\kappa u(\gamma)}$$

or equivalently

$$2u(\gamma) > u_0 + \sqrt{u_0^2 + \frac{2}{\kappa}(1-\cos\gamma)}.$$

But the second summand on the right side is exactly $u(\gamma)$, using (38) again. This proves that $v(\gamma) < u(\gamma)$. q.e.d.

Now we prove the following inclusion result.

Theorem 5.5. *Let $\kappa > 0$. Let $0 < \gamma \leqslant \pi/2$ and let S be a κ-cylindrical surface supported on a horizontal plane and making a contact angle γ. Let $\mathcal{V}$ be its volume. Then every κ-cylindrical surface S with smaller volume and making the same contact angle can be translated rigidly so that it lies strictly interior into S.*

Proof. Assume that S is given by the solution $u(r; u_0)$, $u_0 > 0$. From (46), if the volume is smaller, then the new surface is given by a solution of (4) for an initial value greater than u_0, namely, $u_0 + \delta$, with $\delta > 0$. Consider the solution $u^\delta = u(r; u_0 + \delta)$, with $\delta > 0$, and let $\psi^\delta(r)$ be the corresponding angle function with respect to the r-axis. From (15),

$$\sin\psi^\delta(r) - \sin\psi(r) = \kappa\int_0^r (u^\delta(t) - u(t))\,\mathrm{d}t.$$

As $(u^\delta - u)(0) = \delta > 0$, then $\sin\psi^\delta(r) > \sin\psi(r)$. By (15) and (21), we have $(u^\delta - u)'(r) > 0$. It follows that $u^\delta(r) - u(r) > \delta$ or $u^\delta(r) - \delta > u(r)$. Then if we move downwards the curve u^δ a distance δ, then it lies above the curve u except at the single point $(0, u_0)$. The result is proved if for any $\delta > 0$, $u^\delta(r) - \delta < u(r)$ at the points where the angle γ is achieved. Given γ,

$$(u^\delta - u - \delta)(\gamma) = \int_0^\delta (\dot{u}(\gamma; u_0) - 1)\,\mathrm{d}u_0. \tag{55}$$

Since $\dot{u}(0; u_0) = 1$, an integration from $\psi = 0$ to $\psi = \gamma$ leads to

$$\dot{u}(\gamma) = 1 + \int_0^\gamma \frac{\mathrm{d}\dot{u}}{\mathrm{d}\psi}\,\mathrm{d}\psi.$$

As seen in the proof of theorem 5.2, $\dot{u} < 0$ and (49) implies that the above integrand is negative. Thus $\dot{u}(\gamma) - 1 < 0$, which says that the integrand in (55) is negative. This proves the result. q.e.d.

We end the section obtaining new estimates of a sessile liquid channel, with special attention to the fact if the contact angle lies in the range $[\pi/2, \pi]$. Recall $R = r(\pi/2)$, that is, the r-coordinate where u is vertical.

Theorem 5.6. *Let S be a κ-cylindrical surface supported on Π and $\kappa > 0$. Suppose that $u = u(\psi)$ is the profile of S, where ψ denotes the inclination angle with respect to the r-axis. Then in the range $0 < \gamma \leqslant \pi$ there holds*

$$u(\gamma) - u_0 < \sqrt{\frac{2(1-\cos\gamma)}{\kappa}}.$$

In the range, $\pi/2 \leqslant \gamma \leqslant \pi$, we have

$$R - r(\gamma) < \frac{1}{\sqrt{\kappa}}\left(\sqrt{2} + \log\left(\tan\frac{\pi}{8}\right)\right) - 2\cos\frac{\psi}{2} - \log\left(\tan\frac{\gamma}{4}\right),$$

$$u(\gamma) - u(R) < \frac{\sqrt{2(1-\cos\gamma)} - \sqrt{2}}{\sqrt{\kappa}}.$$

In particular, and setting $\gamma = \pi$,

$$R - r_o < \sqrt{\frac{2}{\kappa}}, \qquad u(\pi) - u(R) < \frac{2-\sqrt{2}}{\sqrt{\kappa}}.$$

Proof. By using (38), we estimate $u(\psi)$ from below as

$$u(\psi) > \sqrt{\frac{2}{\kappa}(1-\cos\psi)}.$$

In combination with (42), we obtain,

$$\frac{\mathrm{d}u}{\mathrm{d}\psi} < \frac{\sin\psi}{\sqrt{2\kappa(1-\cos\psi)}},$$

and for $\pi/2 \leqslant \psi \leqslant \pi$,

$$\frac{\mathrm{d}r}{\mathrm{d}\psi} > \frac{\cos\psi}{\sqrt{2\kappa(1-\cos\psi)}}.$$

The proof finishes by integrating the above two inequalities. q.e.d.

The results obtained in theorem 5.6 could be of interest in physics in the context described in figure 1(*b*), see introduction. Given a strip Ω of hydrophilic type, we completely cover this domain by liquid but confining the liquid to the strip up to its boundary. For this, we assume that the exterior of Ω is made of a solid of hydrophobic character in such a way that the liquid does not contact the exterior of the strip. When the amount of liquid is small, the shape adopted by the liquid–air interface is a graph. As more and more liquid is added to the strip, the interface ceases to be a graph and the contact angle γ with the horizontal substrate lies in the range $\pi/2 \leqslant \gamma \leqslant \pi$. Then there exists a critical volume where the liquid channel has a contact itself. See [8, 12, 17]. The value $2(R - r(\gamma))$ gives the minimum distance which two consecutive hydrophilic strips can approach without being in contact. The following theorem gives new estimates of the above distance.

Theorem 5.7. *Let S be a κ-cylindrical surface supported on the plane Π and $\kappa > 0$. Assume that the contact angle γ satisfies $\pi/2 \leqslant \gamma \leqslant \pi$. Then*

$$\frac{1}{\sqrt{\kappa}} \frac{1 - \sin\gamma}{\sqrt{2(1 - \cos\gamma) + \kappa u_0^2}} < R - r(\gamma) < \frac{1}{\sqrt{\kappa}} \frac{1 - \sin\gamma}{\sqrt{2 + \kappa u_0^2}}.$$

Proof. As $\cos\gamma < \cos\psi < 0$, (38) gives

$$\sqrt{u_0^2 + \frac{2}{\kappa}} < u(\psi) < \sqrt{u_0^2 + \frac{2}{\kappa}(1 - \cos\gamma)}.$$

Substituting into (42), we obtain

$$\frac{1}{\sqrt{\kappa}} \frac{\cos\psi}{\sqrt{2 + \kappa u_0^2}} < \frac{\mathrm{d}r}{\mathrm{d}\psi} < \frac{1}{\sqrt{\kappa}} \frac{\cos\psi}{\sqrt{2(1 - \cos\gamma) + \kappa u_0^2}},$$

and the result follows by integrating from $\psi = \pi/2$ until $\psi = \gamma$. q.e.d.

We do a comparison with the situation of an absence of gravity and pieces of infinite cylinders, whose boundary is $\partial\Omega = L_1 \cup L_2$. For $\pi/2 \leqslant \gamma \leqslant \pi$, the amount $R - r(\gamma)$ is exactly $(1 - \sin\gamma)/(2H)$, where H is the mean curvature of the cylinder.

6. Pendant liquid channels

This section is devoted to the study of κ-cylindrical surfaces when $\kappa < 0$. In theorem 3.5, we have studied its behaviour, see figures 3–5. Let $\alpha(s) = (x(s), 0, z(s))$ be the directrix of such a surface S and without loss of generality, we assume $z_0 = z(0) < 0$. We will describe the shape of these surfaces varying the initial value z_0, as done in theorem 3.5. A first matter that we ask is whether S is a graph on Π, that is, if α is a graph on the x-axis. Theorem 3.5 yields the necessary condition $z_0 > -2/\sqrt{-\kappa}$. However, we have theorem 6.1.

Theorem 6.1. *Let S be a κ-cylindrical surface, $\kappa < 0$. Then S is a graph on Π if and only if*

$$-\sqrt{\frac{2}{-\kappa}} < z_0 < 0. \tag{56}$$

In such a case, we set α as $\alpha(r) = (r, u(r))$, where $u = u(r; u_0)$ is a solution of (4) with $u_0 = z_0 < 0$. Moreover the following properties hold (see figure 5).

(i) The function u is periodic and its domain is $\mathbb{R}$.
(ii) The function u vanishes in an infinite discrete set of points.
(iii) The inflections of u are their zeros.
(iv) We have $u_0 \leqslant u(r) \leqslant -u_0$, the values $\pm u_0$ are attained and they are the only critical points of u.

Proof. As usual, let $\psi(s)$ be the angle that makes $\alpha'(s)$ with the x-axis. From (8),

$$\cos\psi(s) = 1 - \frac{\kappa}{2}(z^2 - z_0^2) \geqslant 1 + \frac{\kappa}{2}z_0^2.$$

Therefore, $\alpha(s)$ has no vertical points if and only if $z_0^2 < -2/\kappa$. In such a case, $x' = \cos\psi >> 0$ and x increases strictly to infinity. Let the new variable $r = x(s)$, defined in $\mathbb{R}$ and put $u(r) = z \circ x^{-1}(r)$. In particular, the derivative $u' = \tan\psi$ is bounded. We apply theorem 3.5 and with the notation used there, let $r_T = x(4s_0)$. Then

$$u(r + r_T) = u(x(s) + x(4s_0)) = u(x(s + 4s_0)) = z(s + 4s_0) = z(s) = u(r).$$

This proves that u is a periodic function. From (4), the inflections agree with the zeros of u. The rest of the properties are a consequence of theorem 3.5. q.e.d.

The behaviour of u is as follows. After $r = 0$, u increases on r until it vanishes at some point R. Theorem 3.3 says that u is symmetric with respect to the point $(R, 0)$. Thus, u increases until the value $r = 2R$, where u takes the value $-u_0$. Again, the symmetry of u with respect to the line $r = 2R$ implies that u decreases until it arrives at $r = 4R$ to the value u_0 again. From this position, the curve u repeats the same behaviour as the periodicity of u.

We write (8) as

$$u(\psi)^2 - u_0^2 = \frac{2}{\kappa}(1 - \cos\psi). \tag{57}$$

In the general case for u_0, given a solution $u(r; u_0)$ of (4)–(14), we can estimate the initial interval of u until the first vertical point.

Lemma 6.1. *Consider $u = u(r; u_0)$ the solution of (4)–(14). Then u can be continued at least until the value $r = 1/(\kappa u_0)$. Furthermore,* $\sin\psi(r) < \kappa u_0 r$.

Proof. Since $(\sin\psi)' = \kappa u$, the function $\sin\psi$ is strictly increasing on r whenever u is negative. Then for $r > 0$ and near to $r = 0$, $\sin\psi = u'/\sqrt{1 + u'^2}$ is positive. As a conclusion, u increases on r near to 0. Given $0 < t < r$, we have $u_0 < u(t) < u(r)$. Because $\kappa < 0$ and by using (16) we have

$$\kappa u(r) < \frac{\sin\psi(r)}{r} < \kappa u_0, \tag{58}$$

and hence $\sin\psi(r) < \kappa u_0 r = 1$. This means that $\psi(r) < \pi/2$. q.e.d.

The next result gives an estimate of the first zero R of u, $u(R) = 0$, in the sense that, fixing the constant κ, the value R remains bounded in some interval, independent of the initial condition u_0.

Theorem 6.2. *Let $\kappa < 0$ and $u(r; u_0)$ a solution of (4)–(14) such that u_0 satisfies (56). Then*

$$\frac{1}{\sqrt{-2\kappa}} < R < \sqrt{\frac{-2e}{\kappa}}. \tag{59}$$

Proof. Since $u(R) = 0$, equation (57) implies that $\cos\psi(R) = 1 + \kappa u_0^2/2$. From (58),

$$R > \frac{\sin\psi(R)}{\kappa u_0} = \frac{1}{\kappa u_0}\sqrt{1 - \cos^2\psi_R} = \frac{-1}{2\kappa}\sqrt{-4\kappa - \kappa^2 u_0^2}.$$

The left side of (59) is then a consequence of this inequality and (56). Now, we show the inequality on the right side of (59). In the region where $u < 0$, $\sin\psi(r)$ increases on r. Let us fix a such that $0 < a < R$. Then $\sin\psi(r) > \sin\psi(a)$. As $\sin\psi < \tan\psi$, we have

$$u' = \tan\psi > \sin\psi \geqslant \frac{a}{r}\sin\psi(a).$$

An integration between a and R gives

$$R < a \exp\left\{\frac{-u(a)}{\sin\psi(a)}\right\}.$$

Again (58) leads to

$$R < a \exp\left\{\frac{-1}{\kappa a^2}\right\}. \tag{60}$$

Since this holds for every $a < R$ and the function on a on the right side of (60) attains a minimum at $a = \sqrt{-2/\kappa}$, we obtain the desired estimate. q.e.d.

We follows the study of pendant cylindrical surfaces with the case

$$z_0 = -\sqrt{\frac{2}{-\kappa}}.$$

This setting can be treated as in theorem 6.1, except that in a discrete set of points, u is vertical. These points are the zeros of u by using (57). Next, let us assume that $z_0 < -\sqrt{-2/\kappa}$.

Theorem 6.3. *Let α the directrix of a κ-cylindrical surface, $\kappa < 0$, such that the initial condition $z(0) = z_0$ satisfies*

$$-\frac{2}{\sqrt{-\kappa}} < z_0 < -\sqrt{\frac{-2}{\kappa}}. \tag{61}$$

Then the function $z(s)$ is periodic. Furthermore,

(i) the directrix α presents exactly four vertical points in each period of $z(s)$,
(ii) each vertical point lies in the segment of α between one extremum and one zero of $z(s)$ and
(iii) the heights of the vertical points are $\pm\sqrt{z_0^2 + 2/\kappa}$.

Proof. By using theorem 3.5, the function $z(s)$ is periodic. By the symmetries of α, it suffices to prove that between $s = 0$ and the first time s_0 where α intersects the r-axis, there exists exactly one vertical point. Since $\theta'(s) = \kappa z(s)$, the function θ increases on r in the interval $(0, s_0)$. We know that θ attains the value $\theta = \pi/2$, the first vertical point, at some point s^*, with $s^* < s_0$. By using again (8) and (61), θ does not reach the value $\theta = \pi$, which shows that it is the unique vertical point. By (8), the height at $s = s^*$ is $-\sqrt{z_0^2 + (2/\kappa)}$. q.e.d.

Remark 6.1. It is possible to determine the region where this first vertical point occurs. For this purpose, one can carry as in the case of pendant liquid drops by using a 'comparison lemma' that compares u with circular arcs and the hyperbolae $ru < 1/(2\kappa)$ and $ru < 1/\kappa$ ([3] and [7, chapter 4.6]). For example, one can show that the directrix α, in the initial region $z < 0$, does not enter the region $ru \leqslant 1/\kappa$.

We now give a complete discussion of shapes of pendant liquid channels under hypothesis (61).

Theorem 6.4. *Let $\alpha(s)$ be the directrix of a κ-cylindrical surface S. Assume that z_0 satisfies (61). Then there exist numbers z_1, z_2, with*

$$-\frac{2}{\sqrt{-\kappa}} < z_2 < z_1 < -\frac{\sqrt{2}}{\sqrt{-\kappa}}$$

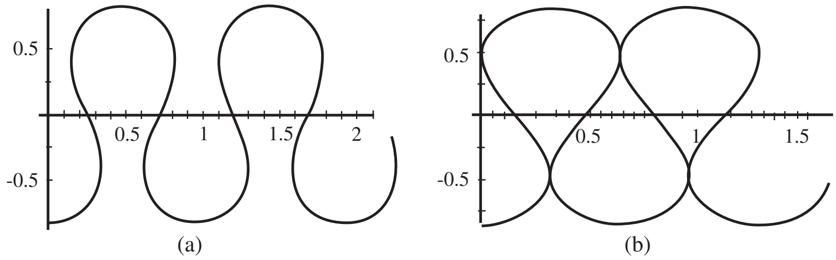

Figure 7. Directrix of a κ-cylindrical surface, with $\kappa = -4$. Case (a) $z_0 = -0.82$; case (b) $z_0 \sim -0.855$.

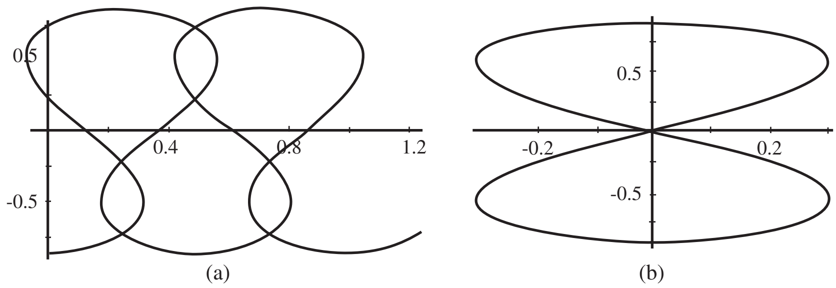

Figure 8. Directrix of a κ-cylindrical surface, with $\kappa = -4$. Case (a) $z_0 = -0.87$, case (b) $z_0 \sim -0.908$.

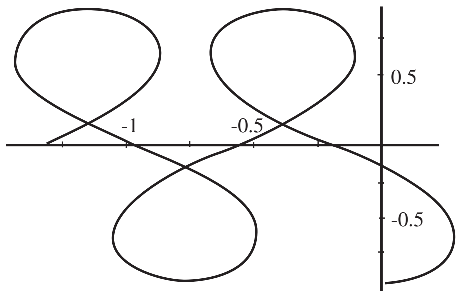

Figure 9. Directrix of a κ-cylindrical surface, with $\kappa = -4$. Here $z_0 = -0.95$.

and such that the following properties hold.

(i) If $z_1 < z_0$, then α has no double points and x goes to ∞, figure 7(a).

(ii) If $z_0 = z_1$, then α has double points, where α tangentially meets itself. Moreover α lies in the halfplane $\{x \geqslant 0\}$ and x goes to ∞, figure 7(b).

(iii) If $z_2 < z_0 < z_1$, α has double points, meeting at these points transversally and x goes to ∞, figure 8(a).

(iv) If $z_0 = z_2$, then α is a closed curve with self intersection at the origin, figure 8(b).

(v) If $z_0 < z_2$, α has double points, where α meets itself transversally and x goes to $-\infty$, figure 9.

Proof. Denote $r(\psi_0)$ and $r(\pi/2)$ the x-coordinates of the first point at which α meets the x-axis and the first vertical point, respectively. We know that $r(\pi/2) > r(\psi_0)$. q.e.d.

Claim 1. $r(\pi/2) \leqslant \pi/(2\sqrt{-2\kappa})$, independent of the initial value z_0.

Proof (Of the claim 1). For each $0 \leqslant \psi \leqslant \pi/2$ and by using (57) and (61), we obtain

$$\kappa u(\psi) > -\kappa\sqrt{\frac{-2\cos\psi}{\kappa}}.$$

Since $\cos\psi > 0$, we have from (42) that

$$\frac{\mathrm{d}r}{\mathrm{d}\psi} < \frac{\sqrt{\cos\psi}}{\sqrt{-2\kappa}} < \frac{1}{\sqrt{-2\kappa}}.$$

An integration of these inequalities from $\psi = 0$ to $\psi = \pi/2$ proves claim 1. q.e.d.

Let us denote $r(\psi_0; z_0)$ and $r(\pi/2; z_0)$ to emphasize the dependence on z_0.

Claim 2. There exists a continuous function $\varphi = \varphi(z_0)$, strictly decreasing on z_0, such that $r(\psi_0; z_0) < \varphi(z_0)$ and

$$\lim_{z_0 \to -2/\sqrt{-\kappa}} \varphi(z_0) = -\infty.$$

Proof (Of the claim 2). Consider $\psi \in [\pi/2, \psi_0]$. Since $u(\psi_0) = 0$, by (57), we have

$$1 - \cos\psi_0 = -\frac{\kappa}{2}z_0^2.$$

This proves that as $z_0 \to -2/\sqrt{-\kappa}$, the angle ψ_0 at which the directrix α meets the r-axis goes to $\psi = \pi$. As $z_0^2 < -4/\kappa$, again (57) leads to

$$u(\psi) \geqslant -\sqrt{\frac{-2(1+\cos\psi)}{\kappa}}.$$

Then (42) implies

$$\frac{\mathrm{d}r}{\mathrm{d}\psi} < \frac{1}{\sqrt{-2\kappa}}\frac{\cos\psi}{\sqrt{1+\cos\psi}}.$$

An integration from $\psi = \pi/2$ until $\psi = \psi_0$ yields

$$r(\psi_0) - r(\pi/2) < \frac{2}{\sqrt{-\kappa}}\left(\sin\frac{\psi_0}{2} - \operatorname{arctanh}\left(\tan\frac{\psi_0}{4}\right) - \frac{\sqrt{2}}{2} + \operatorname{arctanh}\left(\tan\frac{\pi}{8}\right)\right).$$

From claim 1, $r(\pi/2; z_0)$ is bounded. Then, upto a constant C,

$$r(\psi_0; z_0) < \frac{2}{\sqrt{-\kappa}}\left(\sin\frac{\psi_0}{2} - \operatorname{arctanh}\left(\tan\frac{\psi_0}{4}\right)\right) + C := \varphi(z_0).$$

Finally, as $z_0 \to -2/\sqrt{-\kappa}$, $\psi_0(z_0) \to \pi$ and then $\varphi \to -\infty$. q.e.d.

We know from (57) that for two values of z_0, namely, a and b, if $a < b$, then $\cos\psi_0(a) < \cos\psi_0(b)$. Moreover, claim 3 follows

Claim 3. The function $r(\psi_0; z_0)$ is strictly decreasing on z_0.

Proof (Of claim 3). Consider $a < b$ and denote α_a and α_b the corresponding directrix curves, respectively, for these initial conditions. Reasoning similar that in theorem 5.5 proves that if $\delta < 0$, $u(r; z_0 + \delta) + \delta > u(r; z_0)$ (by lemma 6.1, $r(\pi/2; z_0 + \delta) < r(\pi/2; z_0)$). This shows that if we move α_a upwards until we arrive at the point $(0, b)$, α_a lies over α_b at least until the first vertical point of α_a. Then α_a, in the new position, lies over α_b at least until both α_a and

α_b meet at the r-axis. If $\bar{r}$ is the first zero of (the displaced curve) α_a, we have

$$r(\psi_0(a)) < \bar{r} < r(\psi_0(b)). \qquad \text{q.e.d.}$$

Now, we sketch the proof of the theorem and we omit the details. Take z_0 varying from $z_0 = -\sqrt{-2/\kappa}$ until $z_0 = -2/\sqrt{-\kappa}$. With the notation of (11), the period of $z(s)$ is $T = 4s_0$, with $x(s_0) = r(\psi_0)$. Let z_1 and z_2 be the unique numbers, $z_2 < z_1$, such that, in the notation of (11), there hold

$$x\left(s_0 + x^{-1} \circ r\left(\frac{\pi}{2}\right); z_1\right) = 0 \qquad \text{and} \qquad r(\psi_0; z_2) = 0.$$

This means that z_1 is the initial condition where the x-coordinate of the second vertical point vanishes and z_2 is the initial condition in (14) where the x-coordinate of the first zero of $z(s)$ is 0. The existence is given by claim 2 and the uniqueness by claim 3. By (11), the direction, the left or the right one, that α takes depends on the sign of $x(4s_0)$. The critical time occurs when $x(4s_0) = x(s_0) = 0 = r(\psi_0; z_2)$, where α is a closed curve. Moreover, since z increases in the interval $(0, 2s_0)$ (and z goes from z_0 to $-z_0$), the curve α does not intersect itself in this range of values of s. Then the results follow using the symmetries of α according to theorems 3.2 and 3.3. q.e.d.

Here we discuss the behaviour of the directrix depending on the initial value z_0, with $-2/\sqrt{-\kappa} \leqslant z_0 < 0$. For the rest of the values of z_0, that is, when $z_0 < -2/\sqrt{-\kappa}$, the solution α was studied in theorem 3.5.

Remark 6.2. The results obtained here show the contrast of behaviour between pendant liquid channels and pendant liquid rotational drops. In the latter, the profile curve α is a graph on the x-axis when $z_0 < 0$ is sufficiently close to 0. Letting $z_0 \to -\infty$, there comes a time when they appear as vertical points. The number of vertical points indefinitely increases ([3]). In contrast, in our setting, we have proved that the function $z(s)$ is periodic. Moreover, we show that in each period of $z(s)$, the number of vertical points is at most four. This depends on what range the value z_0 lies in, going from zero to four, one and two points when z_0 goes from $z_0 = 0$ to $-\infty$: see theorems 6.1, 6.3 and 3.5.

7. Summary and conclusions

Motivated by possible applications in the field of microfluidic systems, we have studied the shapes of solutions of the capillary equation $\text{div}(Tu) = \kappa u$, $\kappa \neq 0$, on a strip $\Omega = \{(x, y), |x| < a\}$. We search solutions invariant in the y-direction, that is, $u(x) = u(x, y)$. Such surfaces $z = u(x, y)$ are models of liquid–air interfaces that appear in two settings. The first one consists of introducing two parallel plates in a reservoir of liquid and vertically positioned at $\partial\Omega$. During the course of this work, we have tried to analyse the shapes of the capillary meniscus depending on the sign of κ. We have proved that the height coordinate of the surface is a periodic function in the sessile case ($\kappa > 0$) and in almost all cases in the pendant situation ($\kappa < 0$). For this analysis, we used a first integration obtained for the height coordinate of the surface. When $\kappa > 0$, the surface is invariant by horizontal translations orthogonal to the vertical plates and it lies completely on one side of the reference level $\Pi = \{z = 0\}$. Moreover, the velocity vector of its directrix curve indefinitely rotates around the origin. In the pendant case, the morphologies of the meniscus depend on the range in which the initial condition $u(0) = u_0$ lies. If u_0 is near 0, then the surface is a graph on Π with points to both sides of this plane. But if we let $u_0 \to -\infty$, the capillary surface ceases to be a graph, with vertical points appearing until that it comes completely below Π. Under this situation, the surface has similar properties as in the sessile case. We have obtained exact control of the number of these vertical points (see remark 6.2). We point out that this contrasts with the shapes of rotational pendant liquid drops, where the number of vertical points grows indefinitely as $u_0 \to -\infty$.

Together with this analysis of the morphologies, we present and develop a set of estimates of capillary shapes. Our basic tool to calculate these estimates has been the comparison of the interface with pieces of cylinders horizontally positioned on Ω and whose axis is parallel to the y-axis. For example, we estimate the centre height of the meniscus, similar to the classical formula obtained by Laplace for rotational liquid drops.

The second setting that we have considered has been the study of a liquid channel confined on a chemical wetting strip Ω embedded in a solid substrate, which is non-wetting. We assume that the liquid is homogeneous in the y-direction. A first interest has been the analysis of the volume for a sessile liquid channel in relation to the boundary data of the capillary surface. Here, by volume of the liquid channel we mean its volume per unit of length. We have proved the existence and the uniqueness of a liquid channel for the given values of volume and contact angle with the support plane Π. Using comparisons with halfcylinders again, we have obtained upper and lower bounds of the volume. In this sense, our work could be applied in the context of spreading liquid on patterns made by a set of parallel hydrophilic strips sandwiched by a hydrophobic substrate. The liquid is driven by wettability and guided along the hydrophilic strips. When the amount of liquid grows, two consecutive strips can touch. The results obtained show, at least from a theoretical point of view, that it is possible to control the distance between two consecutive liquid channels on hydrophilic strips without touching, in terms of data such as the contact angle, the capillary constant or the centre height.

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